UNIVERSITI PUTRA MALAYSIA

DEFORMED HEISENBERG GROUP FOR A PARTICLE ON NONCOMMUTATIVE SPACES VIA CANONICAL GROUP QUANTIZATION AND EXTENSION

MOHD FAUDZI BIN UMAR

DEFORMED HEISENBERG GROUP FOR A PARTICLE ON NONCOMMUTATIVE SPACES VIA CANONICAL GROUP QUANTIZATION AND EXTENSION

By

MOHD FAUDZI BIN UMAR

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfillment of the Requirements for the Degree of Doctor of Philosophy

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DEDICATIONS

To all of my love;
Umar & Halimah (Allahummaghfirlahuma, warhamhuma, wa'afihima, wa'fu'anhuma)
Ruhanizah, Khairunnisa', Raudhah & 'Umar 'Abdussalam

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Doctor of Philosophy

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November 2020

Chairman : Hishamuddin bin Zainuddin, PhD
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The first part of this work focuses on the canonical group quantization approach applied to non-commutative spaces, namely plane $\mathbb{R}^2$ and two-torus T^2. Canonical group quantization is a quantization approach that adopts the group structure that respects the global symmetries of the phase space as a main ingredient. This is followed by finding its unitary irreducible representations. The use of noncommutative space is motivated by the idea of quantum substructure of space leading to nontrivial modification of the quantization. Extending to noncommuting phase space includes noncommuting momenta that arises naturally in magnetic background as in Landau problem. The approach taken is to modify the symplectic structures corresponding to the noncommutative plane, noncommutative phase space and noncommutative torus and obtain their canonical groups. In all cases, the canonical group is found to be central extensions of the Heisenberg group. Next to consider is to generalize the approach to twisted phase spaces where it employs the technique of Drinfeld twist on the Hopf algebra of the system. The result illustrates that a tool from the deformation quantization can be used in canonical group quantization where the deformed Heisenberg group $\mathtt{H}^2_\theta$ is obtained and its representation stays consistent with the discussion in the literature. In the second part, the two-parameter deformations of quantum group for Heisenberg group and Euclidean group are studied. Both can be achieved through the contraction procedure on $SU(2)_{q,p}$ quantum group. The study also continues to develop (q,p)-extended Heisenberg quantum group from the previous result. As conclusion, it is shown that the extensions of Heisenberg group arise from quantizing noncommutative plane, noncommutative phase space, noncommutative two-torus, and twisted phase space. The work on two-parameter deformation of quantum group also further shows generalizations of the extension of Heisenberg group.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

KUMPULAN HEISENBERG TERCANGGA BAGI SATU ZARAH KE ATAS RUANG TAK KALIS TUKAR TERTIB MELALUI PENGUANTUMAN KUMPULAN BERKANUN DAN LANJUTAN

Oleh

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November 2020

Pengerusi : Hishamuddin bin Zainuddin, PhD
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Bahagian pertama kajian ini memfokuskan kepada kaedah penguantuman kumpulan berkanun yang digunakan untuk ruang tak kalis tukar tertib, iaitu satah $\mathbb{R}^2$ dan dua-torus T^2. Penguantuman kumpulan berkanun ialah satu kaedah penguantuman yang mengambil struktur kumpulan berkaitan simetri global ruang fasa sebagai satu elemen utama. Seterusnya diikuti dengan mendapatkan perwakilan tertutrun unitari. Penggunaan ruang tak kalis tukar tertib dimotivasikan dengan idea sub struktur kuantum ruang yang memacu kepada pengubahsuaian remeh penguantuman. Melanjutkan ruang fasa tak kalis tukar tertib merangkumi momentum tak kalis tukar tertib yang muncul secara semula jadi dalam latar belakang magnet seperti dalam masalah Landau. Pendekatan yang diambil adalah bagi mengubahsuai struktur simpletik yang sepadan dengan satah tak kalis tukar tertib, ruang fasa tak kalis tukar tertib dan torus tak kalis tukar tertib, dan mendapatkan kumpulan berkanunnya. Dalam semua kes, kumpulan berkanun ditemui adalah perluasan berpusat daripada kumpulan Heisenberg. Seterusnya mempertimbangkan bagi mengitlak kaedah pada ruang fasa terpulas, di mana ia menggunakan teknik pulasan Drinfeld ke atas algebra Hopf sesuatu sistem. Keputusan menunjukkan bahawa satu alat daripada penguantuman canggaan boleh digunakan dalam penguantuman kumpulan berkanun di mana kumpulan Heisenberg tercangga $\mathtt{H}^2_\theta$ adalah diperolehi dan perwakilannya tetap konsisten dengan perbincangan dalam literatur. Di bahagian kedua, canggaan dua parameter kumpulan kuantum bagi kumpulan Heisenberg dan Euclidean adalah dikaji. Kedua-duanya boleh dicapai melalui prosedur pengecutan ke atas kumpulan kuantum $SU(2)_{q,p}$. Kajian ini juga diteruskan bagi membangunkan kumpulan kuantum (q,p)-Heisenberg terlanjut daripada keputusan sebelum. Sebagai kesimpulan, ia telah menunjukkan bahawa lanjutan kumpulan Heisenberg muncul daripada menguantumkan satah tak kalis tukar tertib, ruang fasa tak kalis tukar tertib, dua-torus

tak kalis tukar tertib, dan ruang fasa terpulas. Kajian ke atas canggaan dua parameter kumpulan kuantum selanjutnya juga menunjukkan pengitlakkan lanjutan kumpulan Heisenberg.

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This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Doctor of Philosophy. The members of the Supervisory Committee were as follows:

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LIST OF ABBREVIATIONS

QM	Quantum Mechanics
NCQM	Noncommutative Quantum Mechanics
NCPS	Noncommutative Phase Space
QFT	Quantum Field Theory
CGQ	Canonical Group Quantization
HamVF	Hamiltonian vector field
GQ	Geometric Quantization
UIRs	Unitary Irreducible Representations
CCR	Canonical Commutation Relation
ACR	Affine Commutation Relation
UV	Ultraviolet
NC	Noncommutative
NCG	Noncommutative Geometry
AQG	Affine Quantum Gravity
Riem	Riemannian
UEA	Universal Enveloping Algebra
SHO	Simple Harmonic Oscillators
SUSY	Supersymmetry
SUSYQM	Supersymmetric Quantum Mechanics

CHAPTER 1

INTRODUCTION

In the 1900s, there are two major theories in the development of physics namely relativity and quantum theory. Physics of relativity, especially general relativity historically developed a new perspective about gravitation through the geometry of spacetime, and the theory was first introduced by Albert Einstein. Meanwhile, quantum mechanics was pioneered by Werner Heisenberg, Max Born, and Pascual Jordan, in which the theory gave us an understanding of the nature of the atomic scale. In addition, quantum mechanical system can be obtained via a procedure known as "quantization", and this idea fundamentally given by Heisenberg (1925) is about noncommutative algebra of quantum operators, instead of commutative of classical observables in his paper "*Uber quantentheoretische Umdeutung kinematischer und mechaniseher Beziehungen*" (see Van Der Waerden, 2007). In the same year, with Born and Jordan, they developed the matrix mechanics formulation of quantum mechanics based on Heisenberg's quantum operators (Born et al. (1926); the paper was received in November 1925).

Quantization is mathematically a transition of commutative algebra of classical observables into noncommutative algebra of quantum operators. The classical observables are functions on phase space, while the quantum operators are self-adjoint operators acting on a Hilbert space. We can describe the classical mechanics via phase space which is given by the cotangent bundle of the configuration space $\mathbb{R}^n$, and it provides the set of canonical observables, (q^i, p_i). Quantization programs, in literature, have often been developed from classical mechanics which is formulated on symplectic manifold given by cotangent bundle. Symplectic manifold is a manifold $\mathscr{M}$ which is equipped with a closed nondegenerate two-form ω (known as symplectic structure) and is denoted as a pair, $(\mathscr{M}, \omega)$. The symplectic structure of the phase space provides the following set of Poisson bracket

$$\left\{q^i, q^j\right\} = 0 = \left\{p_i, p_j\right\}, \quad \left\{q^i, p_j\right\} = \delta^i_j, \tag{1.1}$$

where $i = 1, 2, 3..., n$, while q^i and p_i are respectively positions and momenta as classical observables, and δ^i_j is the Kronecker delta.

The quantization rules, in brief, are based on the following conditions; *linearity*, *identity*, *irreducibility* of the self-adjoint quantum operators and *commutator* via quantization map; $\hat{}$ (e.g. Dirac, 1925). Thus corresponding brackets (1.1) respectively transform to the following canonical commutation relations (CCR):

$$[\hat{q}^i, \hat{q}^j] = 0 = [\hat{p}_i, \hat{p}_j], \quad [\hat{q}^i, \hat{p}_j] = i\hbar\delta^i_j \mathbb{1}, \tag{1.2}$$

where $\hbar$ is the reduced Planck constant and δ^i_j is the Kronecker delta. The relation

(1.2) is also known as Heisenberg algebra which leads to Heisenberg uncertainty principle. Moreover, some quantized systems also can be reduced to corresponding classical systems via an appropriate classical limit, $\hbar \to 0$, and is called *dequantization*.

Note that CCR (1.2) has been obtained through a canonical quantization framework but some quantization approaches are used by mathematicians and physicists to develop CCR only as outcomes of their approaches for linear spaces (Ali and Englis, 2005). Nevertheless, there are three issues we shall highlight here. First and foremost, quantization via CCR faces a problem with classical systems arising from nonlinear configuration spaces such as circle S^1, n-sphere S^n and n-torus T^n. In addition, the second issue of quantization is when it comes to polynomials of classical observables; q and p with degree $n \geq 3$. The commutator algebra and irreducibility of quantization rule are not consistent with such polynomials and this is known as "Groenewold-van Hove No-Go Theorem", that revolves around operator ordering issues; see Groenewold (1946); Zainuddin et al. (2007). Third issue is on how we generalize the noncommutativity on positions or/and momenta. Therefore, in the present thesis, we use canonical group quantization (CGQ) as proposed by Isham (1984); Isham and Kakas (1984a,b) to investigate noncommutative algebra in the standard quantum mechanics (1.2). Literally, CGQ is a quantization program is based on the unitary irreducible representations of the Lie group that describes the symmetries of the phase space. The approach considers the action of a Lie group on the phase space by relating classical observables and vector fields with Lie algebra without assuming CCR. CGQ uses symmetry of the system as a basis for quantization of systems on nonlinear and nontrivial configuration space; see Isham and Linden (1988); Zainuddin (1989); Sumadi and Zainuddin (2014); Bouketir (2000); Bojowald and Strobl (2000); Benavides and Reyes-Lega (2010); Jung (2012). Therefore, in this work, we will also apply the quantization program for noncommutative configuration spaces.

Mathematically, there is a correspondence between spaces and algebras, where the commutative algebras will be replaced by noncommutative algebra implying noncommutative generalization of geometries (also known as noncommutative geometry). This issue has been often discussed by means of spectral triples (e.g Connes, 1994; Connes and Marcolli, 2008). Noncommutative geometry (NCG) is a plausible model that has influenced in many disciplines in physics especially quantum mechanics. Quantum mechanics in noncommutative space or noncommutative quantum mechanics (NCQM) can be studied via several formulations (Gouba, 2016). Most often discussed and used in literature is by modification of the CCR from the standard quantum mechanics (1.2). The idea of NCQM was first introduced by Snyder (1947), when he studied the quantized space-time to formalize Heisenberg's idea. Noncommutativity of the positions implies non-standard structure of space at very tiny scale, where below Planck scale, localization of space-time has no operational meaning (Doplicher et al., 1995). The noncommutativity of the momenta implies the adoption of a gauge field in the momentum operator in Landau problem. However, a phenomenon occurs at the lowest-Landau level requires Haldane (2018) to introduce Heisenberg description of the noncommutative torus of guiding centers.

NCQM can be achieved by introducing new coordinates, Moyal $\star$-products, Bopp shift, and Seiberg-Witten map (Seiberg and Witten, 1999). In addition, operator theoretic formulation has also attracted the attention of many authors such as Scholtz et al. (2009). They proposed the set of Hilbert-Schmidt operators acting on the noncommutative configuration space, which is isomorphic to boson Fock space. When one also considers noncommutativity of momenta, one finds the representation of noncommutative phase spaces such as the one studied by Li et al. (2005). The representations used by Balogh et al. (2015) for instance, to study deformed Hermite polynomials leads to the family of biorthogonal polynomials. Another aspect of NCQM which is closely related to this thesis is by studying via its group-theoretic structure. Such studies using the group-theoretic approach can be seen in two well-known methods *i.e.* Souriau's and Kirillov's orbit method where they both use coadjoint orbit in their formulation. Those formulations in NCQM have been summarized in Table 1.1. In this work, we propose an alternative approach using Isham's canonical group quantization, which identifies directly the symmetries of the underlying phase space as key feature of the quantization.

Table 1.1 Formulations of NCQM

Formulation	Methods	Studied by
Canonical formulation	New coordinates	Chaichian et al. (2001)
	Moyal $\star$-product	Gamboa et al. (2001a,b)
	Bopp shift	Li and Sayipjamal (2010)
	Seiberg-Witten map	Seiberg and Witten (1999)
Operator-theoretic	Systematic approach	Scholtz et al. (2009)
	Representation theory	Li et al. (2005)
	Polynomial	Balogh et al. (2015)
Group-theoretic	Coadjoint orbit method	Ngendakumana et al. (2011)
		Duval and Horvathy (2000)
		Vanhecke et al. (2006)
		Chowdhury and Ali (2014)

1.1 Motivation

Canonical group quantization is a quantization approach that is geometrical in nature with the group structure as a main ingredient in the scheme and it has been used for quantization of nonlinear systems such as gravity (Isham and Kakas, 1984a,b; Isham, 1984), string on tori (Isham and Linden, 1988) and particle on torus in a constant magnetic field background (Zainuddin, 1989). The group of symmetries of the phase space identified is called the canonical group, which reminds us of the use of the canonical commutation relation based on a Lie algebra. It is of interest to consider whether the procedure can be extended to non-commuting position coordinates, and thus producing a new set of commutation relations reflecting noncommutativity of the position. It is to be noted that another group-theoretic approach for noncommutative systems have been considered via the coadjoint orbit method *i.e.* the group's action on the dual space of its Lie algebra (known as Kirillov-Kostant-Souriau's method; Kirillov (2004); Kostant (1970); Souriau (1997)), and this is illustrated in Table 1.1. Noncommutative phase space, where both positions and momenta are no longer commute, suggests the existence of the non-standard structure of space at very short distances, with the associated Landau problem. There is a strong motivation to examine closely the symmetries of the phase space of the system and along with its representations.

In literature, the noncommutative torus is often discussed using spectral triples and deformation theory, and the attempt made is to approach this matter through the group-theoretic approach *e.g.* CGQ. The quantization on torus has first been considered through the canonical group approach in Isham (1984), and more application can be seen in Isham and Linden (1988) where they considered string quantization on the torus. In Zainuddin (1989), he considered the quantization on the torus with and without the magnetic field background. The system with the magnetic field background showed the noncommutativity of momenta with a Landau gauge choice. The canonical group obtained is $\tilde{E}^2 \rtimes (\tilde{E}^2 \times U(1))$, where $\tilde{E}^2$ is the universal cover of the two-dimensional Euclidean group whose subgroup $SO(2)$ is being replaced by $\mathbb{R}$. This standpoint suggests that we can continue to study the quantization on torus with noncommutativity of positions q^i, q^j, and this is considered in Chapter 5. It would be interesting to quantize the nonlinear configuration space of T^2 whose angular coordinates do not commute, so that we can comprehend the nature of the symmetries of the noncommutative two-torus.

In this thesis we propose two approaches namely extended and deformed methods to make generalization of Isham's method for the case of a noncommutative system as underlying phase space. For the first approach, we modify the symplectic structure of the phase space and investigate the symmetries that preserve this new symplectic structure. As a result, we have the noncommutative algebra with extended observables and operators. For the deformed method, we will utilize the Drinfeld twist on Hopf algebra of the system to obtain the deformed system *i.e.* deformed observables and operators, symplectic structure and canonical group. Here, we seek to reconcile the canonical group quantization with some basic ingredients of deformation quantization namely Moyal $\star$-product, and we also pursue possible ramifications.

The canonical groups for plane and torus with the natural symplectic form are respectively Heisenberg group and Euclidean group. Another possible related topic is to consider the quantum group of such groups. Both groups were studied by Celeghini et al. (1990, 1991) using contraction procedure on $SU_q(2)$ group. In a similar piece of the work, the group $SU(2)$ will be deformed with another deformation parameter and this is known as a two-parameter quantum group. Closely imitating the similar procedure of Celeghini et al. (1990, 1991) the contraction method can be applied to the group $SU(2)_{q,p}$ to obtain (q,p)-Heisenberg and (q,p)-Euclidean groups. Due to the fact that the extended Heisenberg group arises from the non-commutative plane, it will be of interest how the extended Heisenberg group can be deformed or generalized further to its quantum group counterparts.

1.2 Problem Statements and Objectives

This thesis comprises two parts, the first part mainly discusses the quantization program. Recent studies show that the quantization program can be used to explore the noncommutative system. Among all the formulations introduced in the Table 1.1, this work is inspired by the efforts of Ngendakumana et al. (2011, 2014); Duval and Horvathy (2000); Chowdhury and Ali (2013, 2014); Vanhecke et al. (2006) who studies NCQM via coadjoint orbit method of geometric quantization. Alternatively, in this thesis we use CGQ which was proposed by Isham (1984); Isham and Kakas (1984a,b) to study such noncommutative systems. The canonical group for the conventional case $\mathbb{R}^n$ with noncommutative space requires us to extend the Heisenberg group. We proceed with the case by including the magnetic field background to the system when momenta no longer commute, and this case has been often discussed in Landau effect literature. Nonetheless, in this case, our main research questions here are, "What is the canonical group to describing the symmetries of the phase space with the noncommutative plane with and without magnetic field?" and of course "How to apply them to classical and quantum mechanics?"

In the next part we consider quantization on one nonlinear configuration space since CGQ was very successful for nonlinear configuration spaces such Zainuddin (1989); Bouketir (2000); Sumadi and Zainuddin (2014). Hence it is of interest, how can the quantization be adapted to a nonlinear configuration space whose coordinates do not commute and in this case, the torus. Noncommutative torus has been studied often in literature, based on the spectral triples theory (Connes, 1987; Connes and Landi, 2001) but they tend to be obscure. However, it is not easy to overcome this difficulty *i.e.* nonlinear configuration space with noncommutativity system.

In literature, some authors generalized $\star$-product originally comes from Moyal (1949) to develop noncommutativity of the positions in a two-dimensional plane (Gouba, 2016). Therefore, we also attempt to generalize the scheme (CGQ) to accommodate the noncommutative quantum mechanics with $\star$-product. The idea of this part is to deform Hopf algebra structures with the Drinfeld twist in the sense of deformation quantization. A deformed Hopf algebra leads to the noncommutative algebra of the phase space, and this agrees with what is in the literature.

The second part of the thesis is about the quantum group (or Hopf algebra). In Celeghini et al. (1990, 1991), Heisenberg and Euclidean quantum group have been contracted from the $SU(2)_q$ quantum group. However, our contribution is to explore the contraction method from the two-parameter quantum group namely $SU(2)_{q,p}$ to obtain the two-parameter deformation of Heisenberg $H^1_{q,p}$, deformed Heisenberg $H^2_{\theta q,p}$ and Euclidean $E^2_{q,p}$ quantum group.

The problems which are highlighted here will be elaborated in detail from Chapters 4 until 7 respectively. Finally, one summarizes the work in the Chapter 8. The objectives of this thesis are:

1. To quantize the system of a particle moving on noncommutative plane $\mathbb{R}^2_\theta$ with, and without the effect of an external magnetic field.
2. To extend the quantization on nonlinear configuration space namely noncommutative two-torus.
3. To develop the quantization program for twisted phase space, particulaly $\mathbb{R}^2_\theta \times \mathbb{R}^2$.
4. To explore the Heisenberg and Euclidean quantum group with two-parameter deformation.

1.3 Organization

The organization of the thesis will be as follows:

Chapter 2: We will review some literature that are related to our work, namely canonical group quantization (CGQ) and noncommutative systems. We discuss the foundations of quantum theory and general quantization prescription as well as outlines of some quantization approaches. This is followed by various approaches to noncommutative quantum mechanics. (NCQM).

Chapter 3: In this chapter, we present the mathematical tools and theoretical background that will be used throughout the thesis related to CGQ as our main method. We first discuss the symplectic manifold as classical phase space. Since CGQ approach is based on two steps; canonical group, and its unitary irreducible representations, we will review Lie group and algebra, as well as group representation and their irreducibility and unitary conditions. We also will cover Hopf algebra, and twist elements that are used to study the deformed system in Chapter 6 and quantum group in Chapter 7. This is followed by our methodology is our main methodology namely CGQ. The method will be reviewed in detail, and some related examples of quantization on $\mathbb{R}^2$, T^2, S^2, and $\mathbb{R}^2_+$ will be given.

Chapter 4: This chapter is the backbone of the thesis, whose purpose is to quantize the noncommutative plane by using canonical group quantization. This quantization work is applied to a particle that propagates through a noncommutative plane where the positions no longer commute. The canonical group for the phase space with modified symplectic structure is obtained, followed by its unitary irreducible representation. We then extend this work, to study the quantum system using Landau and symmetric gauges of the magnetic field, and three-dimensional noncommutative system. The chapter studies the noncommutativity of momenta which also enlarges the algebra, and modifies some symplectic structures that finally we have some of the groups and their representations. We apply noncommutative phase space to study classical and quantum mechanics, with the examples of such systems are also given, respectively. We also apply the representations in supersymmetric quantum mechanics.

Chapter 5: In this chapter, we study the quantization on nonlinear configuration space namely two-torus T^2. Following Chapter 4, we also modify symplectic form with an additional term in order to obtain noncommutativity of ϕ^i coordinates. The nonclosure of the algebra becomes a problem there, and therefore, will be resolved using Inönü-Wigner contraction procedure. As a result, we have found the extended Heisenberg group from Chapter 4 as the canonical group for the noncommutative torus. This also agrees with the Heisenberg algebra of the NCG of guiding centers (Haldane, 2018).

Chapter 6: Quantization on twisted phase space will be done using Hopf algebra with Drinfeld twist to obtain noncommutative classical mechanics based on Aschieri et al. (2008); Aschieri (2009). Thereafter, the work continues with its group representations.

Chapter 7: The second part of our work is to study the two-parameter deformation of quantum group where it can be associated with Hopf algebra. The chapter basically uses the Celeghini et al. (1991, 1990)'s contraction method on $SU(2)_{q,p}$ to obtain Heisenberg and Euclidean quantum group with two-parameter deformation, respectively are $\mathtt{H}^i_{q,p}$ and $E^2_{q,p}$. The discussion starts with $SU(2)_{q,p}$ quantum group which is the generalization of the work of Biedenharn (1989) to develop the representation of $SU(2)_{q,p}$ quantum group that is based on deformed boson operators ((q,p)-boson operator in our case). We then extend the work to develop (q,p)-extended Heisenberg quantum group from the extended group in Chapter 4.

Chapter 8: The chapter ends with some concluding remarks, and further outlook.

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LIST OF PUBLICATIONS

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