UNIVERSITI PUTRA MALAYSIA

CLASSICAL ASPECT OF UNCERTAINTY PRINCIPLE FOR SPIN ANGULAR MOMENTUM IN GEOMETRIC QUANTUM MECHANICS

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By

UMAIR BIN ABDUL HALIM

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfillment of the Requirements for the Doctor of Philosophy

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DEDICATIONS

This thesis is dedicated to
all my family members
especially
my beloved father Abdul Halim
my dear mother Nor Hayati
my lovely wife Zarifth Shafika
including my lecturers

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfillment of the requirement for the degree of Doctor of Philosophy

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Chairman: Associate Professor Hishamuddin Zainuddin, PhD
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Quantum mechanics is one of two foundational parts of modern physics. Along with relativity, quantum mechanics plays a central roles in explaining the nature and behavior of matter on the microscopic level. It is regarded as most successful theory ever developed in history of physics. However it is difficult to make a smooth connection between classical mechanics and quantum mechanics since classical mechanics is based on geometry and some of the systems are non-linear whereas quantum mechanics is intrinsically algebraic and linear. The fact that classical mechanics, general relativity and others are highly geometrical inspired some physicists to cast quantum mechanics in geometrical language in order to better understand the quantum-classical transition. Within this framework the states are represented by points of a symplectic manifold with a compatible Riemannian metric, the observables are real valued functions on the manifold, and the quantum evolution is governed by a symplectic flow that is generated by a Hamiltonian function. In this research, the properties of spin $\frac{1}{2}$, spin 1 and spin $\frac{3}{2}$ particles in geometric quantum mechanics framework have been studied. Generally the Robertson-Schrödinger uncertainty principle for these systems has been demonstrated varies along any Hamiltonian flows. This work was done by calculating the evolution of symplectic area and component of Riemannian metric under the flows. Besides, the correspondence between Poisson bracket and commutator for these systems was showed by explicitly computed the value of commutator of spin operators and compared it with the Poisson bracket of the corresponding classical observables. This study was extended by comparing the Casimir operator and its classical counterpart. The results showed that there exist correspondence between classical and quantum Casimir operator at least for the case of spin $\frac{1}{2}$. This research might be a good step toward inserting the aspect of symplectic topology such as non-squeezing theorem and clearly showed the limit of classical notion to describe the purely quantum concept.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

ASPEK-ASPEK KLASIK MOMENTUM SUDUT SPIN DI DALAM GEOMETRI KUANTUM MEKANIK

Oleh

UMAIR BIN ABDUL HALIM

April 2021

Pengerusi: Profesor Madya Hishamuddin Zainuddin, PhD
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Kuantum mekanik merupakan satu daripada dua cabang utama fizik moden bersama dengan relativiti memainkan peranan penting dalam menerangkan sifat semulajadi bahan di peringkat atom dan sub-atom. Ia dianggap di antara teori yang paling berjaya pernah dihasilkan oleh ahli fizik sehingga kini. Namun begitu asas pembinaan teori kuantum yang berdasarkan algebra dan bersifat linear menyukarkan ahli fizik untuk membuat hubungan secara langsung dengan mekanik klasik yang berorentasikan geometri dan bersifat tidak linear. Oleh itu sebahagian daripada ahli fizik cuba untuk menghasilkan kuantum mekanik yang berasaskan geometri untuk lebih memahami hubung kait teori ini dengan klasikal mekanik. Keadaan sesuatu sistem dalam kerangka ini diwakili oleh titik di dalam manifold simplektik yang dilengkapi dengan metrik Riemannian. Selain dari itu, kuantiti yang boleh dicerap adalah merupakan fungsi nyata di dalam manifold ini dan evolusi kuantum sistem adalah ditentukan oleh aliran Hamiltonian yang dihasilkan oleh fungsi Hamiltonian. Dalam tesis ini, ciri-ciri spin $\frac{1}{2}$, spin 1 dan spin $\frac{3}{2}$ telah dikaji menggunakan kerangka geometri kuantum mekanik. Secara umumnya prinsip ketidakpastian Robertson-Schrödinger telah ditunjukkan adalah sentiasa berubah disepanjang mana-mana aliran Hamiltonian. Hal ini dapat dilakukan dengan mengira luas simplektik dan komponen metrik Riemannian disepanjang aliran tersebut. Selain dari itu, hubung kait diantara kurungan Poisson dan komutator telah ditunjukkan dengan mengira nilai komutator bagi operator spin dan bandingkan ia dengan nilai kurungan Poisson bagi kuantiti klasiknya. Kemudian perbandingan diantara operator Casimir dan versi klasiknya telah dilakukan. Hasil kajian menunjukkan bahawa sekurang-kurangnya wujud hubungan diantara kuantum dan klasik operator Casimir untuk kes spin $\frac{1}{2}$. Ujikaji ini adalah dianggap penting dalam usaha untuk mengaitkan geometri kuantum mekanik dengan topologi simplektik dan juga menunjukkan secara jelasnya had fahaman klasik dalam menghuraikan konsep kuantum yang tulen.

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This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Doctor of Philosophy. The members of the Supervisory Committee were as follows:

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CHAPTER 1

INTRODUCTION

1.1 The Difference Structures of Classical and Quantum Mechanics

For over 200 years, physicists believe that classical mechanics is a fundamental rule of explaining most of physical phenomena since all the important laws in thermodynamic, physical optic and kinetic theory have to be compatible with this theory. It is used to predict the motion of objects on macroscopic level, from molecular dynamics to the motion of celestial bodies such as moons, planets, and solar system. In this domain, this theory provides very accurate results and become one of the important subject in science, engineering and technology. However, inability to predict certain behaviour of matter discovered in late 19th century such as Black Body Radiation shows the incompleteness of classical mechanics. This fact leads to revolution in physics world that give birth to new era which we now refer as modern physics. Quantum mechanics is one of two foundational parts of modern physics, along with relativity play a central roles to explains the nature and behavior of matter at the microscopic level. It is regarded as the most successful theory ever developed in history of physics. This theory provides significant contribution in development of modern technology including lasers, CDs, solar cells and many others. Besides, it act as theoretical basis for some related field such as nanotechnology, condensed matter physics, quantum chemistry, structural biology, particle physics, and electronics.

In quantum mechanics, the state of the system corresponds to points in complex projective Hilbert space $P(\mathscr{H})$ i.e. the Hilbert space $\mathscr{H}$ modulo scalars multiplication and the observables are represented by self-adjoint linear operators on $\mathscr{H}$. Furthermore, similar to classical description, the space of observables is a real vector space equipped with two algebraic structures called the Jordan product and Lie product. From the latter, the space of observables is endowed with the structure of Lie algebra. The measurement theory, on the other hand is clearly different compare with the classical mechanics. According to Copenhagen interpretation, the measurement of observable $\hat{A}$ in a state $|\Psi\rangle$ yield eigenvalue a and the state instantly change into the corresponding eigenstate. Similar to classical theory, the observable $\hat{A}$ also gives rise to a flow on the state space. Specifically the flow in quantum is generated by one-parameter group $\exp^{i\hat{A}t}$ that preserves the linearity of $\mathscr{H}$. While classical mechanics is regarded as an approximation of quantum mechanics at macroscopic scale, this theory is formulated based on different physical idea. In classical framework the states are represented by the points on phase space which is a symplectic manifold M. The set of real-valued and smooth functions on this manifold are space of observables. In addition this space is equipped with a commutative and associative algebra structure. The measurement of an observable f in any state $p \in M$ is equal to the value $f(p)$. The results are completely certain and the state remains unchanged after measurement. Furthermore, an observable f is closely related with a vector field X_f

which generates flows on the phase space. In term of dynamics, the observables are said to be Hamiltonian H, where the flows generated by X_H describe the evolution of the state on phase space. Although quantum mechanics and classical mechanics have several points in common, they are quite different in several aspects. The most striking one is the classical mechanics is based on geometry and some of the systems are non-linear whereas quantum mechanics is intrinsically formulated as algebraic and linear. The linearity seems to be necessary condition since none of standard quantum mechanics postulate can be stated without referring to it. This distinction is quite strange since in general, linear structure in physics arises as approximations to more accurate non-linear ones, but in this case the situation happens in opposite way. Thus it is difficult to make a smooth connection between classical mechanics and quantum mechanics.

1.2 The Quantum-Classical Correspondence Problem

The connection between classical and quantum representation is one of the greatest problems in understanding microscopic systems. While there is no question that quantum and classical descriptions are doing well in their own implementation scales, one would consider a smooth transition between these two descriptions to be feasible, at least at theoretically. It is well-known however, that this is not a trivial problem. To explain such a transformation a field known as quantization has been intensively developed.

In physics, quantization has been regarded as a process of transition from the classical system which is generally speaking, something that involves macroscopic objects and which one is familiar with from daily life to the corresponding quantum system, which involves microscopic objects where things are subject to more complex laws. The latter can be reduced to the previous, as the scale of the objects becomes greater, that is, as the Planck constant which mathematically refers to the magnitude where the quantum effects are important, appears to be zero. This process is not only become the basic tool used to establish important theories of physics including quantum optics, particle and nuclear physics but is also of central interest in the philosophy of physics. Besides that the quantization method has contributed into mathematical knowledge influencing the fields of group representation theory and symplectic geometry.

However throughout history, it is clear that mathematically and also physically, such a concept is not entirely adequate. From a physical perspective, it is more relevant to consider quantization just as a correspondence of classical and quantum systems since there might be quantum systems that have no classical analog as well as various quantum systems that belong to the same classical system. Moreover from the mathematical perspective, one may face a different type of difficulties, that is the quantization procedure could not be extended to the full algebra of observable, for example it can only be applied in position observable q and momentum observable

p up to second order polynomials. This problem has brought us to the well-known Groenewold-van Hove (no-go) theorem. The theorem states that there is no quantization map Q acting on polynomial of degree not exceeding four that follows the bracket condition

$$Q_{\{f,g\}} = \frac{1}{i\hbar}[Q_f, Q_g] \tag{1.1}$$

for all functions f and g of degree not exceeding three. As a consequence, the presence of several types of quantization that have been faced ranging from geometric quantization, deformation quantization and numerous associated operator-theoretic quantizations to Feynman path integrals and many others.

Besides, according to the Kochen-Specker Theorem, given a premise of noncontextuality certain sets of quantum observables cannot consistently be assigned values at all. The theorem shows that there is a contradiction between two basic assumptions of the hidden-variable theories intended to reproduce the results of quantum mechanics: that all hidden variables corresponding to quantum-mechanical observables have definite values at any given time, and that the values of those variables are intrinsic and independent of the device used to measure them. Originally, a set of 117 different projection operators on Hilbert space with dimension 3 has been found that there was impossible to consistently assign values to these projection operators without reaching the contradiction. It turns out that it is not possible to simultaneously combine all the commuting subalgebras of the algebra of such observables into one commutative algebra, believed to reflect the classical form of the hidden-variable theory, by considering the dimension of Hilbert space is at least three.

1.3 Geometric Quantum Mechanics

The fact that classical mechanics, general relativity and others are highly geometrical inspired some physicists to cast quantum mechanics in geometrical language in order to better understand the quantum-classical transition. The deeper investigation shows that the Hilbert space $\mathscr{H}$ is not the true space of states, since any two state vectors $\Psi, \Phi \in \mathscr{H}$ such that $\Psi = \alpha\Phi$ are physically equivalent ($\Psi \backsim \Phi$). Thus the proper quantum space of pure states is the set of rays through the origin in $\mathscr{H}$, i.e. $P(\mathscr{H}) := \mathscr{H} / \backsim$ which is known as the complex projective Hilbert space or the quantum phase space for both finite and infinite dimensional $\mathscr{H}$. Furthermore, the existence of Hermitian inner product in $\mathscr{H}$ endows $P(\mathscr{H})$ with the structure of a Kähler manifold (ω, g, j) where ω is non-degenerate, closed symplectic two-form, g is Riemannian metric and j is the compatible complex structure satisfying $j^2 = -1$. Thus, similar to classical mechanics, the correct quantum state space is also can be regarded as a symplectic manifold. In term of self-adjoint operator on $\mathscr{H}$, via its expectation value, one can obtain a real valued function on $\mathscr{H}$ which has well defined projection h to $P(\mathscr{H})$. Note that every phase space function induced a flow along its Hamiltonian vector field X_h. Hence on Hilbert space, the flow is certainly defined by Schrödinger equation of the quantum theory. In other words, Schrödinger evolution is exactly the Hamiltonian flow on quantum phase space $P(\mathscr{H})$. Here one can directly see that classical mechanics and quantum mechanics have many similarities. However, the fact that Riemannian metric in quantum phase space is closely related to the notion of probability comes up with several main futures that are missing in classical mechanics such as uncertainty principle and state vector reduction in quantum measurement processes.

1.4 The Classical Uncertainty Principle

Despite the success of quantum mechanics in term of application, the true nature of this theory is still far from being understood. In other words, some of its principles and concepts are clearly counter-intuitive and very difficult to explain in simple language since most of them do not have classical analogue. One of the famous examples that describe the weirdness of quantum mechanics is the uncertainty principle. This principle introduced by German physicist Werner Heisenberg in 1927 states that certain pairs of physical properties of a particle known as complementary variables, such as position and momentum, cannot be simultaneously measured with arbitrarily high precision. In other words the more precisely the position is known the more uncertain the momentum is and vice versa.

It is generally accepted that uncertainty principle is a purely quantum concept and cannot be described using classical mechanics. However this statement has shown not to be entirely true when recently one has successfully shows the uncertainty principle can naturally arise from the structure of classical mechanics (de Gosson, 2004). This is achieved using a topological tool known as symplectic capacity together with notion of quantum blob. As it is known, Heisenberg uncertainty principle is a minimum for the product of the uncertainties of position and momentum measurements. This is consistent with the property of symplectic camel which asserts that it is not possible to shrink a cross-section defined by conjugate coordinates like x and p_x to zero showing that a minimum cross-sectional area within a given volume that cannot shrink further. The minimum area is referring to the notion of quantum blob which is a symplectically invariant replacement of cubic quantum cell frequently used in statistical quantum mechanics. In other words, it is regarded as a smallest unit in phase space which is invariant under symplectic transformation. Technically, this notion is the image of a phase space ball, B with radius $\sqrt{h}$ by linear symplectic transformation. Both symplectic capacity and quantum blob are consequence of the Gromovs non-squeezing theorem means that they are invariant under Hamiltonian flow. In particular, the area of quantum blob is preserved under this flow.

1.5 Problem Statements and Objectives

As mention above, de Gosson shows that the uncertainty principle can be constructed in classical framework by utilizing a topological notion known as quantum blob. This notion which based on famous non-squeezing theorem proof that the minimum of Heisenberg uncertainty principle in this framework is invariant under Hamiltonian flows. Therefore, motivated from this work, the possibility of the uncertainty principle in geometric quantum mechanics is invariant under the Hamiltonian flows has been demonstrated.

Besides, the examination on the correspondence between quantum and classical aspects of geometric quantum mechanics has been focused and the classical properties of the observables has been studied as the literature was identified did not discussed in deeper level. This study is important in order to identify the limitation of classical notion of geometric quantum mechanics to describes the purely quantum concept such as commutator of two spin operators and Casimir operator.

The objectives of this research are:

1. to compute the evolution of geometric uncertainty principle for the case of spin $\frac{1}{2}$ and spin 1 particles.
2. to examine the correspondence between Poisson bracket and commutator for the case of spin $\frac{1}{2}$, spin 1 and spin $\frac{3}{2}$ particles.
3. to study the difference between classical analogue of Casimir operator and it quantum counterpart.

1.6 Organization

In Chapter 2, some literature that discuss about geometrical idea in quantum mechanics have been reviewed in general followed by focusing on geometric quantum mechanics pursued by Kibble.

The discussions based on the theory and methodology used in this research have been presented in Chapter 3. The discussion have been introduced the preliminaries of the mathematical ingredients used in geometric quantum mechanics such as Hamiltonian dynamics and fibre bundle theory. Hence the fundamental idea of geometric quantum mechanics consist of the notion of quantum phase space as a Kähler manifold, the role of symplectic form in dynamical quantum system and the role of Riemannian metric in quantum kinematics for both finite and infinite dimensional cases have been reviewed.

In Chapter 4 is the authors contribution where the discussion and comparison of the results of the geometric formulation of uncertainty principle for the case of spin $\frac{1}{2}$. spin 1 and spin $\frac{3}{2}$ particles. Besides that, examination on the correspondence between quantum and classical aspects of geometric quantum mechanics has been focused and the classical properties of the observables has been studied.

All the author's research findings and the proposed possible generalizations for future work have been summarized in the final chapter. In term of uncertainty principle, one may possibly extend this works by considering aspects of symplectic topology specifically by rephrasing the uncertainty principle in geometric quantum mechanics based on the famous non-squeezing theorem.

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