UNIVERSITI PUTRA MALAYSIA

CHROMATICITY OF CERTAIN 2-CONNECTED GRAPHS

LAU GEE CHOON

FSAS 2003 48
CHROMATICITY OF CERTAIN 2-CONNECTED GRAPHS

By

LAU GEE CHOON

Thesis Submitted to the School of Graduate Studies,
Universiti Putra Malaysia in Fulfilment of the Requirement
for the Degree of Master of Science

January 2003
Dedication

To

My Wife

Low Li Sia

For Her Great Patience

My Grandmother

From Whom I Learned To Be More Determined

And

In Memory Of My Grandfather

For His Encouragement
Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Master of Science

CHROMATICITY OF CERTAIN 2-CONNECTED GRAPHS

By
LAU GEE CHOON
January 2003

Chairman: Associate Professor Peng Yee Hock, Ph.D.
Faculty: Science and Environmental Studies

Since the introduction of the concepts of chromatically unique graphs and chromatically equivalent graphs, many families of such graphs have been obtained. In this thesis, we continue with the search of families of chromatically unique graphs and chromatically equivalent graphs.

In Chapter 1, we define the concept of graph colouring, the associated chromatic polynomial and some properties of a chromatic polynomial. We also give some necessary conditions for graphs that are chromatically unique or chromatically equivalent.

Chapter 2 deals with the chromatic classes of certain existing 2-connected \((n, n + \tau)\)-graphs for \(\tau = 0, 1, 2\) and 3. Many families of chromatically unique graphs and chromatically equivalent graphs of these classes have been obtained. At the end of the chapter, we re-determine the chromaticity of two families of 2-connected \((n, n + 3)\)-graphs with at least two triangles.
Our main results in this thesis are presented in Chapters 3, 4 and 5. In Chapter 3, we classify all the 2-connected \((n, n + 4)\)-graphs with at least four triangles. In Chapter 4, we classify all the 2-connected \((n, n + 4)\)-graphs with three triangles and one induced 4-cycle. In Chapter 5, we classify all the 2-connected \((n, n + 4)\)-graphs with three triangles and at least two induced 4-cycles. In each chapter, we obtain new families of chromatically unique graphs and chromatically equivalent graphs.

We end the thesis by classifying all the 2-connected \((n, n + 4)\)-graphs with exactly three triangles. We also determine the chromatic polynomial of all these graphs. The determination of the chromaticity of most classes of these graphs is left as an open problem for future research.
Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Master Sains

KEKROMATIKAN GRAF TERKAIT-2 TERTENTU

Oleh

LAU GEE CHOON

January 2003

Pengerusi: Profesor Madya Peng Yee Hock, Ph.D.

Fakulti: Sains dan Pengajian Alam Sekitar

Sejak konsep graf unik kromatik dan setara kromatik diperkenalkan, terdapat banyak famili graf yang unik kromatik dan setara kromatik telah diperolehi. Dalam tesis ini, kami meneruskan pencarian famili graf yang unik kromatik dan setara kromatik.

Dalam Bab 1, kami takrifkan konsep pewarnaan graf, polinomial kromatik yang berkaitan dan beberapa ciri polinomial kromatik. Kami juga mengemukakan beberapa syarat yang diperlukan supaya sesuatu graf itu unik kromatik atau setara kromatik.

Bab 2 adalah berkaitan kelas kromatik graf-(n, n + i) tertentu untuk i = 0, 1, 2 atau 3. Banyak famili graf kelas ini yang unik kromatik atau setara kromatik telah ditemui. Pada akhir bab ini, kami menentukan semula kekromatikan dua famili graf-(n, n + 3) dengan dua atau lebih segitiga.
Keputusan utama tesis ini disampaikan dalam Bab 3, 4 dan 5. Dalam Bab 3, kami klasifikasikan semua graf-(n, n + 4) dengan empat atau lebih segitiga. Dalam Bab 4, kami klasifikasikan semua graf-(n, n + 4) dengan tiga segitiga dan satu 4-kitar terarah. Dalam Bab 5, kami klasifikasikan semua graf-(n, n + 4) dengan tiga segitiga dan dua atau lebih 4-kitar terarah. Dalam setiap bab ini, kami temui pelbagai famili graf unik kromatik dan setara kromatik yang baru.

ACKNOWLEDGEMENTS

First and foremost, I am grateful to my supervisor, Associate Professor Dr. Peng Yee Hock for his continuing support, guidance and thoughtful advice. His patience, constant encouragement and suggestions throughout the course of my study are constructive in completing this thesis. I am also grateful to the members of the supervisory committee, Professor Dr. Kamel Ariffin Mohd Atan and Dr. Mohamad Rushdan Md. Said for their co-operations.

Thanks also go to Tunku Abdul Rahman College, Johore Branch for the understanding in arranging my working hours that enables me to take annual leaves periodically.

I am always indebted to my parents for giving me the total freedom in pursuing my career. Without their understanding and financial support in the early years, I would not be able to choose this toughest subject that I ever like.

Last but not least, my special thanks to my family, colleagues and friends for their support and encouragement.
I certify that an Examination Committee met on 28th January 2003 to conduct the final examination of Lau Gee Choon on his Master of Science thesis entitled “Chormaticity of Certain 2-connected Graphs” in accordance with Universiti Pertanian Malaysia (Higher Degree) Act 1980 and Universiti Pertanian Malaysia (Higher Degree) Regulations 1981. The Committee recommends that the candidate be awarded the relevant degree. Members of the Examination Committee are as follows:

HARUN BIN BUDIN, Ph.D.
Associate Professor
Faculty of Science and Environmental Studies
Universiti Putra Malaysia
(Chairman)

PENG YEE HOCK, Ph.D.
Associate Professor
Faculty of Science and Environmental Studies
Universiti Putra Malaysia
(Member)

KAMEL ARIFFIN M. ATAN, Ph.D.
Professor
Faculty of Science and Environmental Studies
Universiti Putra Malaysia
(Member)

MOHAMAD RUSHDAN MD. SAID, Ph.D.
Lecturer
Faculty of Science and Environmental Studies
Universiti Putra Malaysia
(Member)

SHAMSHER MOHAMAD RAMADILI, Ph.D.
Professor/Deputy Dean
School of Graduate Studies
Universiti Putra Malaysia
Date: 3 MAR 2003
This thesis submitted to the Senate of Universiti Putra Malaysia has been accepted as fulfilment of the requirement for the degree of Master of Science. The members of the Supervisory Committee are as follows:

**PENG YEE HOCK, Ph.D.**
Associate Professor
Faculty of Science and Environmental Studies
Universiti Putra Malaysia
(Chairman)

**KAMEL ARIFFIN M. ATAN, Ph.D.**
Professor
Faculty of Science and Environmental Studies
Universiti Putra Malaysia
(Member)

**MOHAMAD RUSHDAN MD. SAID, Ph.D.**
Lecturer
Faculty of Science and Environmental Studies
Universiti Putra Malaysia
(Member)

__________________________

**AINI IDERIS, Ph.D.**
Professor/Dean
School of Graduate Studies
Universiti Putra Malaysia
Date:
DECLARATION

I hereby declare that the thesis is based on my original work except for quotations and citations which have been duly acknowledged. I also declare that it has not been previously or concurrently submitted for any other degree at UPM or other institutions.

Lau Gee Choon
Date: 3/3/03
TABLE OF CONTENTS

DEDICATION ..................................................... ii
ABSTRACT ......................................................... iii
ABSTRAK ........................................................ v
ACKNOWLEDGMENTS .......................................... vii
APPROVAL ......................................................... viii
DECLARATION .................................................. x
LIST OF TABLES ............................................... xiii
LIST OF FIGURES ............................................. xiv

1 INTRODUCTION .............................................. 1
  1.1 Introduction .............................................. 1
  1.2 The Fundamental Reduction Theorem ...................... 2
  1.3 Some Properties of Chromatic Polynomials .............. 5
  1.4 Chromatically Unique and Chromatically
       Equivalent Graph ...................................... 7

2 (n, n + 2)-GRAPHS AND (n, n + 3)-GRAPHS .................. 10
  2.1 Cycles and The Generalized \( \theta \)-graph .............. 10
  2.2 Certain 2-connected \((n, n + 2)\)-graphs ............... 12
  2.3 Certain 2-connected \((n, n + 3)\)-graphs ............... 22
  2.4 The Chromatic Classes of Two Families of 2-connected
       \((n, n + 3)\)-graphs .................................. 25

3 \((n, n + 4)\)-GRAPHS WITH AT LEAST FOUR TRIANGLES ... 42
  3.1 Classification of Graphs ................................ 42
  3.2 The Chromatic Polynomials of \(G_i\) ...................... 61
  3.3 Proof of The Main Theorem ................................ 67
<table>
<thead>
<tr>
<th>Section</th>
<th>Pages</th>
</tr>
</thead>
<tbody>
<tr>
<td>4 (n, n + 4)-GRAPHS WITH THREE TRIANGLES AND ONE</td>
<td></td>
</tr>
<tr>
<td>INDUCED 4-CYCLE</td>
<td>71</td>
</tr>
<tr>
<td>4.1 Classification of Graphs</td>
<td>71</td>
</tr>
<tr>
<td>4.2 The Chromatic Polynomials of $D_i$</td>
<td>81</td>
</tr>
<tr>
<td>4.3 Proof of The Main Theorem</td>
<td>98</td>
</tr>
<tr>
<td>5 (n, n + 4)-GRAPHS WITH THREE TRIANGLES AND AT LEAST</td>
<td></td>
</tr>
<tr>
<td>TWO INDUCED 4-CYCLES</td>
<td>104</td>
</tr>
<tr>
<td>5.1 Classification of Graphs</td>
<td>104</td>
</tr>
<tr>
<td>5.2 The Chromatic Polynomials of $F_i$</td>
<td>120</td>
</tr>
<tr>
<td>5.3 Proof of The Main Theorem</td>
<td>131</td>
</tr>
<tr>
<td>6 DISCUSSION</td>
<td>137</td>
</tr>
<tr>
<td>6. Types of 2-connected $(n, n + 4)$-graphs With Exactly Three Triangles</td>
<td>137</td>
</tr>
<tr>
<td>6.2</td>
<td>147</td>
</tr>
<tr>
<td>6.</td>
<td>154</td>
</tr>
</tbody>
</table>

BIBLIOGRAPHY

APPENDIX A

APPENDIX B

APPENDIX C

APPENDIX D

VITA
# LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1 Special Cases of $G_t^0(a, b; c, d)$</td>
<td>20</td>
</tr>
<tr>
<td>2.2 Special Cases of $K_4$-homeomorph</td>
<td>20</td>
</tr>
<tr>
<td>3.1 Graphs With Exactly Seven Triangles and a $K_4$</td>
<td>46</td>
</tr>
<tr>
<td>3.2 Graphs With Exactly Four Triangles Without a $K_4$</td>
<td>47</td>
</tr>
<tr>
<td>3.3 2-connected $(n, n + 4)$-graphs With At Least 4 Triangles</td>
<td>59</td>
</tr>
<tr>
<td>4.1 2-connected $(n, n + 4)$-graphs With Three Triangles And One Induced 4-cycle</td>
<td>79</td>
</tr>
<tr>
<td>5.1 Graphs With Three Triangles And Two Induced $C_4$'s</td>
<td>110</td>
</tr>
<tr>
<td>5.2 2-connected $(n, n + 4)$-graphs With Three Triangles And At Least Two Induced 4-cycle</td>
<td>118</td>
</tr>
<tr>
<td>6. Getting All 2-connected $(n, n + 4)$-graphs With Exactly Three Triangles From Graphs in Table 3.3</td>
<td>144</td>
</tr>
<tr>
<td>6.2 2-connected $(n, n + 4)$-graphs With Exactly 3 Triangles</td>
<td>146</td>
</tr>
</tbody>
</table>
# LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1 Chromatic Polynomial By Graph</td>
<td>4</td>
</tr>
<tr>
<td>2.1 The Generalized $\theta$-graph</td>
<td>11</td>
</tr>
<tr>
<td>2.2 2-connected $(n, n+2)$-graphs (1)</td>
<td>13</td>
</tr>
<tr>
<td>2.3 2-connected $(n, n+2)$-graphs (2)</td>
<td>20</td>
</tr>
<tr>
<td>2.4 2-connected $(n, n+2)$-graphs (3)</td>
<td>21</td>
</tr>
<tr>
<td>2.5 2-connected $(n, n+3)$-graphs (1)</td>
<td>23</td>
</tr>
<tr>
<td>2.6 2-connected $(n, n+3)$-graphs (2)</td>
<td>26</td>
</tr>
<tr>
<td>3.1 Certain 2-connected $(n, n+4)$-graphs</td>
<td>42</td>
</tr>
<tr>
<td>3.2 $G_1$ And $G_2$</td>
<td>43</td>
</tr>
<tr>
<td>3.3 Adding A $K_3$ To A $K_4$</td>
<td>43</td>
</tr>
<tr>
<td>3.4 $G_3$ And $G_4$</td>
<td>43</td>
</tr>
<tr>
<td>3.5 Adding Two $K_3$’s To A $K_4$</td>
<td>44</td>
</tr>
<tr>
<td>3.6</td>
<td>45</td>
</tr>
<tr>
<td>3.7 Graphs With Exactly Three $K_3$’s</td>
<td>45</td>
</tr>
<tr>
<td>3.8 Certain Graphs With Seven $K_3$’s</td>
<td>46</td>
</tr>
<tr>
<td>3.9 $G_6$</td>
<td>49</td>
</tr>
<tr>
<td>3.10 $G_7$ And $G_8$</td>
<td>50</td>
</tr>
<tr>
<td>3.11 $G_9$ To $G_{12}$</td>
<td>50</td>
</tr>
<tr>
<td>3.12 $G_{13}$</td>
<td>50</td>
</tr>
<tr>
<td>3.13 $G_{14}$</td>
<td>51</td>
</tr>
<tr>
<td>3.14 $G'$ And $G''$</td>
<td>53</td>
</tr>
<tr>
<td>3.15 $G_{15}$</td>
<td>53</td>
</tr>
<tr>
<td>3.16</td>
<td>56</td>
</tr>
<tr>
<td>3.17 $G_{17}$ And $W_6$</td>
<td>56</td>
</tr>
</tbody>
</table>
4.1 Graphs With Exactly Three $K_3$'s 
4.2 $(D_1)$ To $(D_6)$ 
4.3 $D_1$ To $D_6$ 
4.4 $D_7$ And $D_7$ 
4.5 $D_9$ To $D_{13}$ 
4.6 $D_{14}$ And $D_{15}$ 
4.7 $D_{16}$ To $D_{20}$ 
4.8 $D_{21}$ And $D_{22}$ 
4.9 $D_{23}$ 
4.10 $D_{24}$ 
4.11 $D_{25}$ 
4.12 $D_{26}$ 
4.13 $D_{27}$ 
4.14 $D_{28}$ 

5.1 Creating Two Induced $C_4$'s 
5.2 Some Graphs With Two Induced $C_4$'s 
5.3 $F_1$ To $F_6$ 
5.4 $F_7$ 
5.5 $F_8$ 
5.6 $F_9$ 
5.7 $F_{10}$ 
5.8 $F_{11}$ 
5.9 $F_{12}$ 
5.10 $F_{13}$ 
5.11 $F_{14}$ 
5.12 $F_{15}$ 
5.13 $F_{16}$ And $F_{17}$ 
5.14 $F_{18}$
<table>
<thead>
<tr>
<th>Figure</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.15 Some Graphs With Three Induced $C_4$'s</td>
<td>112</td>
</tr>
<tr>
<td>5.16</td>
<td>113</td>
</tr>
<tr>
<td>5.17 $F_{19}$ And $F_{20}$</td>
<td>113</td>
</tr>
<tr>
<td>5.18 $F_{21}$ And $F_{22}$</td>
<td>113</td>
</tr>
<tr>
<td>5.19 $F_{23}$</td>
<td>114</td>
</tr>
<tr>
<td>5.20 $F_{24}$ And $F_{25}$</td>
<td>114</td>
</tr>
<tr>
<td>5.21 $F_{26}$</td>
<td>115</td>
</tr>
<tr>
<td>5.22 $F_{27}$</td>
<td>115</td>
</tr>
<tr>
<td>5.23 $F_{28}$ And $F_{29}$</td>
<td>116</td>
</tr>
<tr>
<td>5.24 $F_{30}$</td>
<td>116</td>
</tr>
<tr>
<td>5.25 $F_{31}$</td>
<td>117</td>
</tr>
<tr>
<td>6.</td>
<td>138</td>
</tr>
<tr>
<td>6. Subdivide An Edge of $K_4$</td>
<td>139</td>
</tr>
<tr>
<td>6. Contraction of Edges To Form $K_3$</td>
<td>141</td>
</tr>
<tr>
<td>6.</td>
<td>142</td>
</tr>
<tr>
<td>6. Applying Subdivision of Vertex</td>
<td>142</td>
</tr>
</tbody>
</table>
CHAPTER 1

INTRODUCTION

1.1 Introduction

A graph $G$ is a collection of the edge set and the vertex set, denoted $E(G)$ and $V(G)$, respectively. An edge $e$ of $G$ is a line joining two vertices in $V(G)$, say $v_i$ and $v_j$, denoted $v_iv_j$ or $(v_i,v_j)$. A loop of a graph is an edge having the same end vertices. A graph is said to have multiple edge if there exists two vertices joined by at least two edges. A graph without any multiple edges is called a simple graph.

The theory of graph was invented by Euler [7] in trying to solve the famous Königsberg bridge problem. However, it was the problem of map colouring, the famous Four-Colour Conjecture that acts as a prime stimulant for the development of graph theory. In 1912, number of ways to colour a map in an attack on the four-colour conjecture. For any positive integer $\lambda$ and any map $M$, it is desired to colour the map $M$ with at most $\lambda$ colours such that no two adjacent regions are assigned the same colour. The function for number of ways of such colouring, $P(\lambda)$ was then proved to be always a polynomial.
In 1932, Whitney [31] extended Birkhoff’s idea of map colouring to vertex colouring of a graph $G$. He introduced the function $M(\lambda)$ as the number of ways of colouring the vertices of $G$ (not necessarily arising from a map) with at most $\lambda$ colours such that no two adjacent vertices are assigned the same colour. The notation $P(G, \lambda)$, now known as chromatic polynomial, was then used by Birkhoff and Lewis [2] what polynomials were chromatic polynomials of maps. They also proved that $P(G, \lambda)$ is always a polynomial for any graph $G$. The minimum integer $\lambda$ such that $P(G, \lambda)$ is nonzero is called the chromatic number of $G$, denoted $\chi(G)$. An introduction to the theory of chromatic polynomials can be found in [25].

The problem of characterizing the chromatic polynomials is not yet solved till this day. However, it leads to the concepts of chromatically unique graphs and chromatically equivalent graphs that were first introduced by Chao and Whitehead [3], thus opening a new area of research to graph theorists. Since then, many families of chromatically unique and chromatically equivalent graphs have been obtained (see [14]).

Throughout this thesis, we shall denote by $P(G)$ the chromatic polynomial of a graph $G$. All graphs we consider are simple, loopless and undirected, unless otherwise specified. We shall refer to [10] explained in this thesis.

1.2 The Fundamental Reduction Theorem

We first give a more formal definition of chromatic polynomial. Given a graph $G$ and $\lambda$ different colours. If $G$ is a graph of order $n$ and size $m$, we say $G$ is an $(n, m)$-
to the vertices of $G$. If no two adjacent vertices of $G$ are assigned the same colour, the colouring is said to be proper. Functionally, a

$G$ is a mapping

$$f: V(G) = \{v_1, v_2, \ldots, v_n\} \rightarrow \{1, 2, \ldots, \lambda\},$$

such that $f(v_i) \neq f(v_j)$ whenever $v_iv_j \in E(G)$. Two proper $\lambda$-colourings $g$ of $G$ will be considered as different if $f(v_i) \neq g(v_i)$ for some vertex $v_i$ in $G$.

Then, $P(G)$ is just the number of different proper $\lambda$-colourings

An empty graph is a graph that has no edges while a complete graph is a graph where every two vertices of the graph is joined by an edge. Obviously, if $O(n)$ is the empty graph of order $n$, then $P(O(n)) = \lambda^n$; and if $K_n$ is the complete graph of order $n$, then $P(K_n) = \lambda(\lambda - 1) \cdots (\lambda - n + 1)$.

The following are some useful known results and techniques for determining the chromatic polynomial of a graph.

**Theorem 1.1** (Fundamental Reduction Theorem) (Whitney [32]) Let $G$ be an $(n, m)$-graph. Then

$$P(G) = P(G - e) - P(G \cdot e)$$

where $G - e$ is the graph obtained from $G$ by deleting $e$, and $G \cdot e$ is the graph obtained from $G$ by identifying the end vertices of $e$.

The Fundamental Reduction Theorem can also be used in another way. Let $v_i, v_j \in V(G)$ such that $v_iv_j \notin E(G)$. Then

$$P(G) = P(G + v_iv_j) + P(G \cdot v_iv_j)$$
where $G + v_i v_j$ is the graph obtained from $G$ by adding the edge $v_i v_j$ and $G \cdot v_i v_j$ is the graph obtained from $G$ by identifying the vertices $v_i$ and $v_j$. In this way, $P(G)$ can be expressed as a sum of the chromatic polynomials of complete graphs.

It is sometimes more convenient to use a drawing to denote the chromatic polynomial of a graph (Figure 1.1).

![Figure 1.1: Chromatic Polynomial by Graphs](image)

Let $G_1$ and $G_2$ be graphs, each containing a complete subgraph $K_p$ with $p$ vertices. If $G$ is the graph obtained from $G_1$ and $G_2$ by identifying the two subgraphs $K_p$, then $G$ is called a $K_p$-gluing of $G_1$ and $G_2$. Note that a $K_1$-gluing and a $K_2$-gluing are also called a vertex-gluing and an edge-gluing, respectively.

**Lemma 1.1** (Zykov [35]) Let $G$ be a $K_r$-gluing of $G_1$ and $G_2$. Then

$$P(G) = \frac{P(G_1)P(G_2)}{P(K_r)}.$$

Lemma 1.1 implies that all $K_r$-gluing of $G_1$ and $G_2$ will have the same chromatic polynomial.

The following easy lemma provides another shortcut in determining $P(G)$ since the number of ways of colouring $k$ graphs simultaneously is equal to the product of the number of ways of colouring each of the $k$ graphs.

**Lemma 1.2** If $G$ is a graph with $k$ components $G_1, G_2, \ldots, G_k$, then

$$P(G) = P(G_1)P(G_2) \cdots P(G_k).$$
Let $G^{(0)}$ be a given graph which is $K_p$-gluing. Forming another $K_p$-gluing of $G_1$ and $G_2$, we obtain a new graph $G^{(1)}$. Note that $G^{(1)}$ may not be isomorphic to $G^{(0)}$. Clearly, $G^{(1)}$ is a $K_p$-gluing of some graphs, say $H_1$ and $H_2$. Note that $H_1$ and $H_2$ may not be $G_1$ and $G_2$. Forming another $K_p$-gluing forming $G^{(1)}$ from $G^{(0)}$ (or $G^{(2)}$ from $G^{(1)}$) is called an elementary operation. A graph $H$ is called a relative of $G$ if $H$ can be obtained from $G$ by applying a finite sequence of elementary operations. It follows from Lemma 1.1 that if $H$ is a relative of $G$, then $P(H) = P(G)$.

1.3 Some Properties of Chromatic Polynomials

Let $G$ be an $(n, m)$-graph
Let $C$ be a cycle of $G$ and $e$ an edge of $C$ such that $\alpha(e) \geq \alpha(x)$ for all $x \in E(C)$. Then the path $C - e$ in $G$ is called a broken cycle of $G$ induced by $\alpha$. We then have the following theorem.

Theorem 1.2 (Whitney’s broken cycle theorem) (Whitney [31])

$(n, m)$-

$$P(G) = \sum_{i=0}^{n-1} (-1)^i h_i \lambda^{n-i}.$$ where $h_i$ is the number of spanning subgraphs of $G$ that have $i$ edges and that contain no broken cycles induced by $\alpha$.

Note that $h_i$ is independent of the mapping $\alpha$. The following results can then be derived directly from Theorem 1.2.

Lemma 1.3 Let $G$ be an $(n, m)$-graph. Then in the polynomial $P(G)$, the coefficient of
(i) \( \lambda^n \) is 1

(ii) \( \lambda^{n-1} \) is \(-m\)

(iii) \( \lambda^{n-2} = \binom{m}{2} - t_1(G) \), where \( t_1(G) \) is the number of triangles in \( G \)

(iv) \( \lambda^{n-3} = \binom{m}{3} + (m - 2)t_1(G) + t_2(G) - 2t_3(G) \), where \( t_2(G) \) is the number of cycles of order 4 without chords and \( t_3(G) \) is the number of the complete subgraphs \( K_4 \) in \( G \) (see [8]).

We also have the following properties for \( P(G) \).

Lemma 1.4 Let \( G \) be an \((n, m)\)-graph. Then \( P(G) \) is a polynomial in \( \lambda \) such that

(i) \( \text{deg}(P(G)) = n \)

(ii) the absolute value of the coefficient of \( \lambda^{n-1} \) is the number of edges of \( G \)

(iii) all the coefficients are integers

(iv) the leading term is \( \lambda^n \)

(v) the constant term is zero

(vi) the coefficients alternate in sign

(vii) either \( P(G) = \lambda^n \) or the sum of the coefficients in \( P(G) \) is zero.

Lemma 1.4 above can be proved by induction on \( m \) (see [25]).

Note that not all polynomials that satisfy all the above conditions are chromatic polynomials of some graphs. For example, consider
\[ P(x) = x^4 - 4x^3 + 3x^2 \]

Note that the coefficient of \( x^4 \) is 1, the constant term is zero, the coefficients alternate in sign and sum up to zero. However, \( P(x) \) is not the chromatic polynomial of any graph. If it were, the graph would have four vertices and four edges, by Lemma 1.4(i) and (ii). The only two graphs with four vertices and four edges do not have this polynomial.

### 1.4 Chromatically Unique and Chromatically Equivalent Graph

We note by Lemma 1.1 that for any tree \( T \) of order \( n \), \( P(T) = \lambda (\lambda - 1)^{n-1} \). This implies that there exists non-isomorphic graphs which have the same chromatic polynomial. On the contrary, there are graphs like the empty graphs \( O(n) \) such that no other graphs will have the same chromatic polynomial as \( O(n) \). These observations lead to the following definitions.

Let \( P(G) \) denote the chromatic polynomial of a graph \( G \). Two graphs \( G \) and \( H \) are *chromatically equivalent* (\( \chi \)-equivalent), written \( G \sim H \), if \( P(G) = P(H) \). A graph \( G \) is *chromatically unique* (\( \chi \)-unique) if \( P(H) = P(G) \) implies that \( H \cong G \).

Let \( < G > \) denote the equivalence class determined by the graph \( G \) under \( \sim \). Clearly, \( G \) is \( \chi \)-unique if and only if \( < G > = \{ G \} \).

Recall that \( t_1(G), t_2(G), t_3(G) \) and \( \chi(G) \) are respectively the number of triangles, the number of cycles of order 4 without chords, the number of complete subgraphs \( K_4 \) in \( G \) and the chromatic number of \( G \). The results of the following lemma can be derived from Lemma 1.3.
Lemma 1.5  Let $G$ and $H$ be graphs such that $G \sim H$. Then

(i) $G$ and $H$ have the same order

(ii) $G$ and $H$ have the same size

(iii) $t_1(G) = t_1(H)$

(iv) $t_2(G) - 2t_3(G) = t_2(H) - 2t_3(H)$

(v) $\chi(G) = \chi(H)$

(vi) $G$ is connected if and only if $H$ is connected.

Since there is no general methods for constructing families of $\chi$-unique graphs and $\chi$-equivalent graphs, the above conditions are just some necessary conditions for two graphs $G$ and $H$ to be $\chi$-equivalent. (Also see [19] for a method that uses adjoint polynomials in proving the chromatic uniqueness of a graph).

For a graph $G$ containing a cycle, the girth $g(G)$ of $G$ is the length of a shortest cycle in $G$. Let $\sigma_g(G)$ be the number of cycles of length $g(G)$ in $G$. It then follows from Whitney’s broken cycle theorem (Theorem 1.2) that if $G$ and $H$ are $\chi$-equivalent and containing cycles, then $g(G) = g(H)$ and $\sigma_g(G) = \sigma_g(H)$ (also see [27]).

The following results are not difficult to obtain.

Lemma 1.6  Let $G$ be a graph of size $m$. Then $m \geq 1$ if and only if $\lambda(\lambda - 1)|P(G)$.

Proof If $m \geq 1$, then $\chi(G) \geq 2$. This implies that $P(0) = P(1) = 0$. Thus, $\lambda$ and $(\lambda - 1)$ are two factors of $P(G)$. On the contrary, if $\lambda(\lambda - 1)|P(G)$, then $\chi(G) \geq 2$. Thus, $G$ must have at least one edge. Hence, $m \geq 1$. \[\square\]