UNIVERSITI PUTRA MALAYSIA

SOLVING HIGHER ORDER DELAY DIFFERENTIAL EQUATIONS WITH BOUNDARY CONDITIONS USING MULTISTEP BLOCK METHOD

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IPM 2021 15

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By

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Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Doctor of Philosophy

April 2021

DEDICATIONS

To mak and ayah

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfillment of the requirement for the degree of Doctor of Philosophy.

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April 2021

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In this thesis, we derived two numerical methods called two point diagonally multistep block method order four and order five with the approach of predictor-corrector technique to solve higher order delay differential equations (DDEs) with boundary conditions. Shooting technique by using the Newton's like method is implemented to solve the boundary value problems (BVPs). This thesis begins with solving second order DDEs with constant, pantograph and time dependent delay type by using both methods. Then, those methods are extended to solve third order DDEs with constant and pantograph delay type.

The approach used to solve constant delay type is by taking the previously calculated solutions at the delay terms while for pantograph and time dependent delay types, the approaches are by using the Lagrange interpolation to approximate the solutions at the delay terms. The derivatives present in the problems at the delay terms will be approximated by using the finite difference method. The analysis of both methods in terms of order, local truncation error and stability are also investigated. Two stability test equations are used to analyze the stability regions of the block methods.

Several numerical problems are illustrated to solve by using C programming. The accuracy of the methods in terms of maximum and average errors along with the total function calls, total iteration steps, total guessing numbers for shooting technique are discussed and compared with the previous methods.

In conclusion, the higher order DDEs with boundary conditions can be solved by using the proposed block methods based on the analysis of the methods and their numerical results.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

MENYELESAIKAN PERSAMAAN PEMBEZAAN LEWAT PERINGKAT TINGGI DENGAN SYARAT SEMPADAN MENGGUNAKAN KAEDAH BLOK MULTI-LANGKAH

Oleh

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Dalam tesis ini, kita memperolehi dua kaedah berangka yang dipanggil kaedah blok multi-langkah dua titik pepenjuru peringkat empat dan peringkat lima dengan pendekatan teknik peramal-pembetul untuk menyelesaikan persamaan pembezaan lewat (PPL) peringkat tinggi dengan syarat sempadan. Teknik penembakan dengan menggunakan kaedah bak-Newton dilaksanakan untuk menyelesaikan masalah nilai sempadan (MNS). Tesis ini bermula dengan menyelesaikan PPL peringkat kedua dengan jenis lewat malar, pantograf dan bersandar masa dengan menggunakan kedua-dua kaedah. Kemudian, kaedah-kaedah itu diperluaskan untuk menyelesaikan PPL peringkat ketiga dengan jenis lewat malar dan pantograf.

Pendekatan yang digunakan untuk menyelesaikan jenis lewat malar ialah dengan mengambil penyelesaian yang dikira sebelum ini pada terma lewat sementara bagi jenis lewat pantograf dan persandaran masa, pendekatannya adalah dengan menggunakan interpolasi Lagrange untuk menganggarkan penyelesaian pada terma lewat. Terbitan yang hadir dalam masalah pada terma lewat akan dianggarkan dengan menggunakan kaedah perbezaan terhingga. Analisis kedua-dua kaedah dari segi peringkat, ralat pemusnahan tempatan dan kestabilan juga disiasat. Dua persamaan ujian kestabilan digunakan untuk menganalisis kawasan kestabilan kaedah blok.

Beberapa masalah berangka digambarkan untuk diselesaikan dengan menggunakan pengaturcaraan C. Ketepatan kaedah dari segi ralat maksimum dan purata ralat

bersama-sama dengan jumlah panggilan fungsi, jumlah langkah lelaran, jumlah bilangan meneka untuk teknik penembakan dibincangkan dan dibandingkan dengan kaedah sebelumnya.

Sebagai kesimpulan, PPL peringkat tinggi dengan syarat sempadan boleh diselesaikan menggunakan kaedah-kaedah blok yang dicadangkan tersebut berdasarkan analisis kaedah dan hasil berangka.

ACKNOWLEDGEMENTS

My utmost gratitude goes to my supervisor, Prof. Dr. Zanariah Abdul Majid for her diligence, patience and constant support in guiding me through this journey. I am so bless for having Prof as my supervisor through my ups and downs of my journey.

My special thanks goes to my parents, mak, Bashirah Hassan and ayah, Jaaffar Lah, that encouraged me to take further studies and constantly support me throughout this journey. This thesis is personally dedicated to them. My deepest gratitude goes to my friends and labmates that always guiding, helping, comforting and delighting me through this 3 years journey. Not to forget, thanks to my buddies that help me shaping my life more clearly along this 3 years phase of my life. To my brothers, Adin, thank you for cheering me up and Akir, thank you for everything.

Last but not least, I felt thankful to my co-supervisors, Dr. Norazak Senu and Dr. Fadzilah Md Ali for their suggestions in my research.

This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Doctor of Philosophy. The members of the Supervisory Committee were as follows:

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LIST OF ABBREVIATIONS

DDE	Delay Differential Equation
BVP	Boundary Value Problem
IVP	Initial Value Problem
ODE	Ordinary Differential Equation
LMM	Linear Multistep Method
FDDE	Fractional Delay Differential Equation
VIDDE	Volterra Integro Delay Differential Equation
NDDE	Neutral Delay Differential Equation

CHAPTER 1

INTRODUCTION

1.1 Background

Delay differential equations (DDEs) are the mathematical model in various real life problems for example in population dynamic, there are delays in breeding and maturation periods of species. Delay also exists in the time that the signal takes to travels to the controlled object, the reaction time, and the time that the signal takes to return in control circuits. Biological process portrays delay in the cell division time and in cell producing time. Medicine field also possesses delay in the model of chronic granulocytic leukaemia (CGL) by Wheldon et al. (1974). There are several detail studies by Driver (1977) on how to form a mathematical model of real life problems in terms of DDEs.

DDEs in mathematics are the differential equations that take into consideration the functions at the past and present time. The solutions of DDEs can be determined by using the exact solutions of the problems however, not all DDEs acquire the exact solutions. Therefore, the solutions can be computed by using the approximate solutions with the aid of various numerical methods.

The solutions at the delay can be executed by taking the initial functions given in the interval of the past time. The initial functions are the smooth exact solutions of the problems. However, delay can also fall in the outside of the interval of the past time, causes a new approach to execute the solutions. If the delay falls in the interval of previously calculated solutions, there would be easier to just take the solutions but if the delay does not fall in the mentioned interval, interpolation method will be implemented to execute the approximate solutions.

The numerical method used in this research is direct block multistep method in the predictor-corrector scheme. Since the focus of the research is higher order DDEs, hence, direct method is needed to ensure the total function calls are reduced without transforming the higher order DDEs to the first order DDEs. Block method is essential to reduce the total number of steps in the iterations. Based on (Lambert, 1973), the implicit method is preferable to the explicit method due to the former has a smaller error constant and bigger size of the interval of absolute stability. This implicit method is called corrector and is often used in pair with the predictor method (explicit). The predictor is needed to guess the implicit dependent variable as accurate as possible before we corrected it by using the implicit method. Subsequently, with all these approaches eventually reduce the computation time needed for C program to compute while attaining the accuracy of the method.

Besides, the boundary conditions presented in the problems require a particular approach. The Newtons-like method in shooting technique will be used to cater this problem. The boundary value problems (BVPs) are transformed to two initial value problems (IVPs) so that the solutions of these two IVPs will be used in Newtons-like formula to obtain the most accurate value for the unknown initial condition needed.

1.2 Problem Statement

DDEs have been arise in practical problems for such a long time and it happens everywhere in our life that to ignore it is impossible. Some modelers ignore the ”lag” effect and use an ordinary differential equation (ODE) model as a substitute for a DDE model, however, based on Kuang (1993), even small delays can have large effects. The fact that many phenomena frequently modeled by ODEs can be better modeled by DDEs has not escaped the attention of the numerical analysis community (Baker et al., 1995), hence this became the motivation of our studies.

DDE is a differential equation that consist of constant and time dependent delay term. Constant delay exists when the delay term is constant while time dependent delay exists when the delay term is a function of variable x. Different approaches are needed to handle different type of delays. The numerical methods for solving DDEs are adapted from numerical methods for ODEs such as linear multistep method. The proposed block multistep method will be paired with the delay implementation such as Lagrange interpolation method to obtain the solutions.

1.3 Delay Differential Equations

DDEs generally can be illustrated in the following form

$$y^{(n)} = f\Big(x, y(x), y'(x), \ldots, y^{(n-1)}(x), y(x-\tau(x,y(x))), y'(x-\tau(x,y(x))), \ldots, y^{(n-1)}(x-\tau(x,y(x)))\Big).$$

The function $\tau(x,y(x))$ is called the delay, the argument $x-\tau(x,y(x))$ is called the delay terms, $y(x-\tau(x,y(x)))$ and $y'(x-\tau(x,y(x)))$ are respectively called as the solutions of delay terms and it's derivatives. DDEs can be split in three different types which are $\tau \in \Re^{+}$ (constant delay), $\tau(x)$ (time dependent delay) and $\tau(x,y(x))$ (state dependent delay) (Bellen and Zennaro, 2003). The focus of this research is only the constant and time dependent delay type.

The general form of second order DDEs of constant delay type is given by

$$\begin{aligned} y'' &= f\big(x, y(x), y'(x), y(x-\tau), y'(x-\tau)\big), \quad && x \in [a,b], \\ y(x) &= \phi(x), && x \in [a-\tau, a], \end{aligned} \tag{1.1}$$

with boundary conditions $y(a) = \phi(a) = \alpha$ and $y(b) = \beta$ where $\tau \in \Re^+$ and $\phi(x)$ is the smooth initial function given in the problems.

The general form of second order DDEs of time dependent delay type is given by

$$\begin{aligned} y'' &= f\big(x, y(x), y'(x), y\big(x-\tau(x,y(x))\big), y'\big(x-\tau(x,y(x))\big)\big), \quad && x \in [a,b], \\ y(x) &= \phi(x), && x \in [a-\tau, a], \end{aligned} \tag{1.2}$$

Another delay type called pantograph delay has also been focused on this research. Pantograph delay is the subset of time dependent delay type but has special form. The general form of second order DDEs of pantograph delay type is given by

$$y'' = f\big(x, y(x), y'(x), y(qx), y'(qx)\big), \quad x \in [a,b], \tag{1.3}$$

where $0 < q < 1$. Since the delay term qx always falls in the interval $[a,b]$, therefore there is no need for the initial function, $\phi(x)$ given in the problems.

1.4 Boundary Value Problems

BVPs exist in various aspects of real life problems especially in physics and engineering field such as in fluid dynamics, soil problems, shock wave problems, optimal control problems and many more. BVPs in mathematics defined as the differential equations with the solutions are specified at more than one point. Mostly, there are two points which are physically at the boundaries of some regions called as the two points BVPs. These two points are called as the boundary conditions. The general form of BVPs are as the following

$$y^{(n)} = f\big(x, y(x), y'(x), y''(x), ..., y^{(n-1)}(x)\big).$$

This research is only focus on the second and third order of BVPs. The second order BVP needs two boundary conditions while the third order BVP needs three boundary conditions. The boundary conditions for second order BVPs can be defined in three types which are Dirichlet type, Neumann type and Robin type as the following

1. Dirichlet type: The boundary conditions are given at the primary dependent variable such as

$$y(a) = \alpha, \quad y(b) = \beta,$$

where a, b, α and β are constants.

2. Neumann type: The boundary conditions are given at the derivative of the primary dependent variable such as

$$y'(a) = \alpha, \quad y'(b) = \beta,$$

where a, b, α and β are constants.

3. Robin type: The boundary conditions are given at the linear combination of primary dependent variable and it's derivatives such as

$$c_1 y(a) + c_2 y'(a) = \alpha, \quad c_3 y(b) + c_4 y'(b) = \beta,$$

where $c_1, c_2, c_3, c_4, a, b, \alpha$ and β are constants and $c_1, c_2, c_3,$ and c_4 are nonzeros.

This research scopes are the Dirichlet boundary conditions for second order BVPs and Type I, Type II, and Type III boundary conditions for third order BVPs. The three types boundary conditions for the third order BVPs can be defined as the following (Pei See, 2015)

1. Type I: $y(a) = \alpha, \quad y'(a) = \gamma, \quad y(b) = \beta$, where a, b, α, β and γ are constants.
2. Type II: $y(a) = \alpha, \quad y'(a) = \gamma, \quad y'(b) = \beta,$
3. Type III: $y(a) = \alpha, \quad y''(a) = \gamma, \quad y(b) = \beta.$

This research scope is at all the three types boundary conditions mentioned for third order BVPs. The existence and uniqueness theory plays a role in illustrating and analyzing numerical methods for solving BVPs to ensure that the solution of BVPs exists and unique. The following theorem consists of the general conditions for the existence and uniqueness solutions of second order BVPs.

Theorem 1.1 *(Burden and Faires, 2011)*
Suppose the function f in the BVPs of the following

$$y'' = f(x, y, y'), \quad x \in [a, b], \quad y(a) = \alpha, y(b) = \beta,$$

is continuous on the set

$$D = \{(x, y, y') | for \quad x \in [a, b], \quad y \in (-\infty, \infty), \quad y' \in (-\infty, \infty)\},$$

and that the partial derivatives $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial y'}$ are also continuous on D. If

1. $\frac{\partial f}{\partial y}(x, y, y') > 0$, for all $(x, y, y') \in D$, and

2. *a constant M exists, with*

$$\left|\frac{\partial f}{\partial y'}(x,y,y')\right| \le M, \quad for \quad all \quad (x,y,y') \in D,$$

then the BVPs have unique solutions.

The detail proof of this theorem can be found in Keller (2018).

1.5 Linear Multistep Method

Consider the general second order ordinary differential equations (ODEs) as follows

$$y'' = f(x,y,y'), \quad x \in [a,b]. \tag{1.4}$$

Most numerical methods used to seek the approximate solutions of the above differential equations are based on the approach of discretization. Discretization means to seek the solutions not in the continuous interval $[a,b]$ but on the discrete point sets $\{x_i | i = 0,1,2,...,(b-a)/h\}$ where h is the step size chosen. Let y_i be an approximation to the theoretical solution at x_i. If a numerical method to determine the sequence of $\{y_i\}$ is in the form of a linear relationship between $y_{i+j}, f_{i+j}, j = 0,1,..,k$, then this method is named as the linear multistep method (LMM) of stepnumber k. The LMM for second order differential equations can be written

$$\sum_{j=0}^{k} \alpha_j y_{i+j} = h \sum_{j=0}^{k} \beta_j y'_{i+j} + h^2 \sum_{j=0}^{k} \gamma_j f_{i+j}, \tag{1.5}$$

where $f_{i+j} = y''_{i+j}$ for $j = 0,1,2,..,k$. α_j, β_j, and γ_j are constants and also throughout this thesis we shall assume that $\alpha_k = 1$. The method is said to be explicit if $\beta_k = \gamma_k = 0$ and implicit if $\beta_k \neq 0$ or $\gamma_k \neq 0$.

The linear difference operator engaged with the LMM (1.5) is introduced as the following:

$$L\big[y(x);h\big] = \sum_{j=0}^{k} \big[\alpha_j y(x+jh) - h\beta_j y'(x+jh) - h^2\gamma_j y''(x+jh)\big], \tag{1.6}$$

where $y(x)$ is a theoretical solution that is continuously differentiable on $[a,b]$. The reason to introduce this operator is to assume the arbitrary function $y(x)$ to have as many higher derivatives as possible to define the order of accuracy of the operator. Now, the functions $y(x+jh)$, $y'(x+jh)$ and $y''(x+jh)$ are approximated by using

Taylor series about x as follows

$$\begin{aligned}
y(x+jh) &= y(x) + jhy'(x) + \frac{(jh)^2}{2!}y''(x) + \frac{(jh)^3}{3!}y'''(x) + \ldots + \frac{(jh)^p}{p!}y^{(p)}(x) + O(h^{p+1}),\\
y'(x+jh) &= y'(x) + jhy''(x) + \frac{(jh)^2}{2!}y'''(x) + \frac{(jh)^3}{3!}y^{(4)}(x) + \ldots + \frac{(jh)^{p-1}}{(p-1)!}y^{(p)}(x)\\
&\quad + O(h^p),\\
y''(x+jh) &= y''(x) + jhy'''(x) + \frac{(jh)^2}{2!}y^{(4)}(x) + \frac{(jh)^3}{3!}y^{(5)}(x) + \ldots + \frac{(jh)^{p-2}}{(p-2)!}y^{(p)}(x)\\
&\quad + O(h^{p-1}).
\end{aligned}$$

After substituting the above approximations into (1.6) and collecting terms, finally the operator can be defined as (Mohd Nasir et al., 2018)

$$\begin{aligned}
L[y(x);h] = \sum_{j=0}^{k} \Bigg[&\alpha_j y(x) + (j\alpha_j - \beta_j)hy'(x) + \left(\frac{j^2}{2!}\alpha_j - j\beta_j - \gamma_j\right)h^2 y''(x)\\
&+ \left(\frac{j^3}{3!}\alpha_j - \frac{j^2}{2!}\beta_j - j\gamma_j\right)h^3 y'''(x) + \ldots + \left(\frac{j^p}{p!}\alpha_j - \frac{j^{p-1}}{(p-1)!}\beta_j\right.\\
&\left. - \frac{j^{p-2}}{(p-2)!}\gamma_j\right)h^p y^{(p)}(x)\Bigg] + O(h^{p+2}).
\end{aligned} \tag{1.7}$$

Supposed that

$$\begin{aligned}
C_0 &= \sum_{j=0}^{k} \alpha_j,\\
C_1 &= \sum_{j=0}^{k} (j\alpha_j - \beta_j),\\
C_2 &= \sum_{j=0}^{k} \left(\frac{j^2}{2!}\alpha_j - j\beta_j - \gamma_j\right),\\
C_3 &= \sum_{j=0}^{k} \left(\frac{j^3}{3!}\alpha_j - \frac{j^2}{2!}\beta_j - j\gamma_j\right),\\
&\vdots\\
C_p &= \sum_{j=0}^{k} \left(\frac{j^p}{p!}\alpha_j - \frac{j^{p-1}}{(p-1)!}\beta_j - \frac{j^{p-2}}{(p-2)!}\gamma_j\right),
\end{aligned} \tag{1.8}$$

then the linear difference operator (1.7) will be

$$\begin{aligned}
L[y(x);h] &= C_0 y(x) + C_1 hy'(x) + C_2 h^2 y''(x) + C_3 h^3 y'''(x) + \ldots + C_p h^p y^{(p)}(x)\\
&\quad + C_{p+1} h^{p+1} y^{(p+1)}(x) + O(h^{p+2}).
\end{aligned}$$

Definition 1.1 *(Lambert, 1973)*
The LMM (1.5) is said to be of order p if $C_0 = C_1 = C_2 = \ldots = C_{p+1} = 0$ *and* $C_{p+2} \neq 0$.

From Definition 1.1, the first non vanishing coefficient which is C_{p+2} is called an error constant of the LMM.

Now, introducing the first, second and third characteristic polynomials of the LMM are defined as $\rho(R), \sigma(R)$ and $\omega(R)$ respectively as the following

$$\rho(R) = \sum_{j=0}^{k} \alpha_j R^j, \quad \sigma(R) = \sum_{j=0}^{k} \beta_j R^j, \quad \omega(R) = \sum_{j=0}^{k} \gamma_j R^j, \tag{1.9}$$

where $R \in \mathbb{C}$. zero-stability is the stability of the LMM concerning only in the limit as the step size, h approaching zero. A zero stable method means that all the roots of the first characteristic polynomial (because we only consider at $h \to 0$), $\rho(R)$ lie in or on the unit circle, those on the circle being simple hence to why R is defined as complex numbers. Thus, definition of zero stable is as follows,

Definition 1.2 *(Lambert, 1973)*
The LMM (1.5) is said to be zero stable if no root of the first characteristic polynomial, $\rho(R)$ *has modulus greater than one, and if every root with modulus one has multiplicity not greater than two.*

The truncation error, T_{i+k} is local if there is no previous truncation error have been made and this is called localizing assumption. Assume that

$$y_{i+j} = y(x_{i+j}), \quad j = 0, 1, ..., k-1$$

where y_{i+k} is the approximate solution of the LMM (1.5) at x_{i+k}.
From (1.6)

$$\begin{aligned}\sum_{j=0}^{k} \alpha_j y(x_i + jh) =& h \sum_{j=0}^{k} \beta_j y'(x_i + jh) + h^2 \sum_{j=0}^{k} \gamma_j f\big(x_i + jh, y(x_{i+j}), y'(x_{i+j})\big) \\ &+ L[y(x_i); h]\end{aligned} \tag{1.10}$$

where $y(x)$ is considered to be the exact solution of the problems. The approximate solution, y_{i+k} of the method (1.5) satisfies

$$\begin{aligned}y_{i+k} + \sum_{j=0}^{k-1} \alpha_j y_{i+j} =& h \sum_{j=0}^{k-1} \beta_j y'_{i+j} + h^2 \sum_{j=0}^{k-1} \gamma_j f(x_{i+j}, y_{i+j}, y'_{i+j}) + h\beta_k y'_{i+k} \\ &+ h^2 \gamma_k f(x_{i+k}, y_{i+k}, y'_{i+k})\end{aligned} \tag{1.11}$$

where $\alpha_k = 1$. Subtracting (1.11) from (1.10) and using the localizing assumption, eventually gives

$$y(x_{i+k}) - y_{i+k} = h\beta_k[y'(x_{i+k}) - y'_{i+k}] + h^2\gamma_k\big[f\big(x_{i+k}, y(x_{i+k}), y'(x_{i+k})\big)$$
$$-f(x_{i+k}, y_{i+k}, y'_{i+k})\big] + T_{i+k}$$

By using mean value theorem,

$$f\big(x_{i+k}, y(x_{i+k}), y'(x_{i+k})\big) - f(x_{i+k}, y_{i+k}, y'_{i+k}) = [y(x_{i+k}) - y_{i+k}]\frac{\delta f}{\delta y}(\eta_{i+k})$$
$$+[y'(x_{i+k}) - y'_{i+k}]\frac{\delta f}{\delta y'}(\eta_{i+k})$$

where η_{i+k} is in the interval whose endpoints are $\big(x_{i+k}, y(x_{i+k}), y'(x_{i+k})\big)$ and $(x_{i+k}, y_{i+k}, y'_{i+k})$. Thus,

$$y(x_{i+k}) - y_{i+k} = h\beta_k[y'(x_{i+k}) - y'_{i+k}] + h^2\gamma_k\left[\big[y(x_{i+k}) - y_{i+k}\big]\frac{\delta f}{\delta y}(\eta_{i+k})\right.$$
$$\left.+\big[y'(x_{i+k}) - y'_{i+k}\big]\frac{\delta f}{\delta y'}(\eta_{i+k})\right] + T_{i+k}$$

$$T_{i+k} = \big[y(x_{i+k}) - y_{i+k}\big]\big[1 - h^2\gamma_k\frac{\delta f}{\delta y}(\eta_{i+k})\big] - \big[y'(x_{i+k}) - y'_{i+k}\big]\big[h\beta_k + h^2\gamma_k\frac{\delta f}{\delta y'}(\eta_{i+k})\big]$$

Hence, for an explicit method which is $\beta_k = 0$ and $\gamma_k = 0$, the local truncation error (LTE), T_{i+k} is then

$$T_{i+k} = y(x_{i+k}) - y_{i+k}$$

where LTE is the difference between the theoretical solution and the approximate solution of the LMM (1.5). Therefore, the LTE is approaching to this difference when h approaching to zero for an implicit method.

If we consider that theoretical solution $y(x)$ has continuous derivatives of sufficiently high order, then for both explicit and implicit methods, we can deduce that the LTE is as the following

$$T_{i+k} = y(x_{i+k}) - y_{i+k} = C_{p+2}h^{p+2}y^{(p+2)}(x_i) + O(h^{p+3}) \tag{1.12}$$

Thus, the definition of LTE is as follows,

Definition 1.3 *(Lambert, 1973)*
The LTE at x_{i+k} of the method LMM (1.5) is defined to be the linear difference operator, L given by (1.6) when $y(x)$ is the theoretical solution of the problems.

From the LTE, the consistency of the method also can be shown as in definition below

Definition 1.4 *(Lambert, 1991)*
The LMM (1.5) is said to be consistent if the LTE, T_{i+k} satisfies

$$\lim_{h\to 0}\frac{1}{h}T_{i+k} = 0.$$

The LMM (1.5) is said to be convergent if

$$\lim_{h\to 0} y_i = y(x_i),$$

where y_i is the numerical solution and $y(x_i), x \in [a,b]$ is the exact solution of the problem for $i = 0,1,2,\ldots,N = (b-a)/h$. Other than that, Dahlquist equivalence theorem also can be used to determine the convergence of LMM.

Theorem 1.2 *(Dahlquist Equivalence Theorem in Lambert (1973))*
The necessary and sufficient conditions for a LMM (1.5) to be convergent are that it be consistent and zero stable.

Qualitatively speaking, consistency controls the magnitude of the LTE obtained at each stage of the calculation while zero-stability controls the manner in which this error is propagated as the calculation continues, making both properties are important for achieving the convergence (Lambert, 1973).

1.6 Preliminary Mathematical Concepts

Stability plays an important role to study the propagation of error as $i \to \infty$. There are two schemes of stability that will be explored which are zero-stability and stability theory. Zero-stability analyzes the stability of the system just in the limit when step size, $h \to 0$ as $i \to \infty$ while the stability theory examines the stability when h takes a fixed non-zero value as $i \to \infty$. The definition for zero-stability is already established previously, this section will discuss the stability theory for second order DDEs.

The general form of linear second order DDEs is as follow

$$y'' = ay(x) + by(x-\tau) + cy'(x-\tau) + dy'(x), \quad x \in [a,b],$$
$$y(x) = \phi(x), \qquad x \in [a-\tau, b],$$

The study of stability criteria for the above form can be found in Cahlon and Schmidt (2004) while the study of stability regions can be found in Li et al. (2010) for $c = 0$

and Cahlon (1995) for $a = c = 0$.

This thesis will only focus on the stability when $c = d = 0$ and when $d = 0$. For $c = d = 0$, the second order DDEs can be defined as

$$\begin{aligned} y'' &= ay(x) + by(x-\tau), \quad x \in [a,b], \\ y(x) &= \phi(x), \quad x \in [a-\tau, b], \end{aligned} \tag{1.13}$$

Meanwhile, for $d = 0$, the second order DDEs can be written as

$$\begin{aligned} y'' &= ay(x) + by(x-\tau) + cy'(x-\tau), \quad x \in [a,b], \\ y(x) &= \phi(x), \quad x \in [a-\tau, b], \end{aligned} \tag{1.14}$$

where $a, b, c \in \mathbb{C}$, τ is constant delay and $\phi(x)$ is the continuous function.

The stability polynomial for (1.13) and (1.14) based on the first, second and third characteristic polynomials for LMM (1.5) is derived as

$$\pi(R; H_1, H_2) = \rho(R) - h\sigma(R) - (H_1, H_2)\omega(R) \tag{1.15}$$

where $H_1 = ha$, $H_2 = hb$ and c is a fixed value.

Definition 1.5
The LMM (1.5) is said to be absolutely stable for the given H_1 and H_2 if, for that H_1 and H_2, all the roots, R_j of (1.15) satisfy $|R_j| \leq 1, j = 1, 2, \ldots, k$.

Definition 1.6
The stability region of the LMM (1.5) for second order DDEs (1.13) and (1.14) is the set S of pairs of complex numbers (H_1, H_2), $H_1 = ha$, $H_2 = hb$, such that the solutions of DDEs (1.13) and (1.14), $\{y_i\}$ obtained with constant stepsize h under the constraint of

$$h = \frac{\tau}{m}, \quad m \geq 1, \quad m \ \ \textit{integer},$$

satisfies

$$\lim_{i \to \infty} y_i = 0.$$

1.7 Objectives of the Thesis

The main objective of this thesis is to establish the two-point multistep block method of order 4 and order 5 to solve directly the second and third order DDEs of constant, pantograph and time dependent delay type with boundary conditions. This objective can be achieved by:

1. deriving two-point diagonally direct block method of order 4 and order 5 by using the Lagrange interpolation polynomial.
2. analyzing both methods in terms of order, zero-stability, and local truncation error.
3. analyzing the stability regions of both methods i.e. the two-point diagonally direct block method of order 4 and order 5.
4. developing an algorithm based on the derived methods for solving second and third order DDEs with boundary conditions of constant, pantograph, and time dependent delay type.

1.8 Scope of the Study

This study concentrates on the approximate solutions of second and third order DDEs with boundary conditions by using the direct multistep block method of order 4 and order 5. The boundary condition involved in second order DDEs is Dirichlet type while for the third order DDEs, there are three different types involved which are Type I, Type II, and Type III. The analysis of these two methods based on the order, zero-stability, and local truncation error are investigated. There are three delay types considered in this study which are constant, pantograph and time dependent delay type. In dealing with the boundary conditions present in the problems, the shooting technique by using the Newton's-like method is used. The two different types stability regions of both methods are studied for second order DDEs.

1.9 Outline of the Thesis

This thesis consists of seven chapters. Chapter 1 describes the brief introduction of DDEs, BVPs, linear multistep method and several mathematical concepts needed to analyze both the multistep methods and DDEs along with the objectives and scope of this thesis.

Chapter 2 explores the review of previous works that closely related to this study such as the methods used and the problems involved will also be discussed.

Chapter 3 portrays the derivation of the proposed methods which are two-point diagonally block method of order 4 and order 5 based on the implementation of Lagrange interpolation polynomial. Order, zero-stability and local truncation error of the methods are also examined. The consistency of the methods based on the local truncation error is also investigated. The stability regions of the methods are presented in two different forms that are appropriate for the problems.

Chapter 4 illustrates the implementation of solving second order DDEs with boundary conditions of constant delay type by using the proposed methods. There are discussion on the tested problems and it's comparison with other methods.

Chapter 5 discusses on the application of the methods to solve the second order DDEs with boundary conditions of pantograph and time dependent delay type. The implementation of Lagrange interpolation is needed to find the delay in the problems. Several numerical problems are solved and compared with the established methods.

Chapter 6 considers the derivation of the proposed methods for solving third order DDEs. The approaches to solve the constant and pantograph delay type are the same as in Chapter 4 and Chapter 5 however the differences are the implementations of Newton's like method used for shooting technique since the different types of boundary conditions will be used in Chapter 6. Some numerical problems are tested and compared with the previous methods.

Finally, Chapter 7 describes the summary of this thesis and some suggestions on the future works.

REFERENCES

Abazari, N. and Abazari, R. (2009). Solution of nonlinear second-order pantograph equations via differential transformation method. In *Proceedings of World Academy of Science, Engineering and Technology*, volume 58, pages 1052–1056. Citeseer.

Agarwal, R., P. and Chow, Y., M. (1986). Finite difference methods for boundary value problems of differential equations with deviating arguments. *Computational & Mathematics with Applications*, 12A(11):1143–1153.

Ahmad, S. Z., Ismail, F., and Senu, N. (2016). A four-stage fifth-order trigonometrically fitted semi-implicit hybrid method for solving second-order delay differential equations. *Mathematical Problems in Engineering*, 2016.

Akgül, A. and Kiliçman, A. (2015). Solving delay differential equations by an accurate method with interpolation. In *Abstract and Applied Analysis*, volume 2015. Hindawi.

Andargie, A. and Reddy, Y., N. (2013). Parameter fitted scheme for singularly perturbed delay differential equations. *International Journal of Applied Science and Engineering*, 11(4):361–373.

Aykut, A., Çelik, E., and Bayram, M. (2003). The modified two sided approximations method and padé approximants for solving the differential equation with variant retarded argument. *Applied mathematics and computation*, 144(2-3):475–482.

Aykut, A. and Yildiz, B. (1998). On a boundary value problem for a differential equation with variant retarded argument. *Applied mathematics and computation*, 93(1):63–71.

Aziz, N. H. A. and Majid, Z. A. (2013). Solving delay differential equations using modified 2-point block method. In *AIP Conference Proceedings*, volume 1522, pages 791–797. American Institute of Physics.

Aziz, N. H. A. and Majid, Z. A. (2014). Variable step size and order strategy for delay differential equations in pie (cie) s mode. In *AIP Conference Proceedings*, volume 1635, pages 58–64. American Institute of Physics.

Bader, G. (1985). Solving boundary value problems for functional differential equations by collation. In *Numerical Boundary Value ODEs*, pages 227–243. Springer.

Baker, C. T., Paul, C. A., and Willé, D. R. (1995). Issues in the numerical solution of evolutionary delay differential equations. *Advances in Computational Mathematics*, 3(1):171–196.

Bakke, V., L. and Jackiewicz, Z. (1989). The numerical solution of boundary value problems for differential equations with state dependent deviating arguments. *Aplikace matematiky*, 34(1):1–17.

Bartoszewski, Z. (2010). A new approach to numerical solution of fixed-point problems and its application to delay differential equation. *Applied Mathematics and Computation*, 215:4320–4331.

Bartoszewski, Z. and Baranowska, A. (2015). Solving boundary value problems for second order singularly perturbed delay differential equations by e-approximate fixed-point method. *Mathematical Modelling and Analysis*, 20(3):369–381.

Bellen, A. (1979). Cohen's iteration process for boundary value problems for functional differential equations.

Bellen, A. (1985). A runge-kutta-nystrom method for delay differential equations. In *Numerical Boundary Value ODEs*, pages 271–283. Springer.

Bellen, A. and Zennaro, M. (1984). A collocation method for boundary value problems of differential equations with functional arguments. *Computing*, 32(4):307–318.

Bellen, A. and Zennaro, M. (2003). *Numerical Methods for Delay Differential Equations*. Oxford University Press.

Bica, A. M. (2011). The numerical method of successive interpolations for two-point boundary value problems with deviating argument. *Computers & Mathematics with Applications*, 62(10):3829 – 3843.

Bilal, M., Rosli, N., Ahmad, I., and Ullah, S. (2017). Numerical solution of second order delay type differential equation by collocation method via first boubeker polynomials. *Global Journal of Pure and Applied Mathematics*, 13(9):6571–6582.

Burden, R. L. and Faires, J. D. (2011). *Numerical Analysis*. Brooks/Cole, Boston, MA, 9th edition.

Cahlon, B. (1995). On the stability of volterra integral equations with a lagging argument. *BIT Numerical Mathematics*, 35(1):19–29.

Cahlon, B. and Nachman, L. (1988). Numerical solution of differential equations with deviation arguments. *International Journal of Modelling and Simulation*, 8(1):10–14.

Cahlon, B. and Nachman, L. J. (1991). Numerical methods for discontinuous linear boundary value problems with deviation arguments. *Journal of Mathematical Analysis and Applications*, 154:529–542.

Cahlon, B. and Schmidt, D. (2004). Stability criteria for certain second-order delay differential equations with mixed coefficients. *Journal of Computational and Applied mathematics*, 170(1):79–102.

Celik, E., Aykut, A., and Bayram, M. (2004). The modified successive approximations method and pade approximants for solving the differential equation with variant retarded argumend. *Applied mathematics and computation*, 151(2):393–400.

Chakravarthy, P. P., Kumar, S. D., Rao, R. N., and Ghate, D. P. (2015). A fitted numerical scheme for second order singularly perturbed delay differential equations via cubic spline in compression. *Advances in Difference Equations*, 2015(1):300.

Challa, L., S. and Reddy, Y., N. (2017). Numerical integration of singularly perturbed delay differential equations using exponential integrating factor. *Mathematical Communications*, 22:251–264.

Chocholaty, P. and Slahor, L. (1979). A numerical method to boundary value problems for second order delay differential equations. *Numerical Mathematic*, 33:69–75.

Cimen, E. and Uncu, S. (2020). On the solution of the delay differential equation via laplace transform. *Communications in Mathematics and Applications*, 11(3):379–387.

Cryer, C. W. (1973). The numerical solution of boundary value problems for second order functional differential equations by finite differences. *Numerical Mathematics*, 20:288–299.

Driver, R. D. (1977). *Ordinary and delay differential equations*. Springer.

Eloe, P. W., Raffoul, Y. N., and Tisdell, C. C. (2005). Existence, uniqueness and constructive results for delay differential equations. *Electronic Journal of Differential Equations (EJDE)[electronic only]*, 2005:Paper–No.

Evans, D. J. and Raslan, K. (2005). The adomian decomposition method for solving delay differential equation. *International Journal of Computer Mathematics*, 82(1):49–54.

Fang, L., Sun, L., and He, G. (2008). An efficient newton-type method with fifth-order convergence for solving nonlinear equations. *Computational & Applied Mathematics*, 27(3):269–274.

Geng, F. and Qian, S. (2015). A hybrid method for singularly perturbed delay boundary value problems exhibiting a right boundary layer. *Bulletin of the Iranian Mathematical Society*, 41(5):1235–1247.

Grace, S. R. (2015). Oscillation criteria for third order nonlinear delay differential equations with damping. *Opuscula Mathematica*, 35(4).

Grimm, L. J., Schmitt, K., et al. (1968). Boundary value problems for delay-differential equations. *Bulletin of the American Mathematical Society*, 74(5):997–1000.

Henrici, P. (1964). *Elements of numerical analysis*. Wiley.

Isa, N. B. M. and Ishak, F. (2018). Special second order stiff delay differential equations directly solved by using variable stepsize variable order technique. In *AIP Conference Proceedings*, volume 2013, page 020039. AIP Publishing LLC.

Isa, N. B. M., Ishak, F., and Othman, K. I. (2014). Efficient technique for solving stiff delay differential equations in variable stepsize variable order. In *AIP Conference Proceedings*, volume 1605, pages 52–57. American Institute of Physics.

Ishak, F., A. Majid, Z., and Suleiman, M. B. (2013a). Block method with hermite interpolation for numerical solution of delay differential equations. In *AIP Conference Proceedings*, volume 1522, pages 708–714. American Institute of Physics.

Ishak, F. and Ahmad, S. N. (2011). Solving delay differential equations by predictor-corrector method using lagrange and hermite interpolations. In *2011 IEEE Colloquium on Humanities, Science and Engineering*, pages 932–934. IEEE.

Ishak, F., Majid, Z. A., and Suleiman, M. B. (2012). Development of implicit block method for solving delay differential equations. In *Proceedings of the 14th WSEAS International Conference on Mathematical and Computational Methods in Science and Engineering, Malta*, pages 67–71.

Ishak, F., Majid, Z. A., and Suleiman, M. B. (2013b). Efficient interpolators in implicit block method for solving delay differential equations. *Int. J. Math. Comput. Simul*, 7(2):116–124.

Ishak, F. and Suleiman, M. (2010). Parallel block method for solving delay differential equations. In *International Mathematical Forum*, volume 5, pages 2707–2722.

Ishak, F., Suleiman, M., and Omar, Z. (2008). Two-point predictor-corrector block method for solving delay differential equations. *MATEMATIKA: Malaysian Journal of Industrial and Applied Mathematics*, 24:131–140.

Ishak, F., Suleiman, M. B., and Majid, Z. A. (2013c). Block method for solving pantograph-type functional differential equations. In *Proceedings of the World Congress on Engineering*, volume 2.

Ishak, F., Suleiman, M. B., and Majid, Z. A. (2014). Numerical solution and stability of block method for solving functional differential equations. In *Transactions on Engineering Technologies*, pages 597–609. Springer.

Ishak, F., Suleiman, M. B., Majid, Z. A., and Othman, K. I. (2011). Development of variable stepsize variable order block method in divided difference form for the numerical solution of delay differential equations. *International Journal of Mathematical and Computational Sciences*, 5(5):728–733.

Ismail, N. I. N., Majid, Z. A., and Senu, N. (2020). Hybrid multistep block method for solving neutral delay differential equations. *Sains Malaysiana*, 49(4):929–940.

Jafari, H., Mahmoudi, M., and Skandari, M. N. (2021). A new numerical method to solve pantograph delay differential equations with convergence analysis. *Advances in Difference Equations*, 2021(1):1–12.

Kadalbajoo, M. K. and Sharma, K. K. (2004). Numerical analysis of singularly perturbed delay differential equations with layer behavior. *Applied Mathematics and Computation*, 157(1):11–28.

Kanth, A. R. and Murali, M. K. P. (2018). A numerical technique for solving nonlinear singularly perturbed delay differential equations. *Mathematical Modelling and Analysis*, 23(1):64–78.

Keller, H. B. (2018). *Numerical methods for two-point boundary-value problems*. Courier Dover Publications.

Kuang, Y. (1993). *Delay Differential Equations: With Applications in Population Dynamics*. ISSN. Elsevier Science.

Kumar, D. (2013). A computational technique for solving boundary value problem with two small parameters. *Electronic Journal of Differential Equations*, 2013(30):1–10.

Lambert, J. D. (1973). *Computational methods in ordinary differential equations*. Wiley.

Lambert, J. D. (1991). *Numerical methods for ordinary differential systems: the initial value problem*. John Wiley & Sons, Inc.

Li, C. and Zhang, C. (2017). The extended generalised stormer-cowell methods for second order delay boundary value problems. *Applied Mathematics and Computation*, 294:87–95.

Li, W., Huang, C., and Gan, S. (2010). Delay-dependent stability analysis of trapezium rule for second order delay differential equations with three parameters. *Journal of the Franklin Institute*, 347(8):1437–1451.

Mahendran, R. and Subburayan, V. (2019). Fitted finite difference method for third order singularly perturbed delay differential equations of convection diffusion type. *International Journal of Computational Methods*, 16(05):1840007.

Majid, Z. A. (2004). *Parallel block methods for solving ordinary differential equations*. PhD thesis, Universiti Putra Malaysia.

Majid, Z. A. and Suleiman, M. (2007). Two point fully implicit block direct integration variable step method for solving higher order system of ordinary differential equations. In *World Congress on Engineering*, pages 812–815.

Mohd Nasir, N., Abdul Majid, Z., Ismail, F., and Bachok, N. (2018). Diagonal block method for solving two-point boundary value problems with robin boundary conditions. *Mathematical Problems in Engineering*, 2018.

Mohd Nasir, N., Abdul Majid, Z., Ismail, F., and Bachok, N. (2019). Direct integration of boundary value problems using the block method via the shooting technique combined with steffensens strategy. *Mathematics*, 7(11):1075.

Nevers, K., D. and Schmitt, K. (1971). An application of the shooting method to boundary value problems for second order delay equations. *Journal of Mathematical Analysis and Applications*, 36:588–597.

Panghal, S. and Kumar, M. (2021). Neural network method: delay and system of delay differential equations. *Engineering with Computers*, pages 1–10.

Pei See, P. (2015). *Direct Methods Via Multiple Shooting Technique For Solving Boundary Value Problems*. Universiti Putra Malaysia.

Qu, R. and Agarwal, R., P. (1998). A subdivision approach to the contruction of approximate solutions of boundary value problems with deviating arguments. *Computational & Mathematics with Applications*, 35(11):121–135.

Raja, M. A. Z. (2014). Numerical treatment for boundary value problems of pantograph functional differential equation using computational intelligence algorithms. *Applied Soft Computing*, 24:806–821.

Reddien, G. (1978). Difference approximations of boundary value problems for functional differential equations. *Journal of Mathematical Analysis and Applications*, 63(3):678–686.

Reddien, G., W. and Travis, C. C. (1974). Approximation methods for boundary value problems of differential equations with functional arguments. *Journal of Mathematical Analysis and Applications*, 46:62–74.

Reutskiy, S. Y. (2015). A new collocation method for approximate solution of the pantograph functional differential equations with proportional delay. *Applied Mathematics and Computation*, 266:642–655.

Saadatmandi, A. and Dehghan, M. (2009). Variational iteration method for solving a generalized pantograph equation. *Computers & Mathematics with Applications*, 58(11-12):2190–2196.

Sakai, M. (1975). Numerical solution of boundary value problems for second order functional differential equations by the use of cubic splines. *Memoirs of the Faculty of Science, Kyushu University. Series A, Mathematics*, 29(1):113–122.

San, H. C., Majid, Z. A., and Othman, M. (2011). Solving delay differential equations using coupled block method. In *2011 Fourth International Conference on Modeling, Simulation and Applied Optimization*, pages 1–4. IEEE.

Sekar, E. and Tamilselvan, A. (2019). Third order singularly perturbed delay differential equation of reaction diffusion type with integral boundary condition. *Journal of Applied Mathematics and Computational Mechanics*, 18(2).

Seong, H. Y. (2016). *Multistep Blocks Methods For Solving Higher Order Delay Differential Eqautions*. Universiti Putra Malaysia.

Seong, H. Y. and Majid, Z. A. (2013). Direct method for solving second order delay differential equations. In *AIP Conference Proceedings*, volume 1522, pages 676–680. AIP.

Seong, H. Y. and Majid, Z. A. (2014). Fifth order predictor-corrector methods for solving third order delay differential equations. In *AIP Conference Proceedings*, volume 1635, pages 94–98. American Institute of Physics.

Seong, H. Y. and Majid, Z. A. (2015). Solving delay differential equations of pantograph type using predictor-corrector method. In *AIP Conference Proceedings*, volume 1660, page 050059. AIP Publishing LLC.

Seong, H. Y. and Majid, Z. A. (2017). Solving second order delay differential equations using direct two-point block method. *Ain Shams Engineering Journal*, 8(1):59–66.

Seong, H. Y., Majid, Z. A., and Ismail, F. (2013). Solving second-order delay differential equations by direct adams-moulton method. *Mathematical Problems in Engineering*, 2013.

Sezer, M. and Akyüz-Daşcıoglu, A. (2007). A taylor method for numerical solution of generalized pantograph equations with linear functional argument. *Journal of Computational and Applied Mathematics*, 200(1):217–225.

Sirisena, U. and Yakubu, S. (2019). Solving delay differential equations using reformulated block backward differentiation formulae methods. *Journal of Advances in Mathematics and Computer Science*, pages 1–15.

Sujitha, S. S. and Piriadarshani, D. (2019). Numerical computation of delay differential equation using laplace transform and lambert w function. *International Journal of Innovative Technology and Exploring Engineering*, 8.

Suleiman, M. B. and Ishak, F. (2010). Numerical solution and stability of multistep method for solving delay differential equations. *Japan journal of industrial and applied mathematics*, 27(3):395–410.

Syam, M., Jaradat, H., and Alquran, M. (2019). A numerical method by reproducing-kernel approximation for the pantograph equation. *Electronic Journal of Mathematical Analysis and Applications*, 7(1):391–409.

Taiwo, O. and Odetunde, O. (2010). On the numerical approximation of delay differential equations by a decomposition method. *Asian Journal of Mathematics and Statistics*, 3(4):237–243.

Tsamatos, P. C. and Ntouyas, S. (1991). Existence of solutions of boundary value problems for differential equations with deviating arguments, via the topological transversality method. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, 118(1-2):79–89.

Vanani, S. K., Hafshejani, J. S., Soleymani, F., and Khan, M. (2011). On the numerical solution of generalized pantograph equation.

Vilu, S., Ahmad, R. R., and Din, U. K. S. (2019). Variational iteration method and sumudu transform for solving delay differential equation. *International Journal of Differential Equations*, 2019.

Wazwaz, A.-M., Raja, M. A. Z., and Syam, M. I. (2017). Reliable treatment for solving boundary value problems of pantograph delay differential equation. *Rom. Rep. Phys*, 69:102.

Wheldon, T., Kirk, J., and Finlay, H. M. (1974). Cyclical granulopoiesis in chronic granulocytic leukemia: A simulation study. *Blood*, 43(3):379–387.

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LIST OF PUBLICATIONS

The following are the list of publications that arise from this study.

Jaaffar, N. T., Majid, Z. A., and Senu, N. 2021. Numerical Computation of Third Order Delay Differential Equations by Using Direct Multistep Method. *Malaysian Journal of Mathematical Sciences*, 15(3), 369-385.

Jaaffar, N. T., Majid, Z. A., and Senu, N. 2021. Solving Numerically the Singularly Perturbation Problems of Delay Differential Equations. *International Journal of Mathematics and Computer Science*, 16(2021), no. 3, 1003-1016.

Jaaffar, N. T., Majid, Z. A., and Senu, N. 2020. Numerical Approach for Solving Delay Differential Equations with Boundary Conditions. *Mathematics*, 8(7), 1073.

Jaaffar, N. T., Majid, Z. A., and Senu, N. November, 2019. Direct multistep method for solving delay differential equation with boundary conditions. In *Journal of Physics: Conference Series* (Vol. 1366, No. 1, p. 012085). IOP Publishing.

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