UNIVERSITI PUTRA MALAYSIA

AN ALGORITHMIC APPROACH FOR STABILITY OF AN AUTONOMOUS SYSTEM

SHAHARUDDIN BIN CIK SOH

AN ALGORITHMIC APPROACH FOR STABILITY OF AN AUTONOMOUS SYSTEM

By
SHAHARUDDIN BIN CIK SOH

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Master of Science

February **2002**

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfillment of the requirements for the degree of Master of Science.

AN ALGORITHMIC APPROACH FOR STABILITY OF AN AUTONOMOUS SYSTEM

By

SHAHARUDDIN BIN CIK SOH

February 2002

Chairman : Associate Professor Harun bin Budin, Ph.D

Faculty : Science and Environmental Studies

Many phenomena in biology can be modeled as a system of first order differential equations

$$\dot{x} = ax + by,$$

$$\dot{y} = cx + dy.$$

An example of such a system is the prey-predator model. To interpret the results we have to obtain full information on the system of equations such as the stability of the equilibrium points of the system. This requires in depth knowledge of differential equations. The literature often emphasizes on the analytical methods to obtain results regarding the stability of the equilibrium points. This is possible to achieve for small systems such as a 2×2 system.

The non-mathematician researchers often do not have the analytical tools to understand the model fully. Very often what they are interested in is the information regarding the critical points and their stability without going through the tedious mathematical analysis. This calls for user friendly tools for the non-mathematicians to use in order to answer their problem at hand.

The objective of this research is to establish an algorithm to determine the stability of a more general system. By doing so we will be able to help those who are not familiar with analytical methods to establish stability of systems at hand

The following algorithm is employed in developing the software:

L1. Search for critical point is conducted.

L2. Eigenvalues of the linear system are computed. These values are obtained from the characteristic equation $|A - \lambda I| = 0$, where λ is an eigenvalue and

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

For the nonlinear system, linearization process around the critical points are carried out.

L3. Stability of system is determined.

L4. Trajectory of the system is plotted in the phase plane.

To develop the software we use the C programming language.

It is hoped that the software developed will be of help to researchers in the field of mathematical biology to understand the concept of stability in their model.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia
sebagai memenuhi keperluan untuk ijazah Master Sains

**PENDEKATAN ALGORITMA BAGI KESTABILAN
SISTEM BERAUTONOMI**

Oleh

SHAHARUDDIN BIN CIK SOH

Februari 2002

Pengerusi : Professor Madya Dr. Harun Bin Budin

Fakulti : Sains dan Pengajian Alam Sekitar

Banyak fenomena dalam biologi boleh dimodelkan sebagai satu sistem persamaan persamaan pembezaan peringkat

$$\begin{aligned} \dot{x} &= ax + by \\ \dot{y} &= cx + dy \end{aligned}.$$

Satu contoh sistem sedemikian ialah model mangsa-pemangsa. Untuk memberi interpretasi kita perlu mendapat maklumat penuh mengenai sistem persamaan tersebut seperti kestabilan titik genting sistem. Ini memerlukan pengetahuan mendalam tentang persamaan pembezaan. Literatur yang ada menitikberatkan kaedah analisis untuk mendapatkan keputusan mengenai kestabilan titik genting. Ini boleh dilakukan bagi sistem kecil seperti system 2x2.

Penyelidik-penyelidik yang bukan ahli matematik biasanya tidak mempunyai latar belakang analisis yang mencukupi bagi memahami dengan mendalam model tersebut. Juga biasanya mereka hanya berminat untuk mendapat maklumat mengenai titik genting dan kestabilannya tanpa melalui proses analisis matematik. Ini memerlukan satu alat yang mesra pengguna untuk digunakan oleh bukan ahli matematik untuk menjawab persoalan yang dihadapi.

Objektif penyelidikan ini ialah untuk mengadakan satu algoritma untuk menentukan kestabilan sistem yang lebih umum. Dengan berbuat demikian kita akan dapat membantu mereka yang kurang biasa dengan kaedah analisis untuk menentukan kestabilan sistem yang ada ditemui mereka.

Algoritma berikut dijalankan untuk membina perisian:

L1. Penentuan titik genting dilakukan.

L2. Nilai eigen bagi sistem linear dicari. Nilai eigen diperolehi dari persamaan cirian

$|A - \lambda I| = 0$, dengan λ nilai eigen dan $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Untuk sistem tak linear, proses pelinearan di sekitar titik genting dilakukan.

L3. Kestabilan untuk sistem ditentukan.

L4. Trajektori diplot di dalam satah fasa.

Bagi membangunkan perisian ini bahasa pengaturcaraan C digunakan.

Adalah diharapkan bahawa perisian ini dapat menolong penyelidik dalam bidang biologi matematik untuk memahami konsep kestabilan model yang biasa mereka temui.

ACKNOWLEDGEMENTS

Praise is for *Allah Subhanahu Wa Taala*, for giving me the strength, guidance and patience to complete this thesis. My blessings and peace also upon Prophet *Muhammad Sallalahu Alaihi Wasallam*, who was sent for mercy to the world.

I am particularly grateful to my advisor, Assoc. Prof. Dr. Harun bin Budin for his excellent supervision, invaluable guidance, helpful discussions and continuous encouragement. I am grateful for having the opportunity to work under his supervision. I would like to thank the members of my supervisory committee, Assoc. Prof. Dr. Malik Hj. Abu Hassan and Assoc. Prof. Dr. Bachok M. Taib for their invaluable discussions, comments and help.

My thanks are due to all the lecturers of the Department of Mathematics, Universiti Putra Malaysia. Their help and continuous encouragement are highly appreciated.

I also wish to express my thanks to all my friends. In particular I would like to thank all person of the Department of Mathematics, Mara University of Technology; their continuous help, encouragement and support are highly appreciated.

My deepest gratitude and love to my parents, Che Soh bin Abdul Rahman (the late) and Hasani Sulaiman; my brothers; Shamsul, Saiful, Shamsuri and Shahrul; and my only sister; Shaharidah, specially for their prayers for my success. I am especially grateful to my dearest wife, Noor Latiffah, for her support and encouragement during the preparation of this thesis.

The financial support was provided by Universiti Putra Malaysia Graduate Scholarship Program, which is highly appreciated and gratefully acknowledged.

I certify that an Examination Committee met on 7th February, 2002 to conduct the final examination of Shaharuddin Bin Cik Soh on his Master of Science thesis entitled "An Algorithmic Approach for Stability of an Autonomous System" in accordance with Universiti Pertanian Malaysia (Higher Degree) Act 1980 and Universiti Pertanian Malaysia (Higher Degree) Regulation 1981. The Committee recommends that the candidate be awarded the relevant degree. Members of the Examination Committee are as follows:

PENG YEE HOCK, Ph.D
Associate Professor,
Faculty of Science and Environmental Studies,
Universiti Putra Malaysia.
(Chairman)

HARUN BUDIN, Ph.D
Associate Professor,
Faculty of Science and Environmental Studies,
Universiti Putra Malaysia.
(Member)

MALIK HJ. ABU HASSAN, Ph.D
Associate Professor,
Faculty of Science and Environmental Studies,
Universiti Putra Malaysia.
(Member)

BACHOK M. TAIB, Ph.D
Associate Professor,
Faculty of Science and Environmental Studies,
Universiti Putra Malaysia.
(Member)

SHAMSHER MOHAMAD RAMADILI, Ph.D
Professor / Deputy Dean
School of Graduate Studies,
Universiti Putra Malaysia.

Date: 28 MAR 2002

This thesis submitted to the Senate of Universiti Putra Malaysia has been accepted as fulfillment of the degree of Master of Science

AINI IDERIS, Ph.D
Professor/Dean
School of Graduate Studies
Universiti Putra Malaysia.

Date:

DECLARATION

I hereby declare that the thesis is based on my original work except for quotations and citations, which have been duly acknowledged. I also declare that it has not been previously or concurrent submitted for any other degree at UPM or other institutions.

SHAHARUDDIN BIN CIK SOH

Date: 7 FEBRUARY 2002

TABLE OF CONTENTS

LIST OF TABLES

Table

CHAPTER I

INTRODUCTION

Numerical solutions of scientific problems did not become useful and popular proposition in its early days of introduction. The emergence of high-speed computers was the impetus to the change in attitude. The microelectronic revolution has stimulated and inspired work in this area simply because computers with sufficiently large memory can automate most of the work thus reducing the labour involved in the implementation of many of the numerical methods. Since then the study of numerical methods in the solution of ordinary and partial differential equations has enjoyed an intense period of activity. This thesis is developed to study the stability of an autonomous system which has until now been done analytically. The purpose is to build a numerical system that has a relation between all the processes that are involved in solving autonomous system. Emphasis will be on biological processes which can be modeled into a system of an autonomous equations.

Ordinary Differential Equations and Boundary Value Problems

Ordinary differential equations originate naturally in many branches of science. In this research we shall look at examples in biology specifically in the interaction between two species. A thorough knowledge of differential equations is a powerful tool in the hands of an applied mathematician with which to tackle these problems. Some of the general definitions and background materials are given here.

Definition 1.1 An ordinary differential equation is an equation in which an unknown function $y(x)$ and the derivatives of $y(x)$ with respect to x appear. If the n-th derivative of y is the highest derivative in the equation, we say that the differential equation is of the n-th order. The domain of the differential equation is the set of values of x on which $y(x)$ and its derivatives are defined.

The domain is usually an interval I in the real line where I could be a finite interval (a,b), the positive real numbers $(0,\infty)$, the nonnegative real numbers $[0,\infty)$, or even the whole real line $(-\infty,\infty)$. We shall use this notation: a square bracket denoting that the endpoint is included and a round bracket denoting that the endpoint is not included in the interval.

Definition 1.2 A linear differential equation is a differential equation where the dependent variable y and all its derivatives are linear. The general form of a linear differential equation of the n-th order is

$$a_n(x)\frac{d^n y}{dx^n}+\ldots+a_1(x)\frac{dy}{dx}+a_0(x)y=b(x), \qquad [1.1]$$

Apart from the differential equation, the mathematical model will typically also include prescribed values of y and/or the derivatives of y at isolated points in the interval I, usually one or both of the endpoints.

Definition1.3 The prescribed values of y and/or the derivatives of y at the endpoints of the interval are called boundary values. The problem of finding $y(x)$ where y appears in a differential equation as well as in prescribed boundary values is called a boundary value problem. If the independent variable is time and all the boundary values are specified at the left endpoint of the interval, the boundary value problem is called an initial value problem.

Definition 1.4 A solution of a given boundary value problem on the interval I is a function which is continuous on I, which satisfies the differential equation at every point $x \in I$, and agrees with the prescribed boundary values.

Definition 1.5 A function f is said to be piecewise continuous on a finite interval $[a,b]$ if the interval can be subdivided into a finite number of intervals with f continuous on each of these intervals and if the jump in the value of f at each of the endpoints of these intervals is finite.

The existence of a solution is usually guaranteed by finding the solution explicitly with the aid of mathematical techniques. However, there are many differential equations whose solutions cannot be expressed in terms of elementary functions. In these cases numerical techniques must be utilized to calculate the value of the solution approximately at selected points in the interval I.

It is also very important to know how sensitive the solution is to change in the boundary values. The main reason for this is that they are subjected to errors. The crucial question is whether a small error in the boundary values will cause a small error in the solution. If this is the case the solution will be a continuous function of the boundary values.

Mathematical Preliminaries

A profound understanding of the mathematical foundation of any numerical method determines the success of its implementation. In the present computer age not only basic skills in mathematics are necessary but skills in developing well structured algorithms that conform with the current computer technology and architecture is also required in order to develop effective numerical methods and numerical software tools. This section reviews basic differential calculus and matrix algebra, which are associated with the numerical methods.

Basic Differential Calculus

Finite difference calculus is based on the following theorem.

Theorem 1.1 Let $f(x)$ be continuous for $a \leq x \leq b$ and differentiable for $a < \xi < b$ for which

$$f'(\xi) = \frac{f(b) - f(a)}{b - a} \qquad [1.2]$$

Geometrically this means that the curve $f(x)$ has at least one interior point, which are tangential and parallel to the secant line connecting the end points of the curve. In the calculus of finite differences the derivatives

$$f'(x) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \qquad [1.3]$$

may be approximated by

$$f'(x_0) \approx \frac{f(x) - f(x_0)}{h} \qquad [1.4]$$

provided $h = x - x_0$ is finite. This means that under certain conditions there is a point ξ in the interval (a,b) for which the derivative can be calculated directly using [1.4].

Another useful formula that is often used in approximating derivatives is based on Taylor's theorem.

Theorem 1.2 Let $f(x)$ be a function with continuous derivatives of all orders in the interval [a,b] and that n is an arbitrary positive integer. Then

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{(x - x_0)^2}{2!} f''(x_0)$$

$$+ \frac{(x - x_0)^3}{3!} f'''(x_0) + \ldots + \frac{(x - x_0)^n}{n!} f^{(n)}(x_0) + R_{n+1}(\xi) \qquad [1.5]$$

where for $\xi = x_0 + \alpha h$ and $0 < \alpha < 1$

$$R_{n+1} = \frac{(x - x_0)^{n+1}}{(n+1)!} f^{(n+1)}(\xi) \qquad [1.6]$$

$R_{n+1}(x)$ is the remainder that constitutes the remaining terms of the infinite series and is also referred as the truncation error of the expanded function. In many cases it is often sufficient to display the dependence of the error on the parameter h without giving the detailed expression. The "big oh" notation *O(h)*, developed by

Bachmann and Landau (1927) will be used as an indication of the order of the truncation error.

Definition 1.6 A truncation error, which consists of a remainder with $(n+1)^{st}$ order term and $(n+1)^{st}$ derivative term, is conventionally abbreviated as $O(h^{n+1})$.

This means the truncation error goes to zero no slower than h does. Thus higher order truncation errors indicate better precision as more terms are considered in the approximation. However higher order truncation errors sometimes lead to longer computation times thus lowering the efficiency of the method formulated.

Both theorems stated above can be extended for functions with more than one variable. For the expansion of *f(x,y)* in the neighborhood of (x_0, y_0), the rate of change of the function can be due to changes in either of the variables. Thus the derivatives of the function is expressed in terms of its partial derivatives as follows

$$f(x,y) = f(x_0,y_0) + (x-x_0)f_x(x_0,y_0) + (y-y_0)f(x_0,y_0)$$

$$+\frac{1}{2!}\left\{f_{xx}(x_0,y_0)(x-x_0)^2 + 2f_{xy}(x_0,y_0)(y-y_0) + f_{xx}(x_0,y_0)(y-y_0)^2\right\} + \ldots + R_{n+1}(x,y)$$

[1.7]

where

$$R_{n+1}(x,y) = \frac{1}{(n+1)!}\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)^{n+1} f(x_0,y_0) \qquad [1.8]$$

It is often impossible to evaluate the remainder exactly. However it can be estimated based on the function involved in the two-dimensional case.

Some Basic Matrix Algebra

The finite difference approach of solving ordinary differential system problems usually involves solving a system of n simultaneous equations with n unknowns. An effective way of tackling such problems is via matrix algebra. Thus selected topics on matrix algebra that are relevant to the iterative solution of such systems are reviewed.

A convenient way of representing a system of n linear equations with n unknowns such as

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n \end{aligned}\right\} \qquad [1.9]$$

is by the matrix representation

$$A\bar{x} = \bar{b}\ , \qquad [1.10]$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad [1.11]$$

is an $n \times n$ matrix with real elements a_{ij}, and

$$\bar{x} = (x_1, x_2, \ldots, x_n)^T \qquad [1.12]$$

$$\bar{b} = (b_1, b_2, \ldots, b_n)^T \qquad [1.13]$$

Usually the problem is to solve for $\bar{x}$ when A and $\bar{b}$ are known. The problem has a unique solution $\bar{x} = A^{-1}b$ if A is nonsingular. However it is usually difficult to determine whether A is invertible when its order is large. In such cases its structure plus some useful matrix properties help in the decision. Several of the important properties that is useful in determining this factor is listed below. A matrix $A \in R^{n\times n}$ is

positive	if $a_{ij} > 0$ for all i	
nonnegative	if $a_{ij} \geq 0$ for all i	
orthogonal	if $A^{-1} = A^{T}$	
nonsingular	if $\lvert A \rvert \neq 0$	
symmetric	if $A^{T} = A$	
skew-symmetric	if $A^{T} = -A$	[1.14]
positive definite	if $\bar{x}^{T} A\bar{x} > 0$ for $\bar{x} \neq 0,\ \bar{x} \in R^{n}$	
nonnegative definite	if $\bar{x}^{T} A\bar{x} \geq 0,\ \bar{x} \in R^{n}$	
diagonally dominant	if $\lvert a_{ii} \rvert \geq \sum \lvert a_{ij} \rvert$ for all i	
permutation	if $A = \{e_{s1}, \ldots, e_{sn}\}$ where $\{s1, \ldots, sn\}$ is a permutation of $\{1, 2, \ldots, n\}$	

Another matrix property that is important in dealing with iterative methods is irreducibility. By this it simply means that the system of equations cannot be further reduced or decomposed by some partitioning into subsystems of equations. Its formal definition is as follows.

Definition 1.7 A square matrix A of order $n > 1$ is irreducible if for any two nonempty disjoint subsets S_1 and S_2 of $w = \{1,2,\ldots,n\}$, where $S_1 \cup S_2 = w$, there exists $i \in S_1$ and $j \in S_2$ such that $a_{ij} \neq 0$.

This property can also be effectively determined by finite directed graphing. Its usefulness is clear in the following theorem.

Theorem 1.3 A square matrix is irreducible if and only if its directed graph is strongly connected.

Although finite graphing is effective in determining irreducibility, in matrix manipulation a formal relationship such as the following theorem is usually more suitable.

Theorem 1.4 A matrix A is irreducible if and only if there exists a permutation matrix P such that

$$P^{-1}AP = \begin{bmatrix} F & 0 \\ G & H \end{bmatrix} \qquad [1.15]$$

where F and H are square matrices while 0 is the null matrix.

Nonsingular matrices can also be determined by irreducibility and diagonal dominance. This follows from the next theorem.

Theorem 1.5 An irreducibly diagonally dominant matrix A has $\det(A) \neq 0$ with nonvanishing diagonal elements.

The systems encountered in most numerical methods are very large since high accuracy is required. These systems are sometimes reordered properly so that they change into some standard form. If the coefficient matrix of the system has relatively few nonzero entries then the matrix is called *sparse*. Sparse matrices can reduce the tediousness in the implementation of any numerical technique. On top of that a sparse matrix can generally be reordered into a *banded* form where the nonzero entries are confined to a narrow band around the main diagonal. Thus a matrix A is

banded if $a_{ij} = 0$ for $|i - j| > r$ [1.16]

where $2r + 1$ is the bandwidth A. The bandwidth of a banded matrix is the number of the diagonal rows of the main diagonal that have nonempty entries. Some special cases of banded matrices can be described by the following definitions. A square banded matrix A is

diagonal if $a_{ij} = 0$ for $i \neq j$

upper triangular if $a_{ij} = 0$ for $i > j$

lower triangular if $a_{ij} = 0$ for $j > i$ [1.17]

triadiagonal if $a_{ij} = 0$ for $|i - j| > 1$ or $r = 1$

quindiagonal if $r = 2$

These types of matrices can also be extended to a block system of matrices. For example, square matrix A is

$$\text{\textit{Block diagonal}} \qquad \text{if } A = \begin{bmatrix} D_1 & & & \\ & D_2 & & \\ & & \ddots & \\ & & & D_n \end{bmatrix} \qquad [1.18]$$

$$\text{\textit{Block tridiagonal}} \qquad \text{if } A = \begin{bmatrix} D_1 & U_2 & & & \\ L_1 & D_2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & U_{n-1} \\ & & & L_{n-1} & D_n \end{bmatrix} \qquad [1.19]$$

where D_i are square diagonal matrices not necessarily of the same order, whilst U_i and L_i are rectangular matrices. A block tridiagonal matrix such as [1.19] is sometimes called a *T-matrix*. The *block triangular* and *block quindiagonal* matrices can also be defined in a similar way.

Most of the basic concepts above that applies to real matrices can be extended to matrices with complex elements. However unlike the real symmetric matrices, matrices of this class are less common in practice. Due of the most useful class is the Hermitian class of matrices. A square matrix A with complex entries is defined to be

Hermitian	if $A^H = A$	
unitary	if $A^H A = I$	[1.20]
positive definite	if $x^H Ax > 0$ for $x \neq 0$	

where A^H is the conjugate transpose of the complex matrix A.

Theorem 1.6 If a matrix A has positive eigenvalues then A (real or Hermitian) is positive definite.