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ESTIMATION OF MULTIPLE EXPONENTIAL SUMS ASSOCIATED WITH QUARTIC POLYNOMIALS

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By

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ESTIMATION OF MULTIPLE EXPONENTIAL SUMS ASSOCIATED WITH QUARTIC POLYNOMIALS

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March 2018

Chairman : Associate Professor Siti Hasana Binti Sapar, PhD
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Let p be a prime number and $f(x,y)$ be a polynomial in $Z_p[x,y]$. For $\alpha > 1$, the exponential sums associated with f modulo a prime p^α is defined as $S(f;p^\alpha) = \sum_{x,y \bmod p^\alpha} e_{p^\alpha}(f(x,y))$. Estimation of $S(f;p^\alpha)$ has been shown to depend on the cardinality of common roots of the partial derivative polynomials, f_x and f_y of f. Such cardinality then has been shown can be derived from the p-adic orders of common roots of the partial derivative polynomials, f_x and f_y in the neighbourhood of (x_0, y_0). The objective of this research is to arrive at such estimations associated with three different quartic polynomials.

To achieve this objective we employ the Newton polyhedron technique to estimate the p-adic sizes of common zeros of partial derivative polynomials associated with the three quartic polynomials considered. The combination of indicator diagrams associated with the polynomials are examined and analyzed on cases where p-adic sizes of common zeros occur at the intersection point of the indicator diagrams. In addition, we apply certain conditions to ensure the existence of common zeros of partial derivative polynomials associated with the three quartic polynomials considered.

The information obtained above is then applied to estimate the cardinality of the set $V(f_x, f_y; p^\alpha)$. This estimation is then applied in turn to arrive at the estimation of exponential sums for the polynomials considered.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

PENGANGGARAN HASIL TAMBAH EKSPONEN BERGANDA DISEKUTUKAN DENGAN BENTUK KUARTIK

Oleh

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Katakan p suatu nombor perdana dan $f(x,y)$ suatu polinomial dalam $Z_p[x,y]$. Untuk $\alpha > 1$, hasil tambah eksponen yang disekutukan dengan f modulo p^α ditakrifkan sebagai $S(f;p^\alpha) = \sum_{x,y \bmod p^\alpha} e_{p^\alpha}(f(x,y))$ yang dinilaikan bagi semua x dan y di dalam set reja lengkap modulo p^α. Penganggaran $S(f;p^\alpha)$ telah ditunjukkan bersandar kepada kekardinalan pensifar sepunya polinomial terbitan separa, f_x dan f_y bagi f. Kekardinalan tersebut kemudiannya ditunjukkan boleh diterbitkan dari saiz p-adic pensifar sepunya polinomial terbitan separa, f_x dan f_y dalam kejiranan (x_0, y_0). Objektif kajian ini adalah untuk mendapatkan penganggaran hasil tambah eksponen disekutukan dengan tiga polinomial berbentuk kuartik yag berbeza.

Untuk mencapai objektif di atas kami menggunakan teknik polihedron Newton untuk menganggarkan saiz p-adic pensifar sepunya polinomial terbitan separa yang disekutukan dengan tiga polinomial kuartik yang dipertimbangkan. Kombinasi gambar rajah penunjuk yang disekutukan dengan polinomial di atas diperiksa dan dianalisis bagi kes saiz p-adic pensifar sepunya yang berlaku di titik persilangan gambar rajah penunjuk. Di samping itu, kami mengenakan syarat-syarat tertentu bagi memastikan kewujudan pensifar sepunya polinomial terbitan separa yang disekutukan dengan tiga polinomial kuartik yang dipertimbangkan.

Keputusan yang diperolehi digunakan untuk menganggarkan kekardinalan bagi set $V(f_x, f_y; p^{\alpha})$. Penganggaran tersebut kemudiannya digunakan untuk mendapatkan penganggaran hasil tambah eksponen yang disekutukan dengan polinomial kuartik yang dipertimbangkan.

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This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Doctor of Philosophy. The members of the Supervisory Committee were as follows:

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LIST OF SYMBOLS AND ABBREVIATIONS

p	Prime number
α	Exponent of prime numbers
Z	Ring of integers
R	Field of real numbers
Q	Field of rational numbers
Z_p	Ring of p-adic numbers
Q_p	Field of p-adic numbers
$\overline{Q_p}$	Algebraic closure of Q_p
Ω_p	Completion of $\overline{Q_p}$
$\underline{x}$	n-tuple of variable $(x_1,\ldots,x_n)$
$\underline{f}$	n-tuple $(f_1, f_1 \ldots, f_n)$ of polynomials, $n \geq 1$
$\deg(\underline{f})$	Degree of $\underline{f}$
$\Delta(f)$	Discriminant of f
N_f	Newton polyhedron of f
$ord_p a$	Highest power of p which divides a
V	Vertex of N_f
E	Edge of N_f
F	Face of N_f
δ	Determinant factor
max	Maximum
min	Minimum
mod	Modulo
exp	Exponential

$e_k(f(t))$	$e^{2\pi i f(t)/k}$
$\sum$	Summation
det A	Determinant A
$V(\underline{f};p^\alpha)$	Set $\{x \bmod p^\alpha : \underline{f} \equiv 0 \bmod p^\alpha\}$
$N(\underline{f};p^\alpha)$	Cardinality of set $V(\underline{f};p^\alpha)$
$S(f;q)$	Exponential sums of f

CHAPTER 1

INTRODUCTION

1.1 Introduction

To obtain estimates of the multiple exponential sums has been the objective of research of a number of number theorists. In this chapter, we will first go through the background of multiple exponential sums by discussing results and methods of earlier researchers. Subsequently, questions or problems that can be solved will be discussed in the problem statement. This is followed by our research objectives that state the concentration of our research and the methods we will employ. Finally, the summary of thesis will be given.

1.2 Background

Refer to Burton's (2011) book, in year 1782 Waring wrote in his book that each positive integer is expressible as a sum of at most 9 cubes, also a sum of at most 19 fourth powers, and so on. Waring's assertion has been interpreted as for a given k, can a number $g(k)$ be sought such that every $N > 0$ can be represented in at least one way as

$$N = {a_1}^k + {a_2}^k + \cdots + {a_{g(k)}}^k$$

where the a_i are nonnegative integers, not necessarily distinct. The basic tool in the estimation of the numbers of solution is the exponential sums.

The exponential sums can be divided into two types, complete exponential sum and incomplete exponential sum. A complete exponential sum is typically a sum over all residue classes modulo an integer q. An incomplete exponential sum is a sum where the range of summation is restricted by an inequality.

Let $\underline{x} = (x_1, x_2, x_3, \dots, x_n)$ denote a vector in the space Z^n where Z denotes the ring of integers. Let q be a positive integer and f a polynomial in $Z[x]$. The multiple exponential sums associated with f as defined by Loxton and Smith (1982a) is

$$S(f;q) = \sum_{\underline{x} \bmod q} \exp\left(\frac{2\pi i f(x)}{q}\right).$$

where the sum is taken over a complete set of $\underline{x}$ mod q.

Refer to Korobov's (2011) book, in year 1811 Gauss first introduced Gaussian sums in the form

$$\sum \exp\left(\frac{2\pi i r^2}{p}\right)$$

where p is a prime. Weyl then provided the first general method of bounding exponential sums in connection with the study of uniform distribution in year 1916.

In 1919, Hardy and Littlewood proved that

$$|S(f;q)| \leq c(k)q^{1-\frac{1}{k}}$$

where $c(k)$ is a positive constant depending on k, q is an integer > 1 and $S(q, f(x)) = q^{1-\frac{1}{k}}$.

Mordell (1932) proved that for $q = p$ where p is large prime,

$$|S(f;p)| = O\left(p^{1-\frac{1}{k}}\right)$$

where $k \geq 3$ is the degree of a polynomial with integral coefficients and O is a constant that depends only on k.

Let $f(x) = ax^n + bx$ with $(a, q) = 1$. Davenport and Heilbronn (1936) showed that

$$S(ax^n + bx; q) \ll_\varepsilon q^{\theta+\varepsilon}(q, b)$$

where $\ll$ means much less than, $\theta = \frac{2}{3}$ when $n = 3$ and $\theta = \frac{3}{4}$ when $n \geq 4$.

Let $P \in Z[x]$ be a polynomial of degree $k \geq 3$ of the form

$$P(x) = a_n x^n + \cdots + a_1 x + a_0.$$

Suppose $\alpha_1, \alpha_2, \cdots, \alpha_n$ be the roots of P. Then the discriminant Δ of P is defined by

$$\Delta(P) = {a_n}^{2n-2} \prod_{1 \le i < j \le n} \left(\alpha_i - \alpha_j\right)^2.$$

Let $P'(x) = \left(R(x)\right)^t \left(R_1(x)\right)^{t_1} \cdots \left(R_s(x)\right)^{t_s}$, $t \ge t_1 \ge \cdots \ge t_s$ where $R(x)$, $R_1(x)$, $\cdots$ are different irreducible polynomials with integral coefficients. Hua (1938) showed that

$$|S(P(x)\,;q)| < c_1\left(\,\Delta(P)^{2k^2}, q\right) q^{1-\frac{1}{k}+\varepsilon}$$

for any $\varepsilon > 0$, where $c_1$depends only on k and ε.

In 1940, Hua found that for every $\varepsilon > 0$,

$$|S(f;q)| \le c q^{1-\left(\frac{1}{m}\right)+\varepsilon}$$

where c is a constant which depends only on the degree of f, m and ε.

Min (1947) proved that

$$|S(f;q)| = O\left(q^{m\left(1-\frac{1}{n}\right)}\right)$$

where the constant O depends on m, number of variables and n, the degree of polynomial f.

In 1948, Weil proved that

$$|S(f;p)| \le (\deg f - 1) p^{\frac{1}{2}}$$

where p is a prime and $f \notin pZ[X]$.

An algebraic field F is a finite degree field extension of the field of rational numbers Q. Let K be an algebraic field of degree n over the rational field. Hua (1951) worked on exponential sums over an algebraic number field and found that

$$|S(f;q)| = O\left(N(q)^{1-\frac{1}{k}+\varepsilon}\right)$$

where the constant implied by the symbol O depends only on k, n and ε.

As mentioned in paper of Mohd Atan and Loxton (2006) that in year 1974 Deligne showed that for a prime p,

$$|S(f;p)| \leq (m-1)^n p^{\frac{n}{2}}$$

where m denotes the total degree of a polynomial f, when the homogeneous part of f of highest degree is non-singular modulo p.

Chubarikov (1976) proved the general estimate

$$|S(f;p)| \leq e^{7d'n} 3^{n\upsilon(q)} \tau(q)^{n-1} q^{n-1/d'},$$

provided that the content of f is prime to q, where d' is the maximum degree of f in any variable, $\upsilon(q)$ is the number of distinct prime divisor of q and $\tau(q)$ is the number of divisors of q.

In 1977, Chen considered k be an integer ≥ 3 and $f(x) = a_k x^k + \cdots + a_1 x + a_0$ be a polynomial with integral coefficients such that $(a_1, \cdots, a_k, q) = 1$, where q is a positive integer. He showed that

$$|S(f;q)| \leq c_4(k) q^{1-\frac{1}{k}},$$

where
$$c_4(k) = \begin{cases} e^{4k} & \text{for } k \geq 10, \\ e^{c_5(k)k} & \text{for } 3 \leq k \leq 9, \end{cases}$$

and $c_5(3) = 6.1$, $c_5(4) = 5.5$, $c_5(5) = 5$, $c_5(6) = 4.7$, $c_5(7) = 4.4$, $c_5(8) = 4.2$, $c_5(9) = 4.05$.

Chen then replaced the q above by p^l where p is a prime and l is a positive integer while considered the same polynomial and conditions. He obtained that

$$|S(f;p^l)| \leq c_3(k)p^{l\left(1-\frac{1}{k}\right)}$$

where $$c_3(k) = \begin{cases} 1 & \text{for } p \geq (k-1)^{\frac{2k}{k-2}}, \\ k^{\frac{2}{k}} & \text{for } (k-1)^{\frac{2k}{k-2}} > p \geq (k-1)^{\frac{k}{k-2}}, \\ k^{\frac{3}{k}} & \text{for } (k-1)^{\frac{k}{k-2}} > p > k, \\ (k-1)k^{\frac{3}{k}} & \text{for } p \leq k. \end{cases}$$

Smith (1980) proved that if the discriminant $\Delta(F')$ of the derivative F' of F does not vanish. Then

$$|S(f;p)| \leq q^{1/2}(\Delta(F'),q)d_m(q)$$

holds for all $q \geq 1$, where $d_m(q)$denotes the number of representations of q as a product of m positive integers and $(\Delta(F'),q)$ denotes the greatest common divisor of $\Delta(F')$ and q.

Let p be a prime, F be a non-linear polynomial in $Z[x]$ of degree $m+1$ such that $D(\nabla F) \neq 0$ and n number of variables. Loxton and Smith (1982a) showed that for any $\alpha > 1$,

$$|S(f;p^\alpha)| \leq m^n p^{\frac{n\alpha}{2}}(D(\nabla F)^5, p^\alpha)^{\frac{n}{2}}.$$

In the same year, Loxton and Smith (1982b) considered f a polynomial in $Z[x]$ of degree $m+1$ with $m \geq 2$ and f' has exponent e written as

$$f(x) = a_0 \prod_{\xi} (x-\xi)^{e_\xi}$$

where the ξ are the distinct zeros of f, and e_ξ is multiplicity of ξ, so that

$$\sum_{\xi} e_\xi = m.$$

The semi-discriminant Δ of f is defined by

$$\Delta(f) = {a_0}^{2m-2} \prod_{\xi \neq \eta} (\xi - \eta)^{e_\xi e_\eta}$$

where the product is over all ordered pairs ξ, η of zeros of f. Loxton and Smith (1982b) obtained that for any positive integer q,

$$|S(f; p^\alpha)| \leq q^{1-1/2e} (\Delta, q)^{1/2e} d_m(q),$$

where $d_m(q)$ denotes the number of representations of q as a product of m positive integers and $e = e(f')$.

In 1985, Loxton and Vaughan considered a polynomial f of degree $n \geq 2$ with integer coefficients, $\delta = ord_p\big(D(f')\big)$, where $\Delta(f')$ is the intersection of the fractional ideals of K, a submodule of the quotient field of an integral domain generated by the numbers

$$\frac{f^{(e_i)}(\xi_i)}{e_i!}, \; i > 1$$

and

$$\tau = \begin{cases} 1 & \text{if } p \leq n, \\ 0 & \text{if } p > n, \end{cases}$$
$$e = \max_{\xi} e_\xi,$$

where ξ are the distinct zeros of f' and e_ξ is the multiplicity of ξ.

Loxton and Vaughan (1985) obtained

$$|S(f; p^\alpha)| \leq (n-1) p^{(\alpha e + \delta + \tau)/(e+1)}.$$

Define t satisfying $p^t | (ka_k, \cdots, 2a_2, a_1)$ and $p^{t+1} \nmid (ka_k, \cdots, 2a_2, a_1)$. Let $\mu_1, \cdots, \mu_r$ be the different zeros modulo p of the congruence equation

$$p^{-t}f'(x) \equiv 0(\text{mod } p),\ 0 \leq x < p$$

and let $m_1, \cdots, m_r$ be their multiplicities. Let $\max_{1 \leq i \leq r} m_i = M = M(f)$, $m_1 + \cdots + m_r = m = m(f)$. Chalk (1987) showed that for $n \geq 2$, if $r > 0$, then

$$|S(f;p^n)| \leq mkp^{t/(M+1)}p^{n[1-1/(M+1)]}$$

and if $r = 0$, then

$$S(f;p^n) = 0 \text{ for all } n \geq 2(t+1)$$

and otherwise

$$|S(f;p^n)| \leq p^{2t+1} \text{ where } p^t \leq k.$$

Mohd Atan (1986a) showed the existence of a relationship between a Newton polyhedron and zeros of its associated polynomial. Such relationship is used to arrive at the *p*-adic estimates of the zeros. An upper bound to the *p*-adic orders of these zeros can be found using the Newton polyhedron method.

Let p be an odd prime, Z_p be the ring of *p*-adic integers, Q_p the field of *p*-adic numbers, $\overline{Q_p}$ the closure of Q_p and Ω_p to denote the algebraically closed and complete extension of the field $\overline{Q_p}$. Let $f(x,y) = ax^3 + bx^2y + cxy^2 + dy^3 + kx + my + n$ be a polynomial in $Z_p[x,y]$, $\delta = \max\left\{ord_p 3a, \frac{3}{2} ord_p b\right\}$ and and $\alpha > 0$. Mohd Atan (1989) showed that for this polynomial

$$|S(f;p^\alpha)| \leq \min\left\{p^{2\alpha}, 4p^{\frac{3\alpha}{2}+\delta}\right\}.$$

In 1991, Ding showed an improvement to Chalk's estimation of exponential sums by considering $r > 0$ and $M \geq 1$. He proved that

$$|S(f;p^n)| \leq mk^{\frac{1}{2}}p^{t/(M+1)}p^{n[1-1/(M+1)]}.$$

Suppose t is a positive integer satisfying $p^t|(ka_k, \cdots, 2a_2, a_1)$ and $p^{t+1} \nmid (ka_k, \cdots, 2a_2, a_1)$. Let $\mu_1, \cdots, \mu_r$ be different zeros modulo p of the congruence

$$p^{-t}f'(x) \equiv 0 (\text{mod } p), 0 \leq x < p$$

and let $m_1, \cdots, m_r$ be their multiplicities. Put $\max_{1 \leq i \leq r} m_i = M = M(f)$, $m_1 + \cdots + m_r = m = m(f)$. For $n \geq 2$ and $r > 0$, Ding (1997) showed that if $3 \leq p \leq m^{M+1}$, then

$$|S(f; p^n)| \leq mp^{t/(M+1)}p^{n[1-1/(M+1)]}.$$

Also, for case where $p \geq 3$ and $p > m^{M+1}$, he obtained

$$|S(f; p^n)| \leq p^{1/(M+1)}p^{t/(M+1)}p^{n[1-1/(M+1)]}.$$

Heng and Mohd Atan (1999) considered the polynomial

$$f(x, y) = ax^3 + bxy^2 + cx + dy + e$$

in $Z_p[x, y]$ where p be an odd prime, $\alpha > 1$ and $\delta = \max\left\{ord_p 3a, \frac{3}{2} ord_p b\right\}$. If $ord_p bc^2 > ord_p ad^2$, then

$$|S(f; p^\alpha)| \leq \min\left\{p^{2\alpha}, 4p^{\frac{3\alpha}{2}+\delta}\right\}.$$

They then considered the condition $bc^2 - 3ad^2 = 0$ and obtained another result

$$|S(f; p^\alpha)| \leq \min\left\{p^{2\alpha}, 2p^{\frac{3\alpha}{2}+\delta}\right\}.$$

Let f be a polynomial in $Z[x]$. Suppose grad f has rank n where grad f is gradient of f and set $\rho = \rho(\text{grad} f)$ and $\theta = \left[\frac{\alpha}{2}\right]$. Loxton (2000) found that

$$|S(f; p^\alpha)| \leq \begin{cases} p^{n\alpha} & \text{if } 1 < \alpha \leq 2\rho + 1 \\ p^{n(\alpha-\theta+\rho+1)} & \text{if } 2\rho + 2 \leq \alpha \leq 4\rho + 1 \\ (\deg f - 1)^n p^{n(\alpha-\theta+\rho)} & \text{if } \alpha > 4\rho + 1. \end{cases}$$

Let p be an odd prime, Z_p be the ring of p-adic integers and $\alpha > 1$. Let $f(x,y) = ax^3 + bx^2y + cxy^2 + dy^3 + kx + my + n$ be a polynomial in $Z_p[x,y]$ with non-zero coefficients in its cubic segment. Let $\delta = \max\{ord_p 3a, ord_p b, ord_p c, ord_p 3d\}$. Mohd Atan and Loxton (2006) showed that for this polynomial

$$|S(f;p^\alpha)| \leq \min\left\{p^{2\alpha}, 4p^{\frac{3\alpha}{2}+\delta}\right\}.$$

Let p be an odd prime and $\alpha > 1$. Let $N(f_x, f_y; p^\alpha)$ denote the cardinality of the set $V(f_x, f_y; p^\alpha)$ is the number of common solutions of congruence equation

$$f_x(x,y) \equiv 0, f_y(x,y) \equiv 0 \ (\mathrm{mod}\ p^\alpha)$$

in the complete set of residues modulo p^α, where f_x and f_y are the partial derivative polynomials with respect to x and y respectively. Estimation of multiple exponential sums depends on the estimates of cardinality of the set $V(f_x, f_y; p^\alpha)$ and also p-adic sizes of common zeros to the partial derivative polynomials associated with the polynomial f considered.

Mohd Atan (1986b) investigate the estimation of $N(f_x, f_y; p^\alpha)$ involved the polynomial $f(x,y) = ax^3 + bxy^2 + cx + dy + e$ in $Z_p[x,y]$ and obtained

$$N(f_x, f_y; p^\alpha) \leq \begin{cases} p^{2\alpha} & \text{if } \alpha \leq \delta \\ 4p^{\alpha+\delta} & \text{if } \alpha > \delta. \end{cases}$$

where $\alpha > 0$ and $\delta = \max\left\{ord_p 3a, \frac{3}{2} ord_p b\right\}$.

Let $f(x,y) = ax^3 + bx^2y + cxy^2 + dy^3 + kx + my + n$ be a polynomial in $Z_p[x,y]$ and $\delta = \max\{ord_p 3a, ord_p b\}$, Mohd Atan and Abdullah (1992) showed that for this polynomial

$$N(f_x, f_y; p^\alpha) \leq \begin{cases} p^{2\alpha} & \text{if } \alpha \leq \delta \\ 4p^{\alpha+\delta} & \text{if } \alpha > \delta. \end{cases}$$

In 1993, they considered the same cubic form and showed that δ is in fact the p-adic order of at least one of the coefficient of the dominant terms of the cubic form. That is $\delta = \max\{ord_p 3a, ord_p b, ord_p c, ord_p 3d\}$. The form of result obtained is the same as above.

Chan and Mohd Atan (1997) investigated a quartic polynomial $f(x,y) = ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4 + rx + sy + t$ in $Z_p[x,y]$ and gave the p-adic orders of common zeros of partial derivative polynomials associated with $f(x,y)$ by employing Newton polyhedron method. They obtained $ord_p(\xi - x_0), ord_p(\eta - y_0) > \frac{1}{3}(\alpha - \delta)$ where $p > 3$ is a prime, (ξ, η) is the common solution of partial derivative polynomials, $\alpha > 0$ and $\delta = \max\{ord_p a, ord_p b, ord_p c, ord_p d, ord_p e\}$.

Subsequently, they obtained the estimates for $N(f_x, f_y; p^\alpha)$of the polynomial f as follows

$$N(f_x, f_y; p^\alpha) \leq \begin{cases} p^{2\alpha} & \text{if } \alpha \leq \delta \\ 9p^{\frac{2}{3}(2\alpha+\delta)} & \text{if } \alpha > \delta. \end{cases}$$

In 2002, Sapar and Mohd Atan considered polynomial $f(x,y) = ax^2 + bxy + cy^2 + dx + ey + m$ in $Z_p[x,y]$ with p an odd prime, $\alpha > 0$ and $\delta = \max\{ord_p a, ord_p b, ord_p c\}$. They showed that if $(x_0, y_0) \in \Omega_p^2$ such that $ord_p f_x(x_0, y_0), ord_p f_y(x_0, y_0) \geq \alpha > \delta$, then there exists (ξ, η) in ${\Omega_p}^2$such that $f_x(\xi,\eta) = 0$, $f_y(\xi,\eta) = 0$ and $ord_p(\xi - x_0) \geq (\alpha - \delta)$, $ord_p(\eta - y_0) \geq (\alpha - \delta)$.

Subsequently, they obtained the estimates for $N(f_x, f_y; p^\alpha)$ of the polynomial as follows

$$N(f_x, f_y; p^\alpha) \leq \begin{cases} p^{2\alpha} & \text{if } \alpha \leq \delta \\ p^{2\delta} & \text{if } \alpha > \delta. \end{cases}$$

Sapar and Mohd Atan (2002) also investigated a cubic polynomial $f(x,y) = ax^3 + bxy^2 + cx + dy + e$ in $Z_p[x,y]$ with p an odd prime, $\alpha > 0$ and $\delta = \max\{ord_p a, ord_p b\}$. They showed that if $(x_0, y_0) \in \Omega_p^2$ such that $ord_p f_x(x_0, y_0), ord_p f_y(x_0, y_0) \geq \alpha > \delta$, then there exists (ξ, η) in ${\Omega_p}^2$such that $f_x(\xi,\eta) = 0$, $f_y(\xi,\eta) = 0$ and $ord_p(\xi - x_0) \geq \frac{1}{2}(\alpha - \delta)$, $ord_p(\eta - y_0) \geq \frac{1}{2}(\alpha - \delta)$. For this cubic polynomial, the estimates for $N(f_x, f_y; p^\alpha)$ is

$$N(f_x, f_y; p^\alpha) \le \begin{cases} p^{2\alpha} & \text{if } \alpha \le \delta \\ 4p^{\alpha+\delta} & \text{if } \alpha > \delta. \end{cases}$$

Let polynomial $f(x,y) = ax^5 + bx^4y + cx^3y^2 + dx^2y^3 + exy^4 + my^5 + nx + ty + k$ in $Z_p[x,y]$ with $p > 5$. Suppose $\alpha > 0$ and $\delta = \max\{ord_p a, ord_p b, ord_p c, ord_p d, ord_p e, ord_p m\}$, $ord_p b^2 > ord_p ac$ and $ord_p(10cm - 2de)^2 > ord_p(10dm - 4e^2)(2ce - d^2)$. Sapar and Mohd Atan (2006) showed that if $ord_p f_x(0,0), ord_p f_y(0,0) \ge \alpha > \delta$, there exists (ξ, η) in ${\Omega_p}^2$ such that $f_x(\xi,\eta) = 0$, $f_y(\xi,\eta) = 0$ and $ord_p \xi \ge \frac{1}{4}(\alpha - \delta)$, $ord_p \eta \ge \frac{1}{4}(\alpha - \delta)$.

In the same year, Sapar et al. (2006) considered the same polynomial and condition as above and obtained p-adic sizes of common zeros to the partial derivative polynomials associated with the polynomial f in the neighbourhood of (x_0, y_0) as $ord_p(\xi - x_0) \ge \frac{1}{4}(\alpha - \delta)$, $ord_p(\eta - y_0) \ge \frac{1}{4}(\alpha - \delta)$. Subsequently, they obtained the estimates for $N(f_x, f_y; p^\alpha)$ as

$$N(f_x, f_y; p^\alpha) \le \begin{cases} p^{2\alpha} & \text{if } \alpha \le \delta \\ 16p^{\frac{1}{3}(3\alpha+\delta)} & \text{if } \alpha > \delta. \end{cases}$$

Aminudin et al. (2014) investigated $f(x,y) = ax^3 + bx^2y + cxy^2 + dy^3 + \frac{3}{2}ax^2 + bxy + \frac{1}{2}cy^2 + sx + ty + k$ a polynomial in $Q_p[x,y]$ with $p > 3$ is a prime. Suppose $\alpha > 0$ and $\delta = \max\{ord_p a, ord_p b, ord_p c, ord_p d\}$ and $(x_0, y_0) \in \Omega_p^2$. They showed that if $ord_p bc > ord_p ad$ and $ord_p f_x(x_0,y_0), ord_p f_y(x_0,y_0) \ge \alpha > 2\delta$, then there exists (ξ,η) in ${\Omega_p}^2$ such that $f_x(\xi,\eta) = 0$, $f_y(\xi,\eta) = 0$ where

$ord_p(\xi - x_0) \ge \alpha - \delta$ or $ord_p(\xi - x_0) \ge \alpha - \delta - \frac{1}{2}\varepsilon$

and $ord_p(\eta - y_0) \ge \alpha - \delta$ or $ord_p(\eta - y_0) \ge \alpha - \delta - \frac{1}{2}\varepsilon$

or $ord_p(\eta - y_0) \ge \alpha - 2\delta$ or $ord_p(\eta - y_0) \ge \alpha - 2\delta - \frac{1}{2}\varepsilon$ for some $\varepsilon > 0$.

By applying the above results, they obtained the estimates for $N(f_x, f_y; p^\alpha)$ as

$$N(f_x, f_y; p^{\alpha}) \leq \begin{cases} p^{2\alpha} & \text{if } \alpha \leq \delta \\ 4p^{4\delta+\varepsilon} & \text{if } \alpha > \delta. \end{cases}$$

for some $\varepsilon \geq 0$.

Suppose $f(x, y) = ax^7 + bx^6y + cx^5y^2 + sx + ty + k$ be a polynomial in $Q_p[x, y]$ with $p > 7$ is a prime. Let $\alpha > 0$, $\delta = \max\{ord_p a, ord_p b, ord_p c\}$ and $(x_0, y_0) \in \Omega_p^2$. Lasaraiya et al. (2015) showed that If $ord_p b^2 \neq ord_p ac$, $ord_p f_x(x_0, y_0), ord_p f_y(x_0, y_0) \geq \alpha > 7\delta$, then there exists (ξ, η) in ${\Omega_p}^2$ such that $f_x(\xi, \eta) = 0$, $f_y(\xi, \eta) = 0$ where

$$ord_p(\xi - x_0) \geq \frac{1}{6}(\alpha - \delta) - \varepsilon_1 \text{ and}$$

$$ord_p(\xi - x_0) \geq \frac{1}{6}(\alpha - \delta) - \varepsilon_2 \text{ and}$$

$$ord_p(\eta - y_0) \geq \frac{1}{6}(\alpha - 3\delta) - \varepsilon_3 \text{ or}$$

$$ord_p(\eta - y_0) \geq \frac{1}{6}(\alpha - 3\delta) - \varepsilon_4 \text{ or}$$

$$ord_p(\eta - y_0) \geq \frac{1}{6}(\alpha - 4\delta) - \varepsilon_3 \text{ or}$$

$$ord_p(\eta - y_0) \geq \frac{1}{6}(\alpha - 4\delta) - \varepsilon_4 \text{ or}$$

$$ord_p(\eta - y_0) \geq \frac{1}{6}(\alpha - 3\delta) - \frac{2}{3}\varepsilon_0 - \varepsilon_3 \text{ or}$$

$$ord_p(\eta - y_0) \geq \frac{1}{6}(\alpha - 4\delta) - \frac{2}{3}\varepsilon_0 - \varepsilon_4 \text{ or}$$

$$ord_p(\eta - y_0) \geq \frac{1}{6}(\alpha - 4\delta) - \frac{2}{3}\varepsilon_0 - \varepsilon_3 \text{ or}$$

$$ord_p(\eta - y_0) \geq \frac{1}{6}(\alpha - 4\delta) - \frac{2}{3}\varepsilon_0 - \varepsilon_4 \text{ or}$$

$$ord_p(\eta - y_0) \geq \frac{1}{6}(\alpha - 5\delta) - \varepsilon_3 \text{ or}$$

$$ord_p(\eta - y_0) \geq \frac{1}{6}(\alpha - 5\delta) - \varepsilon_4 \text{ or}$$

$$ord_p(\eta - y_0) \geq \frac{1}{6}(\alpha - 6\delta) - \varepsilon_3 \text{ or}$$

$$ord_p(\eta - y_0) \geq \frac{1}{6}(\alpha - 6\delta) - \varepsilon_4 \text{ or}$$

$$ord_p(\eta - y_0) \geq \frac{1}{6}(\alpha - 5\delta) - \frac{2}{3}\varepsilon_0 - \varepsilon_3 \text{ or}$$

$$ord_p(\eta - y_0) \geq \frac{1}{6}(\alpha - 5\delta) - \frac{2}{3}\varepsilon_0 - \varepsilon_4 \text{ or}$$

$$ord_p(\eta - y_0) \geq \frac{1}{6}(\alpha - 6\delta) - \frac{2}{3}\varepsilon_0 - \varepsilon_3 \text{ or}$$

$$ord_p(\eta - y_0) \geq \frac{1}{6}(\alpha - 6\delta) - \frac{2}{3}\varepsilon_0 - \varepsilon_4$$

for some $\varepsilon_0, \varepsilon_2, \varepsilon_4 \geq 0$ and $\varepsilon_1, \varepsilon_3 > 0$. Subsequently, they obtained the estimates for $N(f_x, f_y; p^\alpha)$ as

$$N(f_x, f_y; p^\alpha) \leq \begin{cases} p^{2\alpha} & \text{if } \alpha \leq \delta \\ 36p^{12\delta + 8\varepsilon_0 + 12q} & \text{if } \alpha > \delta. \end{cases}$$

for some $\varepsilon_0, q \geq 0$ where $q = \max\{\varepsilon_3, \varepsilon_4\}$.

Let $f(x,y) = ax^{11} + bx^{10}y + cx^9y^2 + sx + ty + k$ be a polynomial in $Z_p[x,y]$ with $p > 11$ is a prime. Suppose $\alpha > 0$, $\delta = \max\{ord_p a, ord_p b, ord_p c\}$ and $ord_p b^2 = ord_p ac$. Lasaraiya et al. (2016) showed that if $ord_p f_x(X + x_0), ord_p f_y(Y + y_0) \geq \alpha > \delta$, then there exists (ξ, η) in $\Omega_p{}^2$ such that $f_x(\xi, \eta) = 0$, $f_y(\xi, \eta) = 0$ where

$$ord_p(\xi - x_0) \geq \frac{1}{10}(\alpha - \delta) - \frac{1}{20}\varepsilon_0 - \varepsilon_2,$$

$$ord_p(\xi - x_0) \geq \frac{1}{10}(\alpha - \delta) - \frac{1}{20}\varepsilon_0 - \varepsilon_3,$$

$$ord_p(\xi - x_0) \geq \frac{1}{10}(\alpha - \delta) - \frac{3}{20}\varepsilon_0 - \varepsilon_2,$$

$$ord_p(\xi - x_0) \geq \frac{1}{10}(\alpha - \delta) - \frac{3}{20}\varepsilon_0 - \varepsilon_3,$$

$$ord_p(\xi - x_0) \geq \frac{1}{10}(\alpha - \delta) - \frac{3}{20}\varepsilon_0 + \frac{1}{5}\varepsilon_1 - \varepsilon_2,$$

$$ord_p(\xi - x_0) \geq \frac{1}{10}(\alpha - \delta) - \frac{3}{20}\varepsilon_0 + \frac{1}{5}\varepsilon_1 - \varepsilon_3,$$

$$ord_p(\eta - y_0) \geq \frac{1}{10}(\alpha - 9\delta) - \frac{11}{20}\varepsilon_0 - \varepsilon_4,$$

$$ord_p(\eta - y_0) \geq \frac{1}{10}(\alpha - 9\delta) - \frac{11}{20}\varepsilon_0 - \varepsilon_5,$$

$$ord_p(\eta - y_0) \geq \frac{1}{10}(\alpha - 9\delta) - \frac{3}{20}\varepsilon_0 - \varepsilon_4,$$

$$ord_p(\eta - y_0) \geq \frac{1}{10}(\alpha - 9\delta) - \frac{3}{20}\varepsilon_0 - \varepsilon_5,$$

$$ord_p(\eta - y_0) \geq \frac{1}{10}(\alpha - 9\delta) - \frac{3}{20}\varepsilon_0 - \frac{4}{5}\varepsilon_1 - \varepsilon_4,$$

and $ord_p(\eta - y_0) \geq \frac{1}{10}(\alpha - 9\delta) - \frac{3}{20}\varepsilon_0 - \frac{4}{5}\varepsilon_1 - \varepsilon_5$

for certain $\varepsilon_0, \varepsilon_1, \varepsilon_3, \varepsilon_5 \geq 0$ and $\varepsilon_2, \varepsilon_4 > 0$. Subsequently, they obtained the estimates for $N(f_x, f_y; p^\alpha)$ as

$$N(f_x, f_y; p^\alpha) \leq \begin{cases} p^{2\alpha} & \text{if } \alpha \leq \delta \\ 100p^{2\left(9\delta + \frac{3}{2}\varepsilon_0 + 8\varepsilon_1 + 10q\right)} & \text{if } \alpha > \delta. \end{cases}$$

for some $\varepsilon_0, \varepsilon_1, q \geq 0$ where $q = \max\{\varepsilon_4, \varepsilon_5\}$.

In Malaysia, research on exponential sums are focused on complete exponential sums by considering polynomials in $Q_p[x, y]$ and $Z_p[x, y]$ and employing the Newton polyhedron technique.

1.3 Problem Statement

In the earlier works concentration were on estimation of multiple exponential sums associated with polynomials $f(x, y)$ in $Z_p[x, y]$ of odd highest degrees. In our present work we will focus on estimation of multiple exponential sums associated with a similar polynomial where highest degree is even by employing Newton polyhedron technique. The three polynomials considered are $f(x, y) = ax^4 + bx^3y + cxy^3 + dy^4 + rx + sy + t$, $f(x, y) = ax^4 + bx^2y^2 + cxy^3 + dy^4 + rx + sy + t$ and $f(x, y) = ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4 + rx + sy + t$. The *p*-adic orders of common zeros of partial derivative polynomials associated with the polynomials will be obtained to estimate the cardinality and multiple exponential sums associated with each polynomial considered. Focus is given to cases where conditions are needed to ensure the existence of distinct common roots of partial derivative polynomials associated with the quartic polynomials.

1.4 Research Objectives

The main objective of this study are :

- To investigate cases where the *p*-adic orders of common zeros of partial derivatives occur on intersection point of indicator diagrams associated with partial derivative polynomials of three different quartic polynomials.
- To obtain the estimates of cardinality by examine the indicator diagrams.
- To obtain estimates of multiple exponential sums associated with three different quartic polynomials considered.

1.5 Organization of Thesis

In Chapter 2, we discuss Newton polyhedron technique which plays an important role as a tool to obtain the p-adic orders of common zeros of partial derivative polynomials associated with the polynomials considered. We begin by giving the definition of Newton polygon in Section 2.2. Two examples are given to illustrate the construction of Newton polygon and information of roots that can be obtained from it. Subsequently, the definitions and examples of Newton diagram and Newton polyhedron are given in Section 2.3 as Newton polyhedron is an analogue of Newton polygon. In Section 2.4, normal of Newton polyhedron will be discussed. The construction of indicator diagram will be discussed by giving the definition and example in Section 2.5. The existence of common roots to two polynomials and its p-adic orders on the associated indicator diagrams are also discussed.

In Chapter 3, the Newton polyhedron technique is applied to obtain p-adic orders of common zeros of partial derivative polynomials associated with quartic polynomial of the form $f(x,y) = ax^4 + bx^3y + cxy^3 + dy^4 + rx + sy + t$. Firstly, p-adic orders of common zeros of partial derivative polynomials associated with $f(x,y)$ in the neighbourhood of $(0,0)$ are obtained in Section 3.2. Subsequently, p-adic orders of common zeros of partial derivative polynomials associated with $f(x,y)$ in the neighbourhood of (x_0, y_0) are obtained in Section 3.3.

In Chapter 4, the quartic polynomial $f(x,y) = ax^4 + bx^2y^2 + cxy^3 + dy^4 + rx + sy + t$ is investigated and p-adic orders of common zeros of partial derivative polynomials associated with $f(x,y)$ under certain two conditions are obtained by employing Newton polyhedron technique. In Section 4.2, p-adic orders of common zeros of partial derivative polynomials associated with $f(x,y)$ in the neighbourhood of (x_0, y_0) subject to the condition $ord_p ac^2 > ord_p b^3$ are obtained. In Section 4.3, p-adic distance of common zeros of partial derivative polynomials associated with $f(x,y)$ in the neighbourhood of (x_0, y_0) under condition $ord_p b^3 > ord_p ac^2$ is obtained.

In Chapter 5, the complete quartic polynomial $f(x,y) = ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4 + rx + sy + t$ are considered and p-adic orders of common zeros of partial derivative polynomials associated with $f(x,y)$ under six conditions are obtained by employing Newton polyhedron technique. In Section 5.2, the condition $ord_p \frac{b}{c} > ord_p \lambda > ord_p \frac{a}{b}$ are considered in obtaining the p-adic sizes of common zeros in the neighbourhood of (x_0, y_0). In Section 5.3, the estimates of p-adic sizes of common zeros in the neighborhood of (x_0, y_0) are obtained under the condition $ord_p \frac{a}{b} > ord_p \lambda > ord_p \frac{b}{c}$. In Section 5.4, p-adic sizes of common zeros in the neighbourhood of (x_0, y_0) subject to condition $ord_p \lambda > ord_p \frac{b}{c} > ord_p \frac{a}{b}$ are determined. In Section 5.5, the condition $ord_p \lambda > ord_p \frac{a}{b} > ord_p \frac{b}{c}$ is considered in obtaining the p-adic sizes of common zeros in the neighbourhood of (x_0, y_0). In Section 5.6, the estimates of p-adic sizes of common zeros in the neighbourhood of (x_0, y_0) are obtained under the condition $ord_p \frac{b}{c} > ord_p \frac{a}{b} > ord_p \lambda$. In Section 5.7, p-adic sizes of common zeros in the neighbourhood of (x_0, y_0) subject to condition $ord_p \frac{a}{b} > ord_p \frac{b}{c} > ord_p \lambda$ are determined.

In Chapter 6, the cardinality of the set of solutions to congruence equations of partial derivative polynomial associated with three polynomials investigated are obtained. For the first quartic polynomial $f(x,y) = ax^4 + bx^3y + cxy^3 + dy^4 + rx + sy + t$, results from Chapter 3 are applied to obtain cardinality of the set of solutions to congruence equations of partial derivative polynomial associated with $f(x,y)$ in Section 6.2. In Section 6.3, results from Chapter 4 are applied to obtain estimates of cardinality of set of solutions to congruence equations of partial derivative polynomial associated with $f(x,y) = ax^4 + bx^3y + cxy^3 + dy^4 + rx + sy + t$. Lastly, estimates of cardinality of set of solutions to congruence equations of partial derivative polynomial associated with $f(x,y) = ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4 + rx + sy + t$ are obtained in Section 6.4 by applying results from Chapter 5.

In Chapter 7, estimates of multiple exponential sums associated with three quartic polynomials investigated are obtained by applying results from Chapter 6. In Section 7.2, estimates of multiple exponential sums associated with $(x,y) = ax^4 + bx^3y + cxy^3 + dy^4 + rx + sy + t$ are shown. In Section 7.3, the polynomial $f(x,y) = ax^4 + bx^3y + cxy^3 + dy^4 + rx + sy + t$ are considered and the associated estimates of multiple exponential sums under the conditions $ord_p ac^2 > ord_p b^3$ and $ord_p b^3 > ord_p ac^2$ are obtained. Lastly, 13 cases of estimation of multiple exponential sums associated with $f(x,y) = ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4 + rx + sy + t$ are shown in Section 7.4.

Finally, the major results, conclusion and future research are given in Chapter 8.

1.6 Conclusion

In this chapter, we have discussed the background of multiple exponential sums from earlier works until recent research. The estimates of upper bounds of exponential sums associated with the quartic polynomials that can be obtained are discussed in Section 1.3. The focuses of our research and method employed are mentioned in Section 1.4. Lastly, the summary of thesis are given in Section 1.5.

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