UNIVERSITI PUTRA MALAYSIA

CHROMATIC EQUIVALENCE CLASSES AND CHROMATIC DEFINING NUMBERS OF CERTAIN GRAPHS

BEHNAZ OMOOMI

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CHROMATIC EQUIVALENCE CLASSES AND CHROMATIC DEFINING NUMBERS OF CERTAIN GRAPHS

By

BEHNAZ OMOOMI

Thesis Submitted in Fulfilment of the Requirement for the Degree of Doctor of Philosophy in the Faculty of Science and Environmental Studies Universiti Putra Malaysia

March 2001

بسم الله الرحمن الرحيم

Specially Dedicated to
My Husband

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Doctor of Philosophy.

CHROMATIC EQUIVALENCE CLASSES AND CHROMATIC DEFINING NUMBERS OF CERTAIN GRAPHS

By
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March 2001

Chairman: **Associate Professor Peng Yee Hock, Ph.D.**

Faculty: **Science and Environmental Studies**

There are two parts in this dissertation: the chromatic equivalence classes and the chromatic defining numbers of graphs.

In the first part the chromaticity of the family of generalized polygon trees with intercourse number two, denoted by $\mathcal{C}_r(a,b\,;\,c,d)$, is studied. It is known that $\mathcal{C}_r(a,b\,;\,c,d)$ is a chromatic equivalence class if $\min\{a,b,c,d\} \geq r+3$. We consider $\mathcal{C}_r(a,b\,;\,c,d)$ when $\min\{a,b,c,d\} \leq r+2$. The necessary and sufficient conditions for $\mathcal{C}_r(a,b\,;\,c,d)$ with $\min\{a,b,c,d\} \leq r+2$ to be a chromatic equivalence class are given. Thus, the chromaticity of $\mathcal{C}_r(a,b;\,c,d)$ is completely characterized.

In the second part the defining numbers of regular graphs are studied. Let $d(n,r,\ \chi = k)$ be the smallest value of defining numbers of all r-regular graphs of order n and the chromatic number equals to k. It is proved that for a given integer k and each $r \geq 2(k-1)$ and $n \geq 2k$, $d(n,r,\ \chi = k) = k-1$. Next, a new lower bound for the defining numbers of r-regular k-chromatic graphs with $k < r < 2(k-1)$ is found. Finally, the value of $d(n,r,\ \chi = k)$ when $k < r < 2(k-1)$ for certain values of n and r is determined.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah.

KELAS KESETARAAN KROMATIK DAN NOMBOR PENTAKRIF KROMATIK BAGI GRAF TERTENTU

Oleh
BEHNAZ OMOOMI
Mac 2001

Pengerusi: **Profesor Madya Peng Yee Hock, Ph.D.**

Fakulti: **Sains dan Pengajian Alam Sekitar**

Dissertasi ini ada dua bahagian: kelas kesetaraan kromatik dan nombor pentakrif kromatik bagi graf.

Dalam bahagian pertama, kekromatikan famili pokok poligon teritlak dengan nombor hubungan bersamaan dua, diberi lambang $\mathcal{C}_r(a,b;\,c,d)$ dikaji. Famili $\mathcal{C}_r(a,b;\,c,d)$ diketahui merupakan suatu kelas kesetaraan kromatik jika $\min\{a,b,c,d\} \geq r+3$. Kita menyelidiki kekromatikan $\mathcal{C}_r(a,b;\,c,d)$ dengan $\min\{a,b,c,d\} \leq r+2$. Syarat perlu, dan cukup bagi $\mathcal{C}_r(a,b;\,c,d)$ dengan $\min\{a,b,c,d\} \leq r+2$ menjadi kelas kesetaraan kromatik ditemui. Dengan yang demikian, kekromatikan $\mathcal{C}_r(a,b;\,c,d)$ terciri secara lengkap.

Dalam bahagian kedua, nombor pentakrif bagi graf sekata dikaji. Misalkan $d(n,r,\ \chi=k)$ nilai terkecil nombor pentakrif bagi semua graf r-sekata berperingkat n dengan nombor kromatik bersamaan k. Bagi sebarang integer k dan setiap $r \geq 2(k-1)$ dan $n \geq 2k$, kita buktikan $d(n,r,\ \chi=k)=k-1$. Seterusnya, suatu batas bawah baru bagi nombor pentakrif graf r-sekata k-kromatik dengan $k<r<2(k-1)$ telah ditemui. Akhirnya, nilai $d(n,r,\ \chi=k)$ apabila $k<r<2(k-1)$ bagi parameter tertentu juga telah diperolehi.

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CHAPTER 1

INTRODUCTION

In this chapter we refer to some definitions and terminology which will be used throughout this thesis. Since the outline of each chapter is given at the beginning of the chapters, we shall give only a brief outline of the thesis in Section 1.2.

1.1 Preliminaries, Definitions, and Notations

Throughout this thesis, a **graph** G is a finite, nonempty **vertex set** $V(G)$ together with an **edge set** $E(G)$, where each edge in $E(G)$ is an unordered pair of vertices. We write uv for the edge $\{u, v\}$. If $uv \in E(G)$, then u and v are **adjacent**. We write $u \leftrightarrow v$ to mean "u is adjacent to v". The vertices contained in an edge are its **endpoints**. If vertex v belongs to edge e, then v and e are **incident**. The repeated edges or edges with both endpoints the same are called **multiple edges** and **loops**, respectively. Here we consider the graphs without multiple edges and loops. The number $|V(G)|$ and $|E(G)|$ are called the **order** and the **size** of G, respectively. The **complement** of a graph G, written $\overline{G}$, is a graph having the same vertex set as G, such that u, v are adjacent in $\overline{G}$ if and only if u, v are not adjacent in G.

A **subgraph** of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$; we write this as $H \subseteq G$. An **induced subgraph** of G is a subgraph H such that every edge of G contained in $V(H)$ belongs to $E(H)$. If H is an induced subgraph of G

with vertex set S, then we write $H = \langle S \rangle$. Similarly, if F is a nonempty subset of $E(G)$, then the subgraph $\langle F \rangle$ induced by F is the graph whose vertex set consist of those vertices of G incident with at least one edge of F and whose edge set is F. A subgraph H of a graph G is called a **spanning subgraph** if $V(G) = V(H)$.

The **degree** of a vertex v in graph G, written $deg_G(v)$ or $deg(v)$, is the number of edges containing v. The **maximum degree** is $\Delta(G)$; the **minimum degree** is $\delta(G)$. An **isolated vertex** has degree 0. A graph G is **regular** of degree r if for each vertex v of G, $deg(v) = r$; such graphs are also called r-**regular** graphs. The **neighborhood** of v, written $N_G(v)$ or $N(v)$, is $\{x \in V(G) \mid x \leftrightarrow v\}$.

An **independent set** in a graph G is a vertex subset $S \subseteq V(G)$, such that the induced subgraph $\langle S \rangle$ has no edges. A graph is **bipartite** if its vertex set can be partitioned into two independent sets. A graph is k-**partite** if $V(G)$ can be partitioned into k independent sets. The independent sets in a specified partition are **partite sets.**

A **complete graph** is a graph in which every pair of vertices forms an edge. We denote a complete graph of order n by K_n. The complement $\overline{K}_n$ of the complete graph K_n has n vertices and no edges and is referred to as the **empty graph** of order n. A **complete bipartite graph** is a bipartite graph in which the edge set consists of all pairs having a vertex from each of the two independent sets in the vertex partition.

A **path** of length n in a graph, denoted by P_n, is an ordered list of distinct vertices $v_0, \cdots, v_n$ such that $v_{i-1}v_i$ is an edge for all $1 \leq i \leq n$. Similarly, a **cycle** of length n in a graph, denoted by C_n, is an ordered list of distinct vertices $v_1, \cdots, v_n$ such that $v_{i-1}v_i$, $2 \leq i \leq n$, and also v_nv_1 are edges. The **girth** of a graph G, denoted by $g(G)$, is the length of a shortest cycle in G. The first and last vertices of a path are its **endpoints**; and the rest are **interior** vertices. A

(u, v)-**path** is a path with endpoints u and v. A path in graph G is called a **simple** path if the degree of each interior vertex is two in G. A graph G is **connected** if it has a (u, v)-path for each pair $u, v \in V(G)$.

A graph G_1 is **isomorphic** to a graph G_2, written $G_1 \cong G_2$, if there exists a one-to-one mapping ϕ, called an **isomorphism**, from $V(G_1)$ onto $V(G_2)$ such that $uv \in E(G_1)$ if and only if $\phi(u)\phi(v) \in E(G_2)$.

The **union** of graphs G and H, written $G \cup H$, has vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. To specify the **disjoint union** with $V(G) \cap V(H) = \emptyset$, we write $G + H$. The **join** of G and H, written $G \vee H$, is obtained from $G + H$ by adding the edges $\{xy \mid x \in V(G),\ y \in V(H)\}$. If X is an nonempty subset of $E(G)$, then $G - X$ denotes the graph obtained from G by removing the edges in X.

Let G_1 and G_2 be graphs containing subgraphs Q_1 and Q_2, respectively, such that Q_1 and Q_2 are isomorphic to some Q. Then a Q-**gluing** of G_1 and G_2 is a graph obtained from the union of G_1 and G_2 by identifying Q_1 with Q_2. When $Q = K_1$ or $Q = K_2$, the Q-gluing of G_1 and G_2 is called **vertex-gluing** and **edge-gluing**, respectively.

A λ-**colouring** of a graph G is an assignment

$$\begin{array}{cccc} c: & V(G) & \longrightarrow & \{1, \cdots, \lambda\} \\ & v & \longmapsto & c(v) \end{array}$$

such that if $u \leftrightarrow v$, then $c(u) \neq c(v)$. A graph G is λ-**colourable** if it has a λ-colouring. The **chromatic number**, $\chi(G)$, is the minimum λ such that G is λ-colourable. In a given graph G, a set of vertices S with an assignment of colours is said to be a **defining set** (with respect to vertex colouring) for G if there exists a unique extension of the colours of S to a $\chi(G)$-colouring of the vertices of G.

A defining set with minimum cardinality is called a **smallest defining set** (of vertex colouring) and its cardinality is the **defining number**, denoted by $d(G, \chi)$.

The vertices having a given colour in a λ-colouring must form an independent set. Hence G is λ-colourable if and only if G is λ-partite. Two λ-colourings, c_1 and c_2 of G are **different** if and only if $c_1(v) \neq c_2(v)$ for some $v \in V(G)$. The number of distinct λ-colourings of G is denoted by $P(G, \lambda)$ or $P(G)$ if there is no danger of confusion. For any graph G, $P(G, \lambda)$ is in fact a polynomial in λ, called the **chromatic polynomial** of G.

Two graphs G and H are **chromatically equivalent** denoted by $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. A graph G is **chromatically unique** if $G \cong H$ for any graph H such that $H \sim G$. Trivially, the relation '$\sim$' is an equivalence relation on the class of graphs. We shall denote by $\langle G \rangle$ the **chromatic equivalence class** determined by G under '$\sim$'; indeed, $\langle G \rangle$ is the set of all graphs having the same chromatic polynomial $P(G, \chi)$. Thus a graph G is chromatically unique if and only if $\langle G \rangle = \{G\}$ (up to isomorphism). In other words, a set of graphs $\mathcal{S}$ is a chromatic equivalence class if (i) any two graphs in $\mathcal{S}$ are chromatically equivalent and (ii) for any graph H with $H \sim G$, where $G \in \mathcal{S}$, we have $H \in \mathcal{S}$. A property of a graph or a quantity associated with a graph is called **χ-invariant** if it is preserved under the equivalence relation. To study the **chromaticity** of a class $\mathcal{S}$ of graphs means to study the problem of determining the chromatic equivalence classes of graphs in $\mathcal{S}$.

A **generalized polygon tree** is a graph defined recursively as follows. Each cycle C_p, $p \geq 3$, is a generalized polygon tree. If H is a generalized polygon tree containing a simple path P_k, $k \geq 1$, as a subgraph, then every P_k-gluing of H and C_r, where $k \leq r$ is also a generalized polygon tree. Every generalized polygon tree is a graph obtained in this manner within a finite number of steps.

Consider the generalized polygon tree $G_t^s(a, b;\, c, d)$ shown in Figure 1.1. The integers a, b, c, d, s, and t represent the lengths of respective paths between the vertices of degree three, where $s \geq 0$ and $t \geq 0$. Let $r = s + t$. We now form a family $\mathcal{C}_r(a, b;\, c, d)$ of the graphs $G_t^s(a, b;\, c, d)$ where the values of a, b, c, d and r are fixed but the values of s and t vary; that is

$$\mathcal{C}_r(a, b;\, c, d) = \{\ G_t^s(a, b;\, c, d) \mid r = s + t, s \geq 0, t \geq 0\ \}.$$

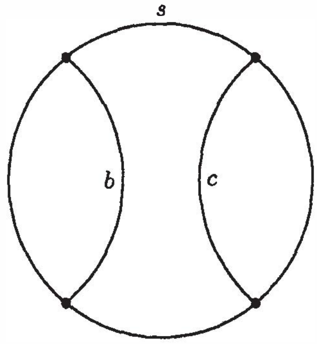

Figure 1.1: $G_t^s(a, b;\, c, d)$.

For example,

$$\mathcal{C}_5(2, 5;\, 3, 6) = \{G_5^0(2, 5;\, 3, 6), G_4^1(2, 5;\, 3, 6), G_3^2(2, 5;\, 3, 6)\}$$

and

$$\mathcal{C}_1(2, 5;\, 3, 6) = \{G_1^0(2, 5;\, 3, 6)\}.$$

In general, there are exactly $\left\lfloor \frac{r}{2} \right\rfloor + 1$ non-isomorphic graphs in the family $\mathcal{C}_r(a, b;\, c, d)$.

The concept of the chromatic polynomial of graphs was first introduced by G.D. Brikhoff [4] in 1912 as a possible means to solve the four-colour problem. For more information about the chromatic polynomial the reader may refer to [35], [37], and [38]. The concept of chromatic uniqueness of graphs was first introduced by Chao and Whitehead [6] in 1978. For expository papers giving a survey on most of the works done on chromatically unique graphs and chromatic equivalence classes, the reader is referred to Koh and Teo in [20] and [21].

1.2 Outline of Chapters

There are two parts in this dissertation: The first part, consisting of Chapters 2 and 3, is about chromatic equivalence classes, and the second part, consisting of Chapters 4, 5, and 6, is about the defining numbers of graphs.

Suppose that H is a graph such that $P(H) = P(G_t^s(a,b;\, c,d))$. Then we know that H is also a generalized polygon tree with intercourse number two (see Theorem 2.2). Thus, $H = G_{t'}^{s'}(a',b';\, c',d')$ where $r' = s' + t'$. The question now is whether or not the graph $G_{t'}^{s'}(a',b';\, c',d')$ is in the family $\mathcal{C}_r(a,b;\, c,d)$. In other words, is $\mathcal{C}_r(a,b;\, c,d) = \langle G_t^s(a,b;\, c,d)\rangle$? Moreover, what is the necessary and sufficient condition for $\mathcal{C}_r(a,b;\, c,d)$ to be a chromatic equivalence class?

In Chapter 2, we first present a brief survey of known results on $\mathcal{C}_r(a,b;\, c,d)$. Then, we consider $\mathcal{C}_r(a,b;\, c,d)$ for $r = 1$. It is clear that for $r = 1$, the family $\mathcal{C}_r(a,b;\, c,d)$ contains only one graph $G_1^0(a,b;\, c,d)$. Thus, the family $\mathcal{C}_1(a,b;\, c,d)$ is a chromatic equivalence class if and only if $G_1^0(a,b;\, c,d)$ is a chromatically unique graph. We shall discuss the chromatic uniqueness of $G_1^0(a,b;\, c,d)$. In [33] it is proved that $G_1^0(a,b;\, c,d)$ is a chromatically unique graph if $\min\{a,b,c,d\} \geq 4$. Also, the chromaticity of $G_1^0(a,b;\, c,d)$ for $\min\{a,b,c,d\} = 1$ is characterized in [44]. We consider the cases $\min\{a,b,c,d\} = 2$ and $\min\{a,b,c,d\} = 3$ in Sections 2.3 and 2.4, respectively, and give a necessary and sufficient condition for $G_1^0(a,b;\, c,d)$ to be a chromatically unique graph.

In Chapter 3, we study the chromaticity of $\mathcal{C}_r(a,b;\, c,d)$ for $r \geq 2$. Peng et al. [34] proved that $\mathcal{C}_r(a,b;\, c,d)$ is a chromatic equivalence class if $\min\{a,b,c,d\} \geq r+3$. The chromaticity of $\mathcal{C}_r(a,b;\, c,d)$ for $\min\{a,b,c,d\} = 1$ is characterized in [44]. We consider the case $\min\{a,b,c,d\} = r+2$ in Section 3.3 and give a characterization theorem for $\mathcal{C}_r(a,b;\, c,d)$ to be a chromatic equivalence class when $r \geq 2$ and

$\min\{a, b, c, d\} = r + 2$. This theorem implies that the conjecture proposed in [34] is not true for $r \geq 2$. In Sections 3.4, we consider $2 \leq \min\{a, b, c, d\} \leq r + 1$ and give a necessary and sufficient condition for $\mathcal{C}_r(a, b\,;\, c, d)$ to be a chromatic equivalence class. Thus, Problem 2 in [21] is completely solved.

Our proofs, roughly, are by comparing polynomials and are lengthy. Apparently, we do not have better methods. Perhaps, because of the nature of the problem, it is not easy to find a shorter proof; for instance, the proof of Theorem 2.12 on page 228 in [17] required more than one hundred pages.

In Chapter 4, we present a review of the concept of defining set in different areas such as latin squares, block designs and graph theory. We also state some related known results which are used in Chapters 5 and 6.

Mahmoodian and Mendelsohn [26] in 1999 studied the defining numbers of regular graphs. Let $d(n, r,\ \chi = k)$ be the smallest value of $d(G,\ \chi)$ for each r-regular graph G of order n and chromatic number k. In Chapter 5, we prove that for a given integer k and each $r \geq 2(k-1)$ and $n \geq 2k$, $d(n, r,\ \chi = k) = k - 1$. Thus, the answer to Question 2 in [26] is in the affirmative.

In Chapter 6, we find a new lower bound for the defining number of r-regular k-chromatic graphs with $k < r < 2(k-1)$. We also determine the value of $d(n, r,\ \chi = k)$ for certain values of n and r.

In Appendix A, we list the papers that were derived from this thesis.

CHAPTER 2

CHROMATIC CHARACTERIZATION OF $\mathcal{C}_1(\mathbf{a}, \mathbf{b}\,;\, \mathbf{c}, \mathbf{d})$

2.1 Chapter Outline

In this chapter, we first review known results on $\mathcal{C}_r(a, b\,;\, c, d)$ which are useful in establishing our theorems. Then we consider the chromaticity of $\mathcal{C}_r(a, b\,;\, c, d)$ for $r = 1$ in Sections 2.3 and 2.4. The chromaticity of $\mathcal{C}_r(a, b\,;\, c, d)$, when $r \geq 2$, will be discussed in Chapter 3. Recall that

$$\mathcal{C}_r(a, b\,;\, c, d) = \{\ G_t^s(a, b\,;\, c, d) \mid r = s + t, s \geq 0, t \geq 0\ \}.$$

It is clear that for $r = 1$, the family $\mathcal{C}_r(a, b\,;\, c, d)$ contains only one graph $G_1^0(a, b\,;\, c, d)$. Thus, the family $\mathcal{C}_r(a, b\,;\, c, d)$ is a chromatic equivalence class if and only if $G_1^0(a, b\,;\, c, d)$ is a chromatically unique graph. We shall discuss the chromatic uniqueness of $G_1^0(a, b\,;\, c, d)$.

Peng in [33] proved that the graph $G_1^0(a, b\,;\, c, d)$ is chromatically unique when $\min\{a, b, c, d\} \geq 4$. Also, in [32], it was shown that the graph $G_1^0(a, b\,;\, c, d)$ is chromatically unique for certain values of a, b, c, and d. We study the chromatic uniqueness of $G_1^0(a, b\,;\, c, d)$ when $2 \leq \min\{a, b, c, d\} \leq 3$ in Sections 2.3 and 2.4. It is proved that $G_1^0(a, b\,;\, c, d)$ with $\min\{a, b, c, d\} > 1$ is a chromatically unique graph except the following five families of graphs: $G_1^0(3, 5\,;\, 5, 8)$, $G_1^0(3, b\,;\, b + 1, b + 3)$ $(b \geq 2)$, $G_1^0(3, c + 3\,;\, c, c + 1)$ $(c \geq 2)$, $G_1^0(3, 3\,;\, c, c + 2)$ $(c \geq 3)$, and $G_1^0(3, b\,;\, 3, b + 2)$ $(b \geq 3)$.

2.2 Introduction and Known Results

Very often, to discover or establish new chromatically unique graphs or chromatic equivalence classes, some χ-invariant properties are required. In the following theorem we list some well-known necessary conditions for chromatic equivalence.

Theorem 2.1 (Whitney [42]) *Let G and H be chromatically equivalent graphs. Then*

(a) $|V(G)| = |V(H)|$;

(b) $|E(G)| = |E(H)|$;

(c) $\chi(G) = \chi(H)$;

(d) $g(G) = g(H)$;

(e) G and H have the same number of shortest cycles.

It follows immediately from Theorem 2.1 that all cycles C_n are chromatically unique. A **chord** of a cycle C_n, $n \geq 4$, is an edge joining a pair of nonadjacent vertices in C_n. A **θ-graph** is a cycle with a chord. Chao and Whitehead [6] showed that every θ-graph is chromatically unique. This result was extended by Loerinc [22]. A graph is called a **generalized θ-graph** if it is obtained by connecting two distinct vertices by three internally disjoint paths. Such a graph is denoted by $\theta(a, b, c)$ if the lengths of the three paths are a, b, and c. Loerinc [22] proved that for any three positive integers a, b, c such that $a \leq b \leq c$ and at most one of them is 1, the generalized θ-graph $\theta(a, b, c)$ is a chromatically unique graph.

Let $s \geq 2$. For any s positive integers $k_1 \leq k_2 \leq \cdots \leq k_s$ with at most one $k_j = 1$, let $\theta(k_1, k_2, \cdots, k_s)$ denote the graph obtained by connecting two distinct vertices with s internally disjoint paths of lengths $k_1, k_2, \cdots, k_s$, respectively. The graph $\theta(k_1, k_2, \cdots, k_s)$ is called a **multi-bridge** or more specifically s-**bridge** graph. Note that for $s = 2, 3$ the graphs are cycles and generalized θ-graphs, respectively, and are known to be chromatically unique graphs.

Definition 2.1 *A* **generalized polygon tree** *is a graph defined recursively as follows. Each cycle C_p, $p \geq 3$, is a generalized polygon tree. If H is a generalized polygon tree containing a simple path P_k, $k \geq 1$, as a subgraph, then every P_k-gluing of H and C_r, where $k \leq r$, is also a generalized polygon tree. Every generalized polygon tree is a graph obtained in this manner within a finite number of steps.*

In the above definition, the value of k may vary from step to step. If we require that $k = 1$ in each step, then such a resulting generalized polygon tree is a **polygon tree**.

Xu [43] investigated the chromaticity of generalized polygon trees and introduced an interesting χ-invariant for them. In [43], it was proved that every generalized polygon tree is a planar graph.

A pair $\{u, v\}$ of nonadjacent vertices of a graph G is called an **intercourse pair** if there are at least three internally disjoint (u, v)-paths in G. let $c(G)$ denote the number of intercourse pairs of vertices in G. Xu [43] showed that the property of being a generalized polygon tree is preserved under '$\sim$' and the quantity $c(G)$ of a generalized polygon tree G is a χ-invariant.

Theorem 2.2 (Xu [43]) *If G is a generalized polygon tree and $H \sim G$, then H is also a generalized polygon tree and $c(H) = c(G)$.*

By using the χ-invariant $c(G)$, Xu [43] also proved that the class of polygon trees is a chromatic equivalence class. This result was obtained earlier by Wakelin and Woodall [41]. Note that a s-bridge, $s \geq 3$, is a generalized polygon tree with one intercourse pair.

Consider the generalized polygon tree $G_t^s(a, b\,;\, c, d)$ with two intercourse pairs shown in Figure 1.1. Recall that

$$\mathcal{C}_r(a, b\,;\, c, d) = \{\ G_t^s(a, b\,;\, c, d) \mid r = s + t, s \geq 0, t \geq 0\ \}.$$

Here we present a survey of works done on chromaticity of the family of graphs $\mathcal{C}_r(a, b\,;\, c, d)$. For $r = 0$, $\mathcal{C}_r(a, b\,;\, c, d)$ is a 4-bridge and the chromaticity of this family was characterized in [44].

Theorem 2.3 (Xu et al. [44]) *The graph $G_0^0(a, b\,;\, c, d)$ is a chromatically unique graph except $G_0^0(1, b\,;\, c, d)$ and $G_0^0(2, b\,;\, b+1, b+2)$.*

Also, Xu et al. in [44] studied the chromaticity of $\mathcal{C}_r(a, b\,;\, c, d)$ for $r \geq 1$ and $\min\{a, b, c, d\} = 1$. In $\mathcal{C}_r(a, b\,;\, c, d)$, without loss of generality, we can assume $\min\{a, b, c, d\} = a$.

Theorem 2.4 (Xu et al. [44]) *The family of graphs*

$$\mathcal{F} = \mathcal{C}_r(1, b\,;\, c, d) \cup \mathcal{C}_{c-1}(1, b\,;\, r+1, d) \cup \mathcal{C}_{d-1}(1, b\,;\, c, r+1),$$

where $r \geq 1$ and $b, c, d \geq 2$, is a chromatic equivalence class except for $r = 2$ and $b = d = c + 1$. Moreover, for $r = 2$ and $b = d = c + 1$ the family of graphs

$$\mathcal{C}_0(2, c\,;\, c+1, c+2) \cup \mathcal{C}_2(1, c+1\,;\, c, c+1) \cup \mathcal{C}_{c-1}(1, c+1\,;\, 3, c+1) \cup \mathcal{C}_c(1, c+1\,;\, c, 3)$$

is a chromatic equivalence class.

Remark 2.1 In the family of graphs

$$\mathcal{F} = \mathcal{C}_r(1, b;\ c, d) \cup \mathcal{C}_{c-1}(1, b;\ r+1, d) \cup \mathcal{C}_{d-1}(1, b;\ c, r+1),$$

if $c = d = r + 1$, then $\mathcal{F} = \mathcal{C}_r(1, b;\ r+1, r+1)$. Therefore by Theorem 2.4, $\mathcal{C}_r(1, b;\ r+1, r+1)$ is a chromatic equivalence class. If $r = 1$, then $G_1^0(1, b;\ 2, 2)$ is a chromatically unique graph (see [40]).

Teo and Koh [40], by considering $\mathcal{C}_r(a, b;\ c, d)$ as a 2-connected graph of order n and size $n+2$ of girth 4, proved that $\mathcal{C}_r(2, 2;\ c, d)$ is a chromatic equivalence class for any integer $r \geq 1$. Chen and Ouyang [9], by considering 2-connected graphs of order n and size $n + 2$ of girth 5, showed that $\mathcal{C}_r(2, 3;\ c, d)$ is a chromatic equivalence class if and only if $(c, d, r) \neq (k, k+2, k+1)$ or $(k+1, k+3, k-1)$, for some $k \geq 2$. In [32], Peng studied the chromaticity of $\mathcal{C}_r(a, b;\ c, d)$ for certain values of a, b, c, d, and r. Peng et al. [34] established that $\mathcal{C}_r(a, b;\ c, d)$ is a chromatic equivalence class if $\min\{a, b, c, d\} \geq r + 3$.

In [7], Chao and Zhao studied the chromatic polynomials of the family $\mathcal{F}$ of connected graphs with k edges and $k-2$ vertices each of whose degree at least two where k at least six. They first divided this family of graphs into three subfamilies $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ according to their chromatic polynomials, and computed the chromatic polynomials for the graphs in each subfamily. Then they discussed the chromatic equivalence of graphs in $\mathcal{F}$. One of their results is Theorem 2.5. Note that the graph $G_t^s(a, b;\ c, d)$ is in $\mathcal{F}_2$.

Theorem 2.5 (Chao and Zhao [7] and Peng et al. [34]) *All the graphs in $\mathcal{C}_r(a, b;\ c, d)$ are chromatically equivalent.*

By Theorem 2.5, we only need to compute $P(G_r^0(a, b;\ c, d))$ for computing the chromatic polynomial of $G_t^s(a, b;\ c, d)$.