UNIVERSITI PUTRA MALAYSIA

PURSUIT DIFFERENTIAL GAME OF MANY PURSUERS AND ONE EVADER IN A CONVEX HYPERSPACE

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By

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Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfillment of the Requirements for the Master of Science

June 2020

DEDICATIONS

To my mother, father, Along, Angah, Adek, Aida, and my future.

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfillment of the requirement for the degree of Master of Science

PURSUIT DIFFERENTIAL GAME OF MANY PURSUERS AND ONE EVADER IN A CONVEX HYPERSPACE

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June 2020

Chairman : **Associate Professor Idham Arif bin Hj Alias, PhD**
Faculty : **Science**

A pursuit differential game of many pursuers and one evader in a nonempty close convex compact hyperspace is studied. The problem is to construct strategies for pursuers to complete the pursuit where movement of the evader can be any. The objective consists of three items. The first objective is to solve the pursuit differential game of three pursuers and one evader in a three-dimensional cube. Secondly is to solve the pursuit differential game of many pursuers and one evader in a convex hyperspace and finally is to find the improved guaranteed pursuit time in the game of many pursuers and one evader in a convex hyperspace. The pursuit problem in a cube is first studied where the method used includes selecting planes on which each pursuer moves using parallel strategy, and we also introduces fictitious pursuer to accomplish pursuit. The problem is then extended to a nonempty close convex compact hyperspace by also using fictitious pursuers, and we proceed to calculate the guaranteed pursuit time of the game. This study constructed the method of applying the parallel strategy of pursuers where the pursuit can be completed in both games mentioned and the guaranteed pursuit time for the game of many pursuers versus one evader in a convex hyperspace space of diameter a is $T = \frac{a}{2}\left(\sqrt{n} + n\sqrt{n-1} + n\right)$, which is an improvement from $O(n^2)$ to $O(n)$.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Master Sains

PERMAINAN PEMBEZAAN PENGEJARAN OLEH RAMAI PENGEJAR DAN SATU PENGELAK DALAM RUANG HIPER CEMBUNG

Oleh

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Permainan pembezaan pengejaran oleh ramai pengejar dan satu pengelak dalam ruang hiper cembung yang tidak kosong, tertutup dan padat telah dikaji. Masalah kajian ini adalah untuk membina strategi untuk pengejar menyelesaikan pengejaran di mana pergerakan pengelak adalah sebarang dan tanpa strategi. Objektif kajian ini terdiri daripada tiga perkara. Objektif pertama adalah untuk menyelesaikan permainan pembezaan pengejaran oleh tiga pengejar dan satu pengelak dalam kuib tiga-dimensi. Ke-dua adalah untuk menyelesaikan permainan pembezaan pengejaran oleh ramai pengejar dan satu pengelak dalam ruang hiper cembung dan akhir sekali untuk mencari jaminan masa pengejaran yang lebih baik untuk permainan pembezaan pengejaran oleh ramai pengejar dan satu pengelak dalam ruang hiper cembung. Kita mulakan penyelesaian untuk permainan pengejaran di dalam ruang kiub terdahulu, di mana kaedah yang digunakan termasuk memilih plana sebagai ruang pergerakan para pengejar untuk menggunakan strategi selari, dan kita memperkenalkan pengelak bayangan untuk mencapai pengejaran yang pasti. Seterusnya, masalah ini dilanjutkan ke ruang hiper cembung yang tidak kosong, tertutup dan padat dengan menggunakan pengelak bayangan juga, dan kita teruskan dengan kiraan jaminan masa pengejaran untuk permainan tersebut. Kajian ini telah membina kaedah untuk mengaplikasikan strategi selari untuk pengejar di mana terbukti bahawa pengejar dapat menangkap pengelak di dalam kedua-dua permainan tersebut dan jaminan masa pengejaran untuk permainan oleh ramai pengejar dan satu pemangsa di dalam ruang hiper cembung yang mempunyai diameter a adalah $T = \frac{a}{2}\left(\sqrt{n} + n\sqrt{n-1} + n\right)$, dimana ianya adalah penambahbaikan dari $O(n^2)$ ke $O(n)$.

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This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Master of Science. The members of the Supervisory Committee were as follows:

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LIST OF ABBREVIATIONS

P	Pursuer
E	Evader
GPT	Guaranteed Pursuit Time
PCC	Pursuit can be completed
EP	Evasion is possible

CHAPTER 1

INTRODUCTION

1.1 Game Theory

Game theory is a branch of mathematics that studies strategic situation for players involve in the game in order to allow each player to achieve their goal. Any game involve at least two players and the strategy constructed must satisfy certain constraint. Differential game was first introduced by Isaacs (1965). In differential game theory, movements of players are described by some differential equations where the state of players depends continuously on time and therefore, $x(t)$, $\dot{x}(t)$, or $x(t)^n$ are variables that play parts in the calculation of the problem. There exists two opposite parties with opposing goals; pursuer and evader. The pursuer's goal is to capture the evader while the evader's goal is to avoid being captured.

We are interested in differential game theory because of their extensive applications in the real world such as military, economy, management, optimal control, engineering, biology, security, search and rescue, robotic, and others. For example in a warfare technology, a strategy is constructed for missiles to move towards any airplane regardless of the movement of the plane. On the other hand, strategy is also formulated for the military plane to avoid any missiles launched in its direction. There are multiple more peaceful applications such as economy which covers the areas such as capital accumulation, industrial organization, marketing and environmental economics. Besides, differential game applications in security is to prevent intruders from performing harmful actions against some persons or territories of strategic importance. This may include the protection of critical infrastructure, transportation systems and international borders. Isaacs (1965) mentioned that some typical models for differential game are certain types of battles, airplanes, dog-fighting, football, a torpedo pursuing a ship, a missile intercepting an aircraft, and a gunner guarding a target against an invader.

In a certain game, a strategy is constructed. A strategy, simply put, is a mapping from state to action instructing the player on what to do in a given situation. As the game progresses, the players choose actions in terms of their control functions rendering changes in the state variables. This goes on until some specified event takes place, which causes the game to terminate.

Aspects which are always considered in a differential game theory are type of game, number of players, space of trajectory and constraints applied on control of the players. In general, there are three types of differential games which are; pursuit differential game, evasion differential game, and pursuit-evasion differential game. Pursuit differential game constructs the pursuer strategy without restriction on the movement of evader. The pursuit game is completed when the position of pursuer and evader coincides at a finite time τ, that is $x(\tau) = y(\tau)$ for some time $\tau \geq 0$ and we say, pursuit can be completed. On the other hand, in evasion differential game, the strategy for the evader is formulated without restriction on the movement of the pursuer. The evader's goal is to avoid being captured by the pursuer indefinitely, that is $x(t) \neq y(t), \forall t \geq 0$ or within a certain time interval $[0,T]$ that is $x(t) \neq y(t), t \in [0,T]$ and we say, the evasion is possible. Furthermore, the pursuit-evasion differential game involves both players in a certain situation, but strategies for pursuit and evasion can be constructed separately for respective pursuers and evaders in the game. The problem can either be solved as a pursuit problem for the pursuer or for the possibility of evasion for the evader depending on the hypothesis of the suggested theorem.

Movement of players in a differential game is described by some differential equations and subjected to at least one of the three types of constraints which are; state constraint, integral constraint and geometric constraint. The state or phase constraint describes the limitation of players position, where movement of players can only occur within an area in a given space, at all time. While the integral constraint is a constraint described in integral form, where usually it refers to the resources of players that could be exhausted by consumption. Geometric constraint usually refers to the speed of players. For example, $|u| \leq 1$ is a constraint on the speed of pursuer which is applied in this paper, and it means that the maximum speed is 1.

In this study, our game is a pursuit differential game with many pursuers and one evader. We apply the state and geometrical constraint on all players. The players can only move within a defined convex set and each player has the maximum speed of 1.

1.2 Some Basic Definitions

There are a few definitions that needs to be taken into consideration for this study.

Definition 1.1 *$G(A,B)$ is a game where all pursuers are in A and evader in B.*

Definition 1.2 *Close set is a set which contains all of its boundary points. (Croft et al. (2012))*

Definition 1.3 *Convex set. Let S be a set. If $\lambda(x_1) + (1-\lambda)x_2 \in S$ for any $x_1, x_2 \in S$ where $\lambda \in [0,1]$, then S is called a convex set. (Croft et al. (2012))*

Definition 1.4 *Compact set. For any subset A of Euclidean space $\mathbb{R}^n$, A is compact if and only if it is closed and bounded. (Croft et al. (2012))*

Definition 1.5 *Control function for pursuer is a measurable function* $u_i(\cdot) = u_i(t), t \geq 0$, *if* $| u_i(t) | \leq 1, t \geq 0$, *and the solution* $x_i(\cdot) = x_i(t), t \geq 0$, *of the initial value problem* $\dot{x}_i = u_i(t), x_i(0) = x_{i0}, i \in \{1, 2, \ldots n\}$ *satisfies the inclusion*

$$x_i(t) \in A.$$

Definition 1.6 *Control function for evader is a measurable function* $v(\cdot) = v(t), t \geq 0$, *if* $| v(t) | \leq 1, t \geq 0$, *and the solution* $y(\cdot) = y(t), t \geq 0$, *of the initial value problem* $\dot{y} = v(t), y(0) = y_0$, *satisfies the inclusion*

$$y(t) \in A.$$

Definition 1.7 *Strategy of the pursuer is a function* $U_i : \mathbb{R}^n \times \mathbb{R}^n \times H(0,1) \to H(0,1)$, *such that* $U_i = U_i(x_i, y, v)$ *for* $i \in \{1, 2, \ldots, n\}$, *for any control of the evader* $v(t)$, *if the initial value problem*

$$\dot{x}_i = U_i(x_i, y, v(t)), \quad x_i(0) = x_{i0}, \tag{1.2.1}$$

$$\dot{y} = v(t), \quad y(0) = y_0, \tag{1.2.2}$$

has a unique absolutely continuous solution $(x_i(t), y(t)), t \geq 0$, *with* $x_i(t), y(t) \in A, t \geq 0$.

Definition 1.8 *Pursuit can be completed at time* τ *by the pursuer* x_i, *if* $x_i(\tau) = y(\tau)$ *for some* $i \in \{1, 2, \ldots, n\}$ *and at some* $\tau \geq 0$.

Definition 1.9 T *is called guaranteed pursuit time if there exist strategies of pursuers* $U_1, U_2, \ldots, U_n$ *such that for any control of the evader* $v(\cdot)$ *an equality* $x_i(\tau) = y(\tau)$ *holds for some* $i \in \{1, 2, \ldots, n\}$ *and* $0 \leq \tau \leq T$ *where* $(\acute{x}_i(t), y(t))$, *is the solution of (1.2.1), (1.2.2) at* $v = v(t), t \geq 0$.

Definition 1.10 *Cube N,*

$$N = \left\{ (q_1, q_2, \ldots, q_n) \mid \frac{-a}{2} \leq q_i \leq \frac{a}{2}, \ i = 1, 2, \ldots, n \right\}.$$

Definition 1.11 d_i *for* $i = 1, 2, \ldots, n$, *is the distance between pursuer* x_i *and origin, in Stage 1. While in Stage 2, it is the distance between pursuer* x_i *and evader* y_i.

Definition 1.12 α_i *is the rate of decreasing of total distance between pursuer* x_i *and evader* y.

Definition 1.13 *$O(n^k)$ is a k-degree expression in n, where k is the highest of the degrees of its monomials (individual terms) of the equation. (Lidl and Niederreiter (1986))*

Definition 1.14 *D is diagonal of cube N of side 1.*

The following are some examples of diagonal of cube N of side 1 in different dimensional space:

Example 1.1 *If $N \subset \mathbb{R}^2$, diagonal of cube, D is*

$$\begin{aligned} D &= 2\left[\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}\right] \\ &= 2\sqrt{\frac{2}{4}} \\ &= \sqrt{2}. \end{aligned} \tag{1.2.3}$$

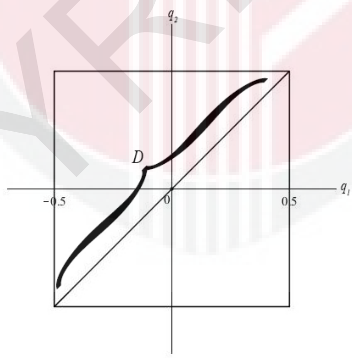

Figure 1.1: Diagonal of cube $N \in \mathbb{R}^2$

Example 1.2 *If $N \subset \mathbb{R}^3$, diagonal of cube, D is*

$$
\begin{aligned}
D &= 2\left[\sqrt{a^2+b^2}\right], \quad \textit{where } a=\frac{1}{2}, b=\frac{1}{\sqrt{2}},\\
&= 2\left[\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2}\right]\\
&= 2\sqrt{\frac{3}{4}}\\
&= \sqrt{3}.
\end{aligned}
\tag{1.2.4}
$$

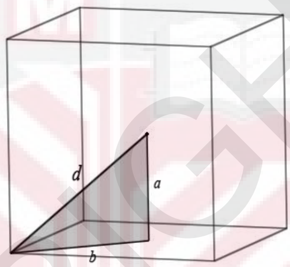

Figure 1.2: Diagonal of cube $N \in \mathbb{R}^3$

Example 1.3 *If* $N \subset \mathbb{R}^n$, *then we have* $D = \sqrt{n}$.

Definition 1.15 *Cube, N in* $\mathbb{R}^n$ *is as follows :*

$$N = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n | x_i \in \mathbb{R}, a \le x_i \le b, \forall i \in 1, 2, ..., n\},$$

where side of cube is $b-a$. *Thus,* $N \subset \mathbb{R}^n$. *If* $n = 1$, *N is an interval, while if* $n = 2$, *N is a square, and if* $n = 3$, *N is a cube. For* $n \ge 4$, *N is a hypercube.*

The following is an example of cube N in different dimensional space:

Example 1.4 *Cube in* $\mathbb{R}^1 = \{x \in \mathbb{R} | a \le x \le b\}$ *is a closed interval of length* $b-a$.

Example 1.5 *Cube in* $\mathbb{R}^2 = \{(x_1, x_2) \in \mathbb{R}^2 | a \leq x_1 \leq b, a \leq x_2 \leq b\}$ *is a closed square of side* $b - a$.

Example 1.6 *Cube in* $\mathbb{R}^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^2 | a \leq x_1 \leq b, a \leq x_2 \leq b, a \leq x_3 \leq b\}$ *is a closed cube of side* $b - a$.

Definition 1.16 *Side of cube in* $\mathbb{R}^n$ *is cube of dimension* $(n-1)$.

The following is an example of side of cube in different dimensional space:

Example 1.7 *Side of cube in* $\mathbb{R}^1$*, is a point such that,*

$$S = \{x \in \mathbb{R} | x = b\}.$$

Example 1.8 *Side of cube in* $\mathbb{R}^2$*, is a line where we have two sides of* S_1 *and* S_2 *such that*

$$S_1 = \{(x_1, x_2) \in \mathbb{R}^2 | x_1 = b, a \leq x_2 \leq b\},$$
$$S_2 = \{(x_1, x_2) \in \mathbb{R}^2 | a \leq x_1 \leq b, x_2 = b\}.$$

Example 1.9 *Side of cube in* $\mathbb{R}^3$*, is a plane where we have three sides of* S_1, S_2 *and* S_3 *such that*

$$S_1 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1 = b, a \leq x_2 \leq b, a \leq x_3 \leq b\},$$
$$S_2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | a \leq x_1 \leq b, x_2 = b, a \leq x_3 \leq b\},$$
$$S_3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | a \leq x_1 \leq b, a \leq x_2 \leq b, x_3 = b\}.$$

Next, we will look at the example of the cube for different dimensional space, with center of origin and edge of legth $2a$, where for each cube we have different side of cube.

Example 1.10 *Cube in* $\mathbb{R}^1 = \{x \in \mathbb{R} | -a \leq x \leq a\}$ *is a line has side* $S = \{x \in \mathbb{R} | x = 0\}$.

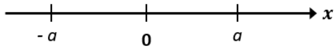

Figure 1.3: Cube in $\mathbb{R}^1$

From Example 1.10, we have cube in 1-dimensional space as a line and the side of cube would be a point $x = 0$.

Example 1.11 *Cube in* $\mathbb{R}^2 = \{(x,y) \in \mathbb{R}^2 | -a \leq x \leq a, -a \leq y \leq a\}$ *is a plane with side*

$$S_1 = \{(x,y) \in \mathbb{R}^2 | x = 0, -a \leq y \leq a\},$$
$$S_2 = \{(x,y) \in \mathbb{R}^2 | -a \leq x \leq a, y = 0\}.$$

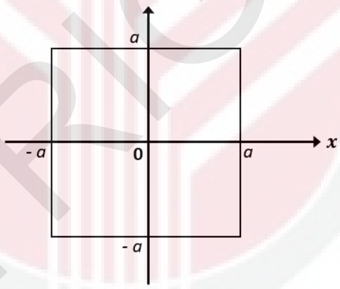

Figure 1.4: Cube in $\mathbb{R}^2$

From Example 1.11, we have cube in 2-dimensional space as a plane and the side of cube would be lines, S_1 and S_2.

Example 1.12 *Cube in* $\mathbb{R}^3 = \{(x,y,z) \in \mathbb{R}^3 | -a \leq x \leq a, -a \leq y \leq a, -a \leq z \leq a\}$ *is a cube with side*

$$S_1 = \{(x,y,z) \in \mathbb{R}^3 | x = 0, -a \leq y \leq a, -a \leq z \leq a\},$$
$$S_2 = \{(x,y,z) \in \mathbb{R}^3 | -a \leq x \leq a, y = 0, -a \leq z \leq a\},$$
$$S_3 = \{(x,y,z) \in \mathbb{R}^3 | -a \leq x \leq a, -a \leq y \leq a, z = 0\}.$$

From Example 1.12, we have cube in 3-dimensional space as a cube and the side of cube would be lines, S_1, S_2 and S_2.

Next, we have a general example of a cube centered at $\left(\frac{b-a}{2}, \frac{b-a}{2}, \frac{b-a}{2}\right)$ with edge $b-a$, for $b>0$:

Example 1.13 *Cube in* $\mathbb{R}^1 = \{x \in \mathbb{R} | a \le x \le b\}$ *has side*

$$S = \left\{x \in \mathbb{R} | x = \frac{b-a}{2}\right\}.$$

Example 1.14 *Cube in* $\mathbb{R}^2 = \{(x,y) \in \mathbb{R}^2 | a \le x \le b, a \le y \le b\}$ *has side*

$$S_1 = \left\{(x,y) \in \mathbb{R}^2 | x = \frac{b-a}{2}, a \le x \le b\right\},$$
$$S_2 = \left\{(x,y) \in \mathbb{R}^2 | -a \le x \le a, y = \frac{b-a}{2}\right\}.$$

Example 1.15 *Cube in* $\mathbb{R}^3 = \{(x,y,z) \in \mathbb{R}^3 | a \le x \le b, a \le y \le b, a \le z \le b\}$ *has side*

$$S_1 = \left\{(x,y,z) \in \mathbb{R}^3 | x = \frac{b-a}{2}, a \le y \le b, a \le z \le b\right\},$$
$$S_2 = \left\{(x,y,z) \in \mathbb{R}^3 | a \le x \le b, y = \frac{b-a}{2}, a \le z \le b\right\},$$
$$S_3 = \left\{(x,y,z) \in \mathbb{R}^3 | a \le x \le b, a \le y \le b, z = \frac{b-a}{2}\right\}.$$

From Example 1.13, 1.14, and 1.15, we can see that a general example of a in n-dimensional cube, has side of $S_1, S_2, \ldots, S_n$, where each side have same centre and length as the cube.

This few examples conclude the definition which is useful for our study. Next, we include some examples of simple game in this chapter, where we look into the game of achilles and tortoise and the game of lion and man.

1.3 Achilles and Tortoise Game

First, we have a look at the game of achilles and tortoise, which is discussed as a pursuit differential game. This game is described in one of Zeno's paradox which can be traced further back from Aristotle's writing. This Zeno's paradox has been discuss in multiple literature, for example we have a study by Black (1951) where he argues using sum of infinite series. In this subchapter, we will prove it from a differential pursuit game context instead. So, we have a two player game with Achilles, A, as the pursuer and Tortoise, T, as the evader. A logical explanation to contradicts this paradox in the context of game theory will be given. Here, we explain the pursuer's strategy as this game is considered as pursuit, while the evader can move freely.

In this game, the players move in a straight line and in the same direction. The initial position of players is illustrated in figure below, where tortoise is further in front of achilles with distance of a. Speed of A is ρ and speed of T is σ where initial distance between the players is a.

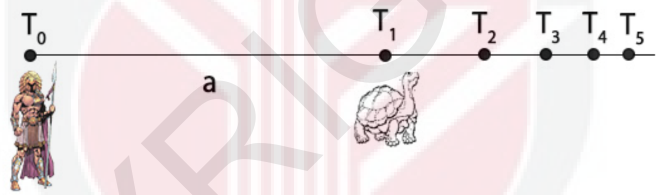

Figure 1.5: Achilles and tortoise game

Whenever A reach point T_i, T is at point T_{i+1}. Thus, A can never catch T. But, since $\rho > \sigma$, pursuit should occur. Thus, describing a paradox. Next, we look into our theorem to show that pursuit can be achieved in the game described above.

Theorem 1.1 *Pursuit can be completed in the game of Achilles and Tortoise.*

Proof:

Here, we would like to show a proof that achilles can catch the tortoise at a certain time T. First, we calculate the total time taken for the game. The table below shows the calculation of the time taken for players to travel on each line segment, (T_{n-1}, T_n).

Table 1.1: Time spent on each line segment.

Line Segment	Distance Travelled		Speed		Time Spent	
A	A	T	A	T	A	T
$(T_0, T_1]$	$T_0T_1 = a$	T_1T_2	ρ	σ	$t_1 = \frac{T_0T_1}{\rho} = \frac{a}{\rho}$	$t_1 = \frac{T_1T_2}{\sigma}$
$(T_1, T_2]$	T_1T_2	T_2T_3	ρ	σ	$t_2 = \frac{T_1T_2}{\rho} = \frac{\sigma}{\rho}t_1$	$t_2 = \frac{T_2T_3}{\sigma}$
$(T_2, T_3]$	T_2T_3	T_3T_4	ρ	σ	$t_3 = \frac{T_2T_3}{\rho} = \left(\frac{\sigma}{\rho}\right)^2 t_1$	$t_3 = \frac{T_3T_4}{\sigma}$
$(T_{n-1}, T_n]$	$T_{n-1}T_n$	T_nT_{n+1}	ρ	σ	$t_n = \left(\frac{\sigma}{\rho}\right)^{n-1} t_1$	$t_n = \frac{T_nT_{n+1}}{\sigma}$

Then, we can easily calculate the total time of the game from the information which we obtained from Table 1.1. Let θ_n be total time spent by players up to t_n, and θ be total time of game.

$$\theta_n = t_1 + t_2 + \ldots + t_n = \sum_{i=1}^{n} t_i$$

Now,

$$\begin{aligned}
\theta &= \lim_{n\to\infty} \theta_n \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} t_i \\
&= \sum_{i=1}^{\infty} t_i \\
&= \sum_{i=1}^{\infty} \left(\frac{\sigma}{\rho}\right)^{i-1} t_1 \\
&= t_1 \sum_{i=1}^{\infty} \left(\frac{\sigma}{\rho}\right)^{i-1} \\
&= t_1 \left[1 + \frac{\sigma}{\rho} + \left(\frac{\sigma}{\rho}\right)^2 + \ldots\right] \\
&= t_1 \left[\frac{1}{1 - \frac{\sigma}{\rho}}\right] \\
&= \frac{a}{\rho} \left[\frac{1}{1 - \frac{\sigma}{\rho}}\right] \\
&= \frac{a}{\rho - \sigma} \\
&< \infty
\end{aligned}$$

Since $\theta < \infty$, we know that the game is terminated at some time. Thus, proving that pursuit can be completed. Finally, we further verify pursuit can be completed by ensuring the distance between players decreases as the game progress. Let $A(\theta_n)$ be the position of A at $\theta_n = T_n$, $T(\theta_n)$ be the position of T at $\theta_n = T_{n+1}$, and $A(\theta_n)T(\theta_n)$ be the distance between A and T at θ_n. Now,

$$\begin{aligned} A(\theta_n)T(\theta_n) &= T_nT_{n+1} \\ &= \sigma t_n \\ &= \sigma\left(\frac{\sigma}{\rho}\right)^{n-1} t_1 \\ &= \sigma\left(\frac{\sigma}{\rho}\right)^{n-1} \frac{a}{\rho} \\ &= \left(\frac{\sigma}{\rho}\right)^{n} a, \end{aligned}$$

and thus,

$$\begin{aligned} \lim_{n\to\infty} A(\theta_n)T(\theta_n) &= \lim_{n\to\infty} \left(\frac{\sigma}{\rho}\right)^{n} a \\ &= a \lim_{n\to\infty} \frac{\sigma^n}{\rho^n} \\ &= a \cdot 0, \quad \text{since } \rho > \sigma, \\ &= 0. \end{aligned}$$

Hence, $A(\theta_n)T(\theta_n) \to 0$ as $n \to \infty$. This shows a contradiction to the paradox describe earlier since the distance approaches to null as time increases. Thus, proving pursuit can be completed. □

Next, we look into the game of lion and man where we consider it as an evasion problem. Unlike the earlier game, we consider lion and man game as an evasion differential game with state and geometric constraint. So we construct the evader's strategy.

1.4 Lion and Man Game

In this section, we look at an example of a simple evasion differential game. This game was made famous by Rado (1973), where the problem is about how a man escaped from a lion while entrapped in a circular arena. The lion plays the role of the pursuer and man as the evader and all players have the same motion capabilities with the maximum speed of each player is 1. Here it is assumed that man moves with maximum speed of 1. We look into the proving of evasion is possible in the game by the study done by Besicovitch (1953).

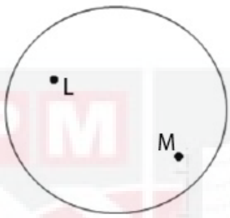

Figure 1.6: Lion and man game

Here, we will discuss the theorem and proving done by Besicovitch (1953) for the evasion game of lion and man.

Theorem 1.2 *Evasion is possible in the game of Lion and Man. (Besicovitch (1953))*

Proof:

In order to proof the statement above, we have divided the proving into three parts; construction of evader's strategy, admissibility of strategy and proof that evasion is possible. All three parts are necessary to complete the proving that evasion is possible.

Construction of the Strategy of Evader

First, we will construct the strategy of evader. At the beginning of the game, the initial position of man could be inside the circle or on the circumference of the circle. If man is on the circumference, we first let man moves towards the origin, on time interval $[0, \tau]$ such that $\tau = \frac{1}{4}L_0M_0$.

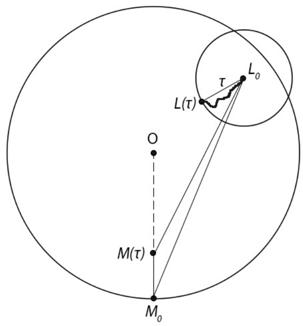

Figure 1.7: Initial position of man and lion

We need to show that lion is yet to capture man on time interval $[0, \tau]$ by proving $M(\tau)L(\tau) > 0$. The illustration of motion for the time interval $[0, \tau]$ is shown in Figure 1.7.

Now,

$$\begin{aligned}
L_0M(\tau) &\geq L_0M_0 - M_0M(\tau) \\
&= L_0M_0 - \frac{L_0M_0}{4}, \text{ since } M_0M(\tau) = \tau = \frac{L_0M_0}{4} \\
&= \frac{3}{4}L_0M_0 \\
&> \frac{L_0M_0}{4} \\
&= \tau
\end{aligned}$$

$$\Rightarrow L_0M(\tau) > \tau = L_0L(\tau)$$

$$\Rightarrow L_0M(\tau) > L_0L(\tau)$$

$$\Rightarrow M(\tau) > L(\tau)$$

$$\Rightarrow M(\tau) \neq L(\tau)$$

Therefore, $M(\tau)$ is inside of circle $L(\tau)$, and the game of evasion start when man is inside the circle.

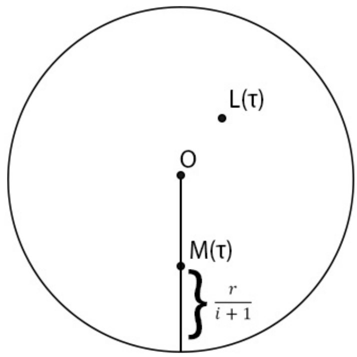

Figure 1.8: l_i is line that passes through O and M_i

From Figure 1.8, R is radius of circle, l_i is line passes through O and M_i, and $\frac{r}{i+1} = r_i$ is distance between M_i and circumference at time t_i. At each t_i, there are three possible cases of the position of Lion and Man. The lion, L_i, would either be on the right, left or on the line l_i. If lion is on the right of l_i, then man moves to the left perpendicularly to l_i. On the other hand, if lion is on the left of l_i, then man will move to the right. Thus, whichever position the lion is, man will move in the opposite direction perpendicularly to l_i. However, if lion is on l_i, then man can move either to the left or right direction perpendicularly to l_i.

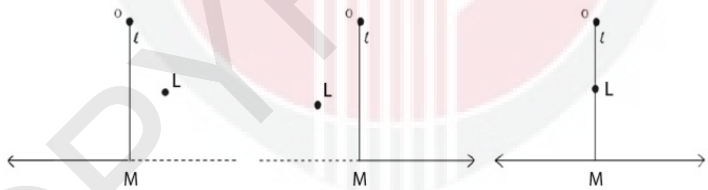

Figure 1.9: Position of lion and man

Without loss of generality, we assume L_i is on the right or on l_i and M_i will move in left direction perpendicularly to l_i.

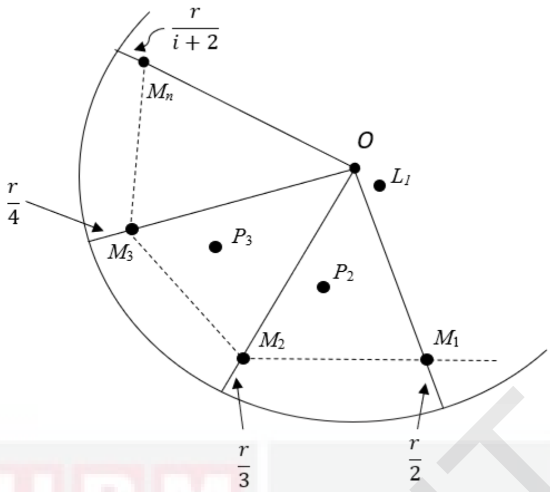

Figure 1.10: Strategy of man

Figure 1.10 shows the strategy of man in this game. First, M_i moves to the left perpendicularly to l_i until it reaches the point M_{i+1}, where distance of M_{i+1} with circumference is $r_{i+1} = \frac{r}{i+2}$. Then, man changes direction perpendicularly to l_{i+1}. The process will repeat itself and thus, trajectory of man is a broken line segment. Next, we look into the admissibility of the strategy constructed.

Admissibility of strategy

We say that strategy of Man is acceptable, if the followings are satisfied :

1. The geometric constraint is $|v(t)| \leq 1$ where $|v(t)|$ is the speed of the man. We have $\alpha(t) = 1 \leq 1$ and thus it satisfies the geometric constraint.
2. Man needs to stay inside the circle and does not reach the circle circumference at all time. Thus we show $OM < R$, where M is the position of man betweeen M_i and M_{i+1}, at any time.

Now,

$$
\begin{aligned}
OM &= \sqrt{\left(R-\frac{r}{i+1}\right)^2+(MM_i)^2} \\
&= \sqrt{\left(R-\frac{r}{i+1}\right)^2+(t'-t_i)^2}, \\
&< \sqrt{\left(R-\frac{r}{i+1}\right)^2+(t_{i+1}-t_i)^2} \\
&= \sqrt{\left(R-\frac{r}{i+1}\right)^2+\left(R-\frac{r}{i+2}\right)^2-\left(R-\frac{r}{i+1}\right)^2} \\
&= \sqrt{\left(R-\frac{r}{i+2}\right)^2} \\
&= R-\frac{r}{i+2} \\
&< R.
\end{aligned}
$$

Proof of Evasion is Possible

First, we proof evasion from lion is possible on each line segment.

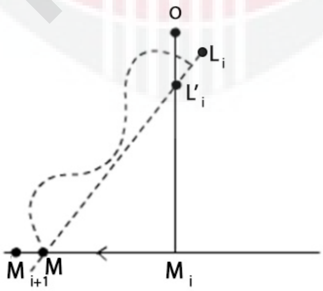

Figure 1.11: Line segment M_iM_{i+1}

Figure 1.11 above shows trajectory of man on each line segment. In order to prove evasion is possible on each line segment, we use proof by contradiction. Assume, $L(t') = M(t') = M \in M_iM_{i+1}$ for some time $t' \in [t_i, t_{i+1}]$.

Indeed,

$$\frac{M_iM}{1} = t' - t.$$

On the other hand,

$$\begin{aligned} L_iM &= \int_{t_i}^{t'} u(t)dt \\ &\le \int_{t_i}^{t'} 1dt \\ &= t' - t_i \\ &= M_iM. \end{aligned}$$

This implies,

$$L_i'M > L_iM, \text{ since } L_i'M > M_iM \ge L_iM,$$

which is a contradiction, since $L_i'M \le L_iM$. Hence, evasion is possible on each M_iM_{i+1}. Next, we calculate the total time of game. Let t_n be total time spent by man to reach M_n.

$$\begin{aligned} t_n &= t_n + 0 \\ &= t_n + (t_{n-1} - t_{n-1}) + (t_{n-2} - t_{n-2}) + \ldots + (t_1 - t_1) + (t_0 - t_0) \\ &= (t_n - t_{n-1}) + (t_{n-1} - t_{n-2}) + \ldots + (t_1 - t_0) + t_0 \\ &= t_0 + \sum_{i=0}^{n-1} t_{i+1} - t_i \\ &= \sum_{i=0}^{n-1} t_{i+1} - t_i, \text{ since } t_0 = 0. \end{aligned}$$

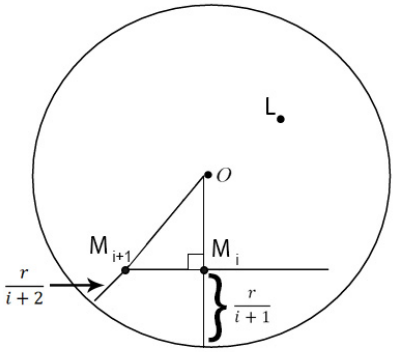

Figure 1.12: Strategy of man

Time spent for man to move from M_i to M_{i+1} is

$$\begin{aligned}
t_{i+1} - t_i &= \frac{M_iM_{i+1}}{1}, \text{ since speed of } M = 1 \\
&= M_iM_{i+1} \\
&= \sqrt{\left(R - \frac{r}{i+2}\right)^2 - \left(R - \frac{r}{i+1}\right)^2} \\
&= \sqrt{R^2 + \left(\frac{r}{i+2}\right)^2 - 2R\frac{r}{i+2} - R^2 - \left(\frac{r}{i+1}\right)^2 + 2R\frac{r}{i+1}} \\
&= \sqrt{\left(\frac{r}{i+2}\right)^2 + \frac{2Rr(i+2) - 2Rr(i+1)}{(i+1)(i+2)} - \left(\frac{r}{i+1}\right)^2} \\
&= \sqrt{\left(\frac{r}{i+2}\right)^2 + \frac{2Rr}{(i+1)(i+2)} - \left(\frac{r}{i+1}\right)^2} \\
&= \sqrt{\left(\frac{r}{i+2}\right)^2 + \left(\frac{r}{i+1}\right)\left[\frac{2R}{i+2} - \frac{r}{i+1}\right]} \\
&= \sqrt{\left(\frac{r}{i+2}\right)^2 + \left(\frac{r}{i+1}\right)\left[\frac{2R(i+1) - r(i+2)}{(i+1)(i+2)}\right]} \\
&> \sqrt{\left(\frac{r}{i+2}\right)^2 + \left(\frac{r}{i+1}\right)\left[\frac{2r(i+1) - r(i+2)}{(i+1)(i+2)}\right]} \\
&= \sqrt{\left(\frac{r}{i+2}\right)^2 + \left(\frac{r}{i+1}\right)\left(\frac{ri}{(i+1)(i+2)}\right)} \\
&> \sqrt{\left(\frac{r}{i+2}\right)^2}
\end{aligned}$$

$$= \frac{r}{i+2}.$$

Thus,

$$\begin{aligned} t_n &= \sum_{i=0}^{n-1} (t_{i+1} - t_i) \\ &> \sum_{i=0}^{n-1} \frac{r}{i+2} \\ &= r \sum_{i=0}^{n-1} \frac{1}{i+2}, \end{aligned}$$

and thus,

$$\lim_{n \to \infty} t_n = \infty.$$

Since the infinite series $\sum_{i=0}^{n-1} \frac{1}{i+2}$ is divergent, then the total time is infinite and thus evasion is possible. □

1.5 Problem Statement

Let S be a nonempty closed bounded convex subset of $\mathbb{R}^n, n \geq 2$. We consider a pursuit differential game described by the following equations:

$$\begin{aligned} &P_i : \dot{x}_i = u_i, \;\; x_i(0) = x_{i0}, \;\; |u_i| \leq 1, \;\; i = 1, 2, \ldots, n, \\ &E : \dot{y} = v, \quad\; y(0) = y_0, \quad |v| \leq 1, \end{aligned} \tag{1.5.1}$$

where $x_i, y, u_i, v, x_{i0}, y_0 \in S$.

The problem is to construct method for the pursuers in applying parallel strategy to capture the evader in a finite time.

1.6 Research Objectives

The main objective of this study is as follows:

1. Solve the pursuit differential game of three pursuers and one evader in three-dimensional cube.
2. Solve the pursuit differential game of many pursuers and one evader in a convex hyperspace, where the game is described in Equation 1.5.1.
3. To find the improved guaranteed pursuit time in the game of many pursuer and one evader in a convex hyperspace.

1.7 Research Methodology

In our study, in order to achieve the objective defined above, we select the planes on which each pursuer moves with parallel strategy and we introduce fictitious pursuer to achieve pursuit. We construct a method for pursuers in using parallel strategy which ensures pursuit can be completed in a game. This is followed by the calculation of guaranteed pursuit time which is expected to be an improvement from that of Alias et al. (2015).

1.8 Outline of Thesis

This thesis covers eight chapters with the following content:

Chapter 1 consists of brief introduction on game theory. Next, we discuss some basic and necessary definitions for this project. Then, some example of simple game are included which is the Achilles and Tortoise game and the Lion and Man game. Finally, the problem statement, research objectives, research methodology and outline of the thesis is included to complete our introduction of thesis.

Chapter 2 discusses the past literature which motivates our study, where we divide them into some categories which is closely related work in differential game theory, differential game theory with integral or mixed constraint, parallel strategy in a differential game theory, game in convex space, and application of differential game theory.

Chapter 3 focuses the parallel strategy which includes the vertical and general parallel strategy.

Chapter 4 covers the pursuit differential game of three pursuers and one evader in a three-dimensional cube, where we construct the method in applying parallel strategy and calculate the guaranteed pursuit time of game.

Chapter 5 discusses our Lemma where we show the guaranteed pursuit time in the game of a convex compact set.

Chapter 6 shows the pursuit differential game of many pursuers and one evader in an n-dimensional space and the calculation for guaranteed pursuit time of the game.

Chapter 7 discusses the proving of pursuit problem of many pursuer and one ecader in a close convex compact set, where we include the guaranteed pursuit time as well.

Finally, the conclusion of this project with some ideas of future work, is included in Chapter 8.

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