UNIVERSITI PUTRA MALAYSIA

CHROMATICITY OF CERTAIN 2-CONNECTED GRAPHS

LAU GEE CHOON

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CHROMATICITY OF CERTAIN 2-CONNECTED GRAPHS

By

LAU GEE CHOON

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Dedication

To

My Wife

Low Li Sia

For Her Great Patience

My Grandmother

From Whom I Learned To Be More Determined

And

In Memory Of My Grandfather

For His Encouragement

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Master of Science

CHROMATICITY OF CERTAIN 2-CONNECTED GRAPHS

By

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January 2003

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Since the introduction of the concepts of chromatically unique graphs and chromatically equivalent graphs, many families of such graphs have been obtained. In this thesis, we continue with the search of families of chromatically unique graphs and chromatically equivalent graphs.

In Chapter 1, we define the concept of graph colouring, the associated chromatic polynomial and some properties of a chromatic polynomial. We also give some necessary conditions for graphs that are chromatically unique or chromatically equivalent.

Chapter 2 deals with the chromatic classes of certain existing 2-connected $(n, n+i)$-graphs for $i = 0$, 1, 2 and 3. Many families of chromatically unique graphs and chromatically equivalent graphs of these classes have been obtained. At the end of the chapter, we re-determine the chromaticity of two families of 2-connected $(n, n+3)$-graphs with at least two triangles.

Our main results in this thesis are presented in Chapters 3. 4 and 5. In Chapter 3, we classify all the 2-connected $(n, n+4)$-graphs with at least four triangles. In Chapter 4, we classify all the 2-connected $(n, n+4)$-graphs with three triangles and one induced 4-cycle. In Chapter 5, we classify all the 2-connected $(n, n+4)$-graphs with three triangles and at least two induced 4-cycles. In each chapter, we obtain new families of chromatically unique graphs and chromatically equivalent graphs.

We end the thesis by classifying all the 2-connected $(n, n+4)$-graphs with exactly three triangles. We also determine the chromatic polynomial of all these graphs. The determination of the chromaticity of most classes of these graphs is left as an open problem for future research.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Master Sains

KEKROMATIKAN GRAF TERKAIT-2 TERTENTU

Oleh

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Pengerusi: **Profesor Madya Peng Yee Hock, Ph.D.**

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Sejak konsep graf unik kromatik dan setara kromatik diperkenalkan, terdapat banyak famili graf yang unik kromatik dan setara kromatik telah diperolehi. Dalam tesis ini, kami meneruskan pencarian famili graf yang unik kromatik dan setara kromatik.

Dalam Bab 1, kami takrifkan konsep pewarnaan graf, polinomial kromatik yang berkaitan dan beberapa ciri polinomial kromatik. Kami juga mengemukakan beberapa syarat yang diperlukan supaya sesuatu graf itu unik kromatik atau setara kromatik.

Bab 2 adalah berkaitan kelas kromatik graf-$(n, n+i)$ tertentu untuk $i = 0$, 1, 2 atau 3. Banyak famili graf kelas ini yang unik kromatik atau setara kromatik telah ditemui. Pada akhir bab ini, kami menentukan semula kekromatikan dua famili graf-$(n, n+3)$ dengan dua atau lebih segitiga.

Keputusan utama tesis ini disampaikan dalam Bab 3, 4 dan 5. Dalam Bab 3, kami klasifikasikan semua graf-$(n, n+4)$ dengan empat atau lebih segitiga. Dalam Bab 4, kami klasifikasikan semua graf-$(n, n+4)$ dengan tiga segitiga dan satu 4-kitar teraruh. Dalam Bab 5, kami klasifikasikan semua graf-$(n, n+4)$ dengan tiga segitiga dan dua atau lebih 4-kitar teraruh. Dalam setiap bab ini, kami temui pelbagai famili graf unik kromatik dan setara kromatik yang baru.

Kami akhiri tesis ini dengan mengklasifikasikan semua graf-$(n, n+4)$ dengan tepat tiga segitiga. Kami juga menentukan polinomial kromatik setiap graf kelas ini. Penentuan kekromatikan kebanyakan kelas graf ini dibiarkan sebagai masalah terbuka untuk kajian masa depan.

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CHAPTER 1

INTRODUCTION

1.1 Introduction

A graph G is a collection of the edge set and the vertex set, denoted $E(G)$ and $V(G)$, respectively. An edge e of G is a line joining two vertices in $V(G)$, say v_i and v_j, denoted v_iv_j or (v_i, v_j). A *loop* of a graph is an edge having the same end vertices. A graph is said to have *multiple edge* if there exists two vertices joined by at least two edges. A graph without any multiple edges is called a *simple* graph.

The theory of graph was invented by Euler [7] in trying to solve the famous Königsberg bridge problem. However, it was the problem of map colouring, the famous Four-Colour Conjecture that acts as a prime stimulant for the development of graph theory. In 1912,
number of ways to colour a map in an attack on the four-colour conjecture. For any positive integer λ and any map M, it is desired to colour the map M with at most λ colours such that no two adjacent regions are assigned the same colour. The function for number of ways of such colouring, $P(\lambda)$ was then proved to be always a polynomial.

In 1932, Whitney [31]

ing of a graph G. He ıntroduced the function $M(\lambda)$ as the number of ways of colouring the vertices of G (not necessarily arising from a map) with at most λ colours such that no two adjacent vertices are assigned the same colour. The notation $P(G, \lambda)$, now known as *chromatic polynomial*, was then used by Birkhoff and Lewis [2]

what polynomials were chromatic polynomials of maps. They also proved that $P(G, \lambda)$ is always a polynomial for any graph G. The minimum integer λ such that $P(G, \lambda)$ is nonzero is called the chromatic number of G, denoted $\chi(G)$. An introduction to the theory of chromatic polynomials can be found in [25].

The problem of characterizing the chromatic polynomials is not yet solved till this day. However, it leads to the concepts of chromatically unique graphs and chromatically equivalent graphs that were first introduced by Chao and Whitehead [3], thus opening a new area of research to graph theorists. Since then, many families of chromatically unique and chromatically equivalent graphs have been obtained (see [14]

Throughout this thesis, we shall denote by $P(G)$ the chromatic polynomial of a graph G. All graphs we consider are simple, loopless and undirected, unless otherwise specified. We shall refer to [10]

explained in this thesis.

1.2 The Fundamental Reduction Theorem

We first give a more formal definition of chromatic polynomial. Given a graph G and λ different colours. If G is a graph of order n and size m, we say G is an (n, m)-

to the vertices of G. If no two adjacent vertices of G are assigned the same colour, the colouring is said to be *proper*. Functionally, a

G is a mapping

$$f: V(G) = \{v_1, v_2, \ldots, v_n\} \longrightarrow \{1, 2, \ldots, \lambda\},$$

such that $f(v_i) \neq f(v_j)$ whenever $v_i v_j \in E(G)$. Two proper λ-

g of G will be considered as different if $f(v_i) \neq g(v_i)$ for some vertex v_i in G. Then, $P(G)$ is just the number of different proper λ-colourings

An empty graph is a graph that has no edges while a complete graph is a graph where every two vertices of the graph is joined by an edge. Obviously, if $O(n)$ is the empty graph of order n, then $P(O(n)) = \lambda^n$; and if K_n is the complete graph of order n, then $P(K_n) = \lambda(\lambda - 1)\cdots(\lambda - n + 1)$.

The following are some useful known results and techniques for determining the chromatic polynomial of a graph.

Theorem 1.1 (Fundamental Reduction Theorem) (Whitney [32]) *Let G be an (n, m)-graph. Then*

$$P(G) = P(G - e) - P(G \cdot e)$$

where $G - e$ is the graph obtained from G by deleting e, and $G \cdot e$ is the graph obtained from G by identifying the end vertices of e.

The Fundamental Reduction Theorem can also be used in another way. Let $v_i, v_j \in V(G)$ such that $v_i v_j \notin E(G)$. Then

$$P(G) = P(G + v_i v_j) + P(G \cdot v_i v_j)$$

where $G + v_i v_j$ is the graph obtained from G by adding the edge $v_i v_j$ and $G \cdot v_i v_j$ is the graph obtained from G by identifying the vertices v_i and v_j. In this way, $P(G)$ can be expressed as a sum of the chromatic polynomials of complete graphs.

It is sometimes more convenient to use a drawing to denote the chromatic polynomial of a graph (Figure 1.1).

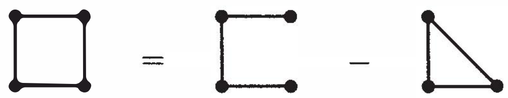

Figure 1.1: Chromatic Polynomial by Graphs

Let G_1 and G_2 be graphs, each containing a complete subgraph K_p with p vertices. If G is the graph obtained from G_1 and G_2 by identifying the two subgraphs K_p, then G is called a $K_p - gluing$ of G_1 and G_2. Note that a $K_1 - gluing$ and a $K_2 - gluing$ are also called a $vertex - gluing$ and an $edge - gluing$, respectively.

Lemma 1.1 (Zykov [35]) *Let G be a K_r-gluing of G_1 and G_2. Then*

$$P(G) = \frac{P(G_1)P(G_2)}{P(K_r)}.$$

Lemma 1.1 implies that all K_r-gluing of G_1 and G_2 will have the same chromatic polynomial.

The following easy lemma provides another shortcut in determining $P(G)$ since the number of ways of colouring k graphs simultaneously is equal to the product of the number of ways of colouring each of the k graphs.

Lemma 1.2 *If G is a graph with k components $G_1, G_2, \ldots, G_k$, then*

$$P(G) = P(G_1)P(G_2)\cdots P(G_k).$$

Let $G^{(0)}$ be a given graph which is K_p-gluing

Forming another K_p-gluing of G_1 and G_2, we obtain a new graph $G^{(1)}$. Note that $G^{(1)}$ may not be isomorphic to $G^{(0)}$. Clearly, $G^{(1)}$ is a K_p-gluing of some graphs, say H_1 and H_2. Note that H_1 and H_2 may not be G_1 and G_2. Forming another K_p-gluing

forming $G^{(1)}$ from $G^{(0)}$ (or $G^{(2)}$ from $G^{(1)}$) is called an *elementary operation*. A graph H is called a *relative* of G if H can be obtained from G by applying a finite sequence of elementary operations. It follows from Lemma 1.1 that if H is a relative of G, then $P(H) = P(G)$.

1.3 Some Properties of Chromatic Polynomials

Let G be an (n, m)-graph

Let C be a cycle of G and e an edge of C such that $\alpha(e) \geq \alpha(x)$ for all $x \in E(C)$. Then the path $C - e$ in G is called a *broken cycle* of G induced by α. We then have the following theorem.

Theorem 1.2 (Whitney's broken cycle theorem) (Whitney [31]

(n, m)-

$$P(G) = \sum_{i=0}^{n-1} (-1)^i h_i \lambda^{n-i}.$$

where h_i is the number of spanning subgraphs of G that have i edges and that contain no broken cycles induced by α.

Note that h_i is independent of the mapping α. The following results can then be derived directly from Theorem 1.2.

Lemma 1.3 *Let G be an (n, m)-graph. Then in the polynomial $P(G)$, the coefficient of* *(i)* λ^n *is* 1

(ii) λ^{n-1} *is* $-m$

(iii) λ^{n-2} *is* $\binom{m}{2} - t_1(G)$, *where* $t_1(G)$ *is the number of triangles in* G

(iv) λ^{n-3} *is* $-\binom{m}{3} + (m-2)t_1(G) + t_2(G) - 2t_3(G)$, *where* $t_2(G)$ *is the number of cycles of order* 4 *without chords and* $t_3(G)$ *is the number of the complete subgraphs* K_4 *in* G (see [8]).

We also have the following properties for $P(G)$.

Lemma 1.4 *Let* G *be an* (n, m)*-graph. Then* $P(G)$ *is a polynomial in* λ *such that*

(i) $deg(P(G)) = n$

(ii) the absolute value of the coefficient of λ^{n-1} *is the number of edges of* G

(iii) all the coefficients are integers

(iv) the leading term is λ^n

(v) the constant term is zero

(vi) the coefficients alternate in sign

(vii) either $P(G) = \lambda^n$ *or the sum of the coefficients in* $P(G)$ *is zero.*

Lemma 1.4 above can be proved by induction on m (see [25]).

Note that not all polynomials that satisfy all the above conditions are chromatic polynomials of some graphs. For example, consider

$$P(x) = x^4 - 4x^3 + 3x^2$$

Note that the coefficient of x^4 is 1, the constant term is zero, the coefficients alternate in sign and sum up to zero. However, $P(x)$ is not the chromatic polynomial of any graph. If it were, the graph would have four vertices and four edges, by Lemma 1.4(i) and (ii). The only two graphs with four vertices and four edges do not have this polynomial.

1.4 Chromatically Unique and Chromatically Equivalent Graph

We note by Lemma 1.1 that for any tree T of order n, $P(T) = \lambda(\lambda - 1)^{n-1}$. This implies that there exists non-isomorphic graphs which have the same chromatic polynomial. On the contrary, there are graphs like the empty graphs $O(n)$ such that no other graphs will have the same chromatic polynomial as $O(n)$. These observations lead to the following definitions.

Let $P(G)$ denote the chromatic polynomial of a graph G. Two graphs G and H are *chromatically equivalent* (χ-equivalent). written $G \sim H$, if $P(G) = P(H)$. A graph G is *chromatically unique* (χ-unique) if $P(H) = P(G)$ implies that $H \cong G$. Let $< G >$ denote the equivalence class determined by the graph G under $\sim$. Clearly, G is χ-unique if and only if $< G >= \{G\}$.

Recall that $t_1(G), t_2(G), t_3(G)$ and $\chi(G)$ are respectively the number of triangles. the number of cycles of order 4 without chords, the number of complete subgraphs K_4 in G and the chromatic number of G. The results of the following lemma can be derived from Lemma 1.3.

Lemma 1.5 *Let G and H be graphs such that $G \sim H$. Then*

(i) G and H have the same order

(ii) G and H have the same size

(iii) $t_1(G) = t_1(H)$

(iv) $t_2(G) - 2t_3(G) = t_2(H) - 2t_3(H)$

(v) $\chi(G) = \chi(H)$

(vi) G is connected if and only if H is connected.

Since there is no general methods for constructing families of χ-unique graphs and χ-equivalent graphs, the above conditions are just some necessary conditions for two graphs G and H to be χ-equivalent. (Also see [19] for a method that uses adjoint polynomials in proving the chromatic uniqueness of a graph).

For a graph G containing a cycle, the *girth* $g(G)$ of G is the length of a shortest cycle in G. Let $\sigma_g(G)$ be the number of cycles of length $g(G)$ in G. It then follows from Whitney's broken cycle theorem (Theorem 1.2) that if G and H are χ-equivalent and containing cycles, then $g(G) = g(H)$ and $\sigma_g(G) = \sigma_g(H)$ (also see [27]).

The following results are not difficult to obtain.

Lemma 1.6 *Let G be a graph of size m. Then $m \geq 1$ if and only if* $\lambda(\lambda - 1)|P(G)$.

Proof If $m \geq 1$, then $\chi(G) \geq 2$. This implies that $P(0) = P(1) = 0$. Thus, λ and $(\lambda - 1)$ are two factors of $P(G)$. On the contrary, if $\lambda(\lambda - 1)|P(G)$, then $\chi(G) \geq 2$. Thus, G must have at least one edge. Hence, $m \geq 1$. □