UNIVERSITI PUTRA MALAYSIA

NON-MARKOVIAN DYNAMICS OF DOUBLE-WELL BOSE-EINSTEIN CONDENSATE-RESERVOIR SYSTEM

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By

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DEDICATION

To all students deprived of resources in the pursuit of advance scientific knowledge

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfillment of the requirement for the degree of Doctor of Philosophy

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Chair : Associate Professor Gafurjan Ibragimov, PhD
Faculty : Science

In this thesis, we report our study on the dynamics of a symmetric double-well Bose-Einstein condensate (BEC)-reservoir system. The mentioned system is well described by total Hamiltonian composed of a sub-Hamiltonians representing the double-well BEC, multi-mode reservoir fields and the interactions of condensate atoms with the reservoir fields. The dynamical equation obtained is in the form of generalized Quantum-Heisenberg-Langevin equation (QHLE). Dissipation kernels of the QHLE determines whether the system operates within Markovian or non-Markovian basis. We found full analytical solution for the interaction free BEC-reservoirs for the Markovian operating system but only partial analytical solution is given for its non-Markovian counterpart. The interacting BEC-reservoirs system (Markovian and non-Markovian) invokes mean-field and noise-correlated models. The set of ordinary differential equations (ODE) of the latter models (mMF, MF, Mark, nonMark) were solved using Matlab ODE-45 solver, an effective tool for solving non-stiff ODEs. Physical quantities such as population imbalance, tunneling current, coherence and entanglement-entropy were computed numerically and analysed. The system operate on the Markovian and non-Markovian basis show distinctive features with respect to applied control parameters. As an overall conclusion, the finding shows the dynamics is more volatile in the Markovian operation in comparison to the non-Markovian operational basis for the mean-field approach especially on its driving from macroscopic quantum self trapping to the quantum tunneling state. For the noise-correlated approach on the other hand, the non-classical behaviour described by its entanglement-entropy is more prominent in the Markovian operational basis in comparison with its non-Markovian counterpart.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

DINAMIK TAK-MARKOVIAN BAGI SISTEM TADAHAN KONDENSASI BOSE-EINSTEIN DUA PERANGKAP

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Tesis ini melaporkan kajian kami terhadap sistem dinamik kondensasi Bose-Einstein dua-perangkap simetri-tadahan. Dinamik sistem ini ditaksirkan oleh Hamiltonian yang merangkumi sub-Hamiltonian BEC dua-perangkap, sub-Hamiltonian tadahan medan multi-modal dan sub-Hamiltonian interaksi atom BEC terhadap medan tadahan. Persamaan dinamik diperolehi adalah berbentuk persamaan Kuantum Heisenberg-Langevin (PKHL). Kernel lesapan pada PKHL menentukan sama ada sistem kita beroperasi berasaskan proses Markovian atau tak-Markovian. Penyelesaian analitik diperolehi bagi sistem BEC tak-beinteraksi-tadahan berasaskan proses Markovian tetapi bagi sistem tak-Markovian pula hanya penyelesaian separa diperolehi. Sistem BEC berinteraksi-tadahan (berasaskan proses Markovian dan tak-Markovian) pula membentuk model mean-field dan model berkorelasi kebisingan. Persaman pembeza biasa (PPB) yang terjana daripada model-model itu (mMF, MF, Mark dan nonMark) diselesaikan secara numerik mengunakan penyelesai Matlab ODE-45 yang efektif dalam menyelesaikan sistem persamaan PPB jenis tak-kaku. Ketakseimbangan populasi, penusukan aruhan arus, koheren dan simpulan-entropi adalah parameter-parameter fizikal yang dikomputasi dan dianalisis. Sistem beroperasi berasaskan proses Markovian dan tak-Markovian memberi keputusan berbeza terhadap parameter kawalan. Secara keseluruhannya, kami dapat menyimpulkan bahawa sistem beasaskan proses Markovian sangat tak menentu berbanding dengan sistem tak-Markovian, khususnya bagaimana ia memandu keadaan perangkap kuantum kendiri makroskopik ke keadaan penusukan kuantum. Bagi model korelasi kebisingan pula, sifat tak-klasikal yang ditentukan oleh simpulan-entropi lebih menonjol pada sistem yang mematuhi proses Markovian berbanding dengan proses tak-Markovian.

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TABLE OF CONTENTS

	Page
ABSTRACT	i
ABSTRAK	ii
ACKNOWLEDGEMENTS	iii
APPROVAL	iv
DECLRATION	vi
LIST OF FIGURES	x
LIST OF ABBREVIATIONS	xiv

CHAPTER

1 **INTRODUCTION** 1
- 1.1 Bose-Einstein Condensate 1
- 1.2 Basic theory of the Bose-Einstein condensate 2
- 1.3 Double-well Bose-Einstein condensate 4
- 1.4 Motivation 13
- 1.5 Problem Statement of the Present Research 13
- 1.6 Objective of the work 14
- 1.7 Structure of the thesis 14

2 **LITERATURE SURVEY** 16
- 2.1 Basis of the Open Quantum System 16
- 2.2 Master equation method for a two-level system 17
- 2.3 Two-level atom model interacting with a multi-mode field (environment) 19
- 2.4 Dissipative Bose-Einstein condensate 21
- 2.5 Experimental BEC dissipation 22
- 2.6 Dissipative two-mode BEC: Master equation approach 25

3 **THE DOUBLE-WELL BEC-RESERVOIRS SYSTEM** 30
- 3.1 The system of interest and its conditions 30
- 3.2 Dynamical equations of the system 32
- 3.3 Transforming Integro-differential equation into Ordinary differential equations 35

4 **INTERACTION FREE BEC-RESERVOIRS SYSTEM** 36
- 4.1 Markovian dynamics 36
- 4.2 Non-Markovian dynamics 42

5 **NON EQUILIBRIUM DYNAMICS AND THE FIXED-POINTS** 48
5.1 The semi-classical approach 48
5.2 Limitation of the mean-field approximation 49
5.3 Markovian *vs* non-Markovian dynamics 50
5.4 Phase-space analysis of the Markovian dynamics 50
5.5 Phase-space analysis of the non-Markovian dynamics 52

6 **NOISE CORRELATED DYNAMICS** 60
6.1 The noise correlated non-Markovian dynamics: nonMark model 60
6.2 The noise correlated Markovian dynamics: Mark model 61
6.3 Numerical solution and discussion 62
6.3.1 Comparison of the models in term of their population imbalance 63
6.3.2 Coherence in the system 64
6.3.3 Entanglement 65

7 **CONCLUSION AND FURTHER RESEARCH** 73
7.1 General overview 73
7.2 Dynamics of the interaction free BEC-reservoir system 74
7.3 Dynamics of the interacting BEC-reservoir system within semi-classical approach 74
7.4 Dynamics of the interacting BEC-reservoir system subjected to noise-dissipation correlation 76
7.5 Limitations and future direction 77

REFERENCES 78
APPENDICES 83
BIODATA OF STUDENT 95
LIST OF PUBLICATIONS 96

LIST OF FIGURES

Figure **Page**

1.1 The schematic picture of a BEC in a asymmetric double-well trap, adapted from Smerzi et al. (1997). 8

1.2 The population imbalance and its corresponding phase-space diagram at weak $\chi = 0.5$ (top panels) and strong $\chi = 2.5$ (lower panels) non-linear interaction strength, solving Eqs. (1.3.7) and (1.3.8). Coloured lines corresponds to initial conditions: blue for $(s(0) = 0.4, \theta(0) = 0)$, green for $(s(0) = -0.4, \theta(0) = 0)$ and red for $s(0) = 0, \theta(0) = 0)$. The top panel (a) indicates Rabi oscillation while panel (b) shows elliptic trajectory around point fixed-point $(0,0)$ is known as plasma oscillation. Lower panel (c) exhibit partial oscillations with time average $< s > \neq 0$ where the fixed-point at $(0,0)$ has been bifurcated into hyperbolic fixed-points, indicating MQST. 9

1.3 Constant energy lines in phase-space (s, θ) using Eq. (1.3.9). Top panel is for weak interaction $\chi = 0.5$, elliptic trajectories around fixed points $(0,0)$, $(0, \pm\pi)$ exhibiting QTS. Lower panel is the contour plot of Eq. (1.3.9) for stronger non-linear interaction $\chi = 2.5$, elliptic fixed point at $(0,0)$ and hyperbolic fixed points at $(0, \pm\pi)$ corresponding to MQST. 10

1.4 Figure reported by Albiez et al. (2005) showing dynamics on a weakly connected Bose-Einstein condensates confined by a symmetric double-well potential as observed from absorption images. Josephson oscillation is shown in the left panel (a). MQST is exhibited on the right panel (b). 11

1.5 Figure reported by Zibold et al. (2010). Experimentally observed phase portraits data showing the dynamics by increasing the non-linear interaction strength parameter Λ. Comparison is made with theoretical result produced by Eqs. (1.3.7)-(1.3.8) [solid line] for the corresponding parameters. The theoretical trajectories are replicated on the Bloch spheres model for clarity. Results on phase diagram are (a) QTS in the Rabi regime observed at $\Lambda = 0.78$, (b) Josephson regime seen at $\Lambda = 1.55$ while stable macroscopic quantum self-trapping obtained for average π phase. The result are verified by the experimental data indicated by green squares and green crosses. 12

2.1 Observation on the scanning electron microscopy of the BEC profile along the weak confining axis of the optical trap reported by Barontini et al. (2013). The integrated column density profile is along the direction of propagation of the electron beam (EB). At origin the density is depleted hence denoting parting of BEC at the center. The numerical solution of Eq. (2.4.5) is represented by red solid line. 24

2.2 Coherence dynamics of the mean-field model with $\Omega = 1.0s^{-1}$, $U = 0.001s^{-1}$, $\varepsilon = 0.2$ and $f = 0.5$ for different initial states $(s_x(0), s_y(0), s_z(0))$: $(0,0,1.0)$ (red line), $(1.0,0,0)$ (green line) and $(0.5,0,0.9)$ (blue line) respectively for (a) $\gamma_p = 0.0$, $T_1^{-1} = 0.0$ (b) $\gamma_p = 1.0$, $T_1^{-1} = 0.0$ (c) $\gamma_p = 0.0$, $T_1^{-1} = 2.0$ and (d) $\gamma_p = 2.0$, $T_1^{-1} = 3.0$. The figure shows inclusion of both noise and dissipation enhances coherence. Dissipation alone does not temper coherence as much in comparison to the induced noise for all the cases considered here. 28

2.3 Purity dynamics of the mean-field model for $\Omega = 1.0s^{-1}$, $U = 0.001s^{-1}$, $\varepsilon = 0.2$ and $f = 0.5$ with different initial states $(s_x(0), s_y(0), s_z(0))$: $(0,0,1.0)$ (red line) and $(1.0,0,0)$ (green line) and $(0.5,0,0.9)$ (blue line) respectively for (a) $\gamma_p = 0.0$, $T_1^{-1} = 0.0$ (b) $\gamma_p = 2.0$, $T_1^{-1} = 0.0$ (c) $\gamma_p = 1.0$, $T_1^{-1} = 2.0$ and (d) $\gamma_p = 2.0$, $T_1^{-1} = 3.0$. It is clear from these figures that dissipation does not temper purity as much compared to noise, a similar behaviour found for the coherence property in Figure (2.2). 29

3.1 The schematic picture of a double-well BEC-reservoir system 30

4.1 Population evolution of trap A as a function of dimensionless time (Ωt) at ω/Ω= 1.0, in variation with $(\Gamma_1/\Omega, \Gamma_2/\Omega)$ and temperatures k_BT/Ω. Initially atom distribution between the traps (A,B) is (70,30). 46

4.2 Caption is the same as in Figure 4.1 for population of trap B. 47

5.1 Population imbalance evolution as a function of dimensionless time (Ωt), for the MF (non-Markovian mean-field) model computed at weak onsite interaction $U/\Omega = 0.01$. Panel (a) is dissipative free system, exhibiting Rabi oscillation while panels (b)-(d) show dissipative dynamics. Weak oscillation is seen for the $(50,50)$ initial atom populations case when system subjected to bias dissipation. Strong dissipation reduces oscillation amplitudes to zero at much faster rate, as seen on panel (d). Fixed parameters here are $\gamma/\Omega = 10.$ and $\omega/\Omega = 5.0$. Coloured lines corresponds to system with initial atom distributions $(n_A(0), n_B(0))$: $(64,36)$ blue line, $(50,50)$ green line, $(36,64)$ red line. 55

5.2 Similar to Figure 5.1 but for stronger non-linear interaction $U/\Omega = 0.1$. Panel (a) shows Macroscopic Self Trapping state (MQST) of the dissipation free system. Panels(b)-(d) are the evolution of dissipative system. Dissipation clearly destroys the MQST phase. Again weak oscillation is observed for the $(50,50)$ initial atom distribution case when bias dissipation applied on the system. The driving from MQST to QTS phase is much regular and stronger for the stronger non-linear interaction as seen on panel (d). 56

5.3 Population imbalance corresponding to its tunnelling current as a function of (Ωt) for the MF model *vs* mMF (Markovian mean-field) model. Top panel represents weak non-linear strength $U/\Omega = 0.01$ whereas lower panel is for the stronger $U/\Omega = 0.1$ case. Dissipation bias $(Q_1 = 0.05, Q_2 = 0.1)$ is applied while other fixed parameters in the calculation are correlation memory time $\gamma/\Omega = 10$, trap frequency $\omega/\Omega = 5.0$. Initial atom number distributions at trap (A,B) is: $(n_A(0) = 64, n_B(0) = 36)$. Coloured line: red corresponds to the mMF and blue line corresponds to the MF model respectively. 57

5.4 The same comaprison as in Figure 5.3 but for equally initial distribution of atoms at trap (A,B): $(n_A(0) = 50, n_B(0) = 50)$. 58

5.5 Evolution of phase diagrams (s,θ) in variation with inter-particle interaction strengths (increasing χ from top to bottom order) subjected to three dissipation strengths $\zeta = 0, 0.2, 0.5$. Initial conditions are in the ranges $-1 \leq s(0) \leq 1$ and $-\pi \leq \theta(0) \leq \pi$ while the trajectories runs for $\Omega t = 3.$. Phases are evolving from Quantum Tunelling state (QTS) (all elliptic fixed points) at $\chi = 0.2$ to Macroscopic Quantum Self-trapping (combination of hyperbolic and elliptic fixed points) at $\chi = 2.5$. Phase difference θ, is plotted in unit radian. 59

6.1 Population imbalance for the non-Markovian and Markovian dynamics as a function of dimensional time (Ωt). The left panels (a) and (c) are for smaller correlation memory time $(\gamma/\Omega = 2)$ whereas larger $(\gamma/\Omega = 25)$ value considered depicted on the right panels (b) and (d). Similarly top and lower panels shows distinguished dynamical features for weak $(U/\Omega = 0.5)$ and stronger $(U/\Omega = 2.5)$ interparticle interaction strength respectively. The trap frequencies, reservoir temperature and dissipation strengths used in the computation are $\omega/\Omega = 5$, $k_BT_1/\Omega = k_BT_2/\Omega = 2$ and $(Q_1 = 0.1, Q_2 = 0.2)$, with initial atoms distribution $n_A(0) = 64$ and $n_B(0) = 36$. Blue line corresponds to nonMark model while green is for the Mark model, also shown on the box legend shown at panel (d). 67

6.2 Population imbalance evolution in unit (Ωt). Comparison is made between MF and nonMark models which display non-Markovian dynamics. In the former model, noise has been suppressed while the nonMark model obeys FDT. Panels (a)-(d) depict the dynamics of system as the inter-particle interaction strength U/Ω is tuned from weak to stronger value. Small $(\gamma/\Omega = 2)$ correlation memory time is considered for system to produce effective non-Mark dynamics. All other fixed parameters are the same as in Figure 6.1. On the figure, blue line corresponds to the nonMark model while red line for MF model, also shown on the legend box shown at panel (b). 68

6.3 Temporal evolution of coherence in unit (Ωt). Again, comparison between small and large correlation memory time (γ/Ω) is shown. The results verify that both nonMark and Mark model yield similar dynamics at large correlation memory time. However, the system shows distinct dynamical features when controlled by non-linear interaction strength (U/Ω). Top panels depicts the coherence dynamics for weak $(U/\Omega = 0.5)$ non-linear interaction strength whereas a stronger one $(U/\Omega = 2.5)$ is shown on the lower panels. Other fixed parameters are the same used in Figure 6.1. Legends shows coloured line correspondence with models. 69

6.4 von Neumann Entropy Eq. (6.3.4) as a function of (Ωt), computed in variation with correlation memory time γ/Ω. Other fixed parameters are the same used in Figure 6.1. Blue line corresponds to nonMark model while green is noise correlated Mark model. Distinctive entropy evolution is observed for the compared models (nonMark and Mark) at small correlation memory time. Same entropy is obtained for the models at large correlation memory time. Top panels depicts entropy for weak $(U/\Omega = 0.5)$ non-linear interaction strength whereas a stronger one $(U/\Omega = 2.5)$ is shown on the lower panels. Higher entropy value yielded by the models at stronger non-linear interaction. 70

6.5 Eigenvalues Eq. (6.3.2)-(6.3.3) of the reduced density matrix Eq. (6.3.1) evolve with dimension-less time (Ωt), computed for small (left panels) and large (right panels) correlation memory time γ/Ω. Top panels depicts eigenvalues for weak $(U/\Omega = 0.5)$ non-linear interaction strength whereas a stronger one $(U/\Omega = 2.5)$ is shown on the lower panels. Fixed parameters are the same with Fig 6.1. Blue solid and dotted lines correspond to eigenvalues pair of the non-Markovian system, while red solid and dotted lines corresponds to eigenvalues pair of the Markovian case. 71

6.6 Temporal evolution of linear entropy with dimension-less time Ωt in variation with small (left panel) and large (right panel) correlation memory time γ/Ω. Other fixed parameters are the same used in Figure 6.1. Blue line corresponds to nonMark model while green is noise correlated Mark model, also shown on the legend box. Top panels depicts linear entropy for weak ($U/\Omega = 0.5$) non-linear interaction strength whereas a stronger one ($U/\Omega = 2.5$) is shown on the lower panels. Linear entropy shows similar characteristics as observed in von Neumann entropy. 72

7.1 The flowchart of our system and their models 73

B.1 Solution of non-stiff Van der Pol equation with $\mu = 1$ obtained using three different ODE solvers. 89

B.2 Solution of stiff van der Pol equation with $\mu = 1000$ obtained using four different ODE solvers. 90

LIST OF ABBREVIATIONS

BEC	Bose-Einstein Condensation
BJJ	Bose Josephson junction
FDT	Fluctuation Dissipation Theorem
GP	Gross-Pitaevskii
GPE	Gross-Pitaevskii Equation
GQLE	Generalized Quantum Langevin Equation
HFB	Hartree-Fock Bogoliubov approximation
MF	Mean Field model
MFA	Mean Field Approximation
MQST	Macroscopic self-trapping
OU	Ornstein-Uhlenbeck
ODE	Ordinary differential equation
PV	Principal value
pdf	Probability density function
QTS	Quantum tunneling state
SI	Smoothing index
SPDM	Single particle density matrix
SR	Stiffness ratio
t	Time
h	Planck's constant $\approx 6.6262 \times 10^{-34}$Js

k_B	Boltzmann constant $\approx 1.3807 \times 10^{-23}$ J/K
m	mass of atom/boson
δ_{ij}	Krönecker delta
$\delta(r - r')$	Dirac delta function
T_a	Temperature of reservoir R_1
T_b	Temperature of reservoir R_2
U	Inter-particle interaction strength
Ω	Josephson tunneling strength
ω	Frequency of trap A and trap B
$n_A(t)$	Number of atoms at trap A at time t
$n_B(t)$	Number of atoms at trap B at time t
$n(t)$	Total number of atoms at time t
$s(t)$	Population imbalance at time t
$\theta(t)$	Phase difference between the two BECs

CHAPTER 1

INTRODUCTION

1.1 Bose-Einstein Condensate

Bose-Einstein condensation was first predicted by Einstein (1924), generalizing the concept of earlier work of Bose (1924) on the quantum statistics of photons to indistinguishable atoms with integral spin (coined as "boson"). In nature, there exist only two types atoms, classified as bosons or fermions based on their intrinsic spin properties. Bosons have integral spin described by symmetric wave function. In contrast, fermions are half-integral spin particles having anti-symmetric wave function property. Pauli's exclusion principles prohibit two fermions to occupy the same quantum state which is the crux of this anti-symmetric property. On the other hand, the symmetric nature of bosons permits them to macroscopically occupy a given state.

The most illustrative boson characteristic is the process of Bose-Einstein Condensation (BEC) where at extremely low temperatures the bosons plunge into occupying in their lowest single-particle quantum state (ground state). It is indeed a quantum phase transition where bosons occupying various higher eigenstates macroscopically crunched into its lowest energy state as temperature swept across a critical temperature. At this stage, the de Broglie waves of neighbouring atoms coalesce to form a giant matter-wave. For the three-dimension (3D) geometry, the condition for attaining BEC is to achieve the criteria $\tilde{\rho}\lambda_{DB}^3 \geq 2.612$ where $\tilde{\rho}$ is dimensionless phase space density and $\lambda_{DB} = h/(2\pi m k_B T)^{1/2}$ is its de Broglie wavelength. It also means, the BEC phase occur when the size of de Broglie wavelength matches the inter-particle distance $d = \rho^{-1/3}$ of atoms.

Pioneering realization of BEC of alkali atoms in the mid 90's (Anderson et al. (1995); Davis et al. (1995); Bradley et al. (1995); Weber et al. (2003); Greiner et al. (2003)), that led to the 2001 Nobel prize in physics boosted further research interest in the (theory and experimental) Bose-Einstein condensate field. Unexpected theoretical features were predicted and experimentally observed thence after. New avenues from interdisciplinary fields (atomic physics, quantum optics, statistical mechanics, and condensed-matter physics), emerges making it a more exciting area of research. Due to the broad scope of the subject, it is quite impossible to give full coverage of the basic theory of Bose-Einstein condensation and its experimental realizations together with its many novel applications. A large series of reviews and books have already emerged elaborating in clear details this novel state of matter. Readers are referred to the work of (Baym and Pethick (1996); Leggett (2001); Parkins and Walls (1998); Griffin et al. (1996)) for basic ideas of the BEC subject matter.

1.2 Basic theory of the Bose-Einstein condensate

Generally speaking, BEC is a phenomena where atoms populate the lowest quantum state with zero momentum at large scale during a phase transition. Momentum distribution shows a peak at this condensation process. Penrose and Onsager (1956) suggested a criterion for this condensation process in which the BEC can be recognised with the emergence of long-range off-diagonal order in the one-body density matrix. In any statistical mechanics literature, one could find at thermal equilibrium, a uniform gas consist of N atoms obeys the well-known Bose-Einstein distribution

$$f(\varepsilon_{\mathbf{k}},T) = \frac{1}{\exp((\varepsilon_{\mathbf{k}}-\mu)/k_BT)-1}, \tag{1.2.1}$$

where k_B is the Boltzmann constant, T is the temperature and μ corresponds to the chemical potential of the system. The chemical potential is the energy required by a system to exchange one particle with its environment. The total number of particles in the un-condensed (excited) states is determined by the sum $N_{ex} = \sum_{\mathbf{k}} f(\varepsilon_{\mathbf{k}},T)$ which specifies the value of the chemical potential μ. The ground state ($\varepsilon_o = 0$) population is calculated to be $N_0 = [\exp(-\mu/k_BT)-1]^{-1}$ which becomes macroscopic even for a small negative μ. Therefore, one can separate out the population of the ground state and exited state by the relation $N = N_o + N_{ex}$.

Bose-Einstein condensate is a phenomena where a phase transition occur at a critical temperature T_c, reducing temperature below it will cause all particles in the excited states crunched into its ground (lowest energy) state. The critical temperature T_c is the highest temperature where a Bose condensate can still exist (or a vice versa). At this temperature (T_c) the chemical potential is zero. Hence, one can use this condition $N = N_{ex}(T = T_c, \mu = 0)$ to find the critical temperature and show the relation:

$$N\left[1-(\frac{T}{T_c})^{\alpha}\right] = N_0, \tag{1.2.2}$$

where T_c is the BEC transition temperature while α depends on the confinement, for instance, α=3/2 for box potential or α=3 for harmonic trap potential. The value of critical temperature T_c is shifted for the interacting BEC case. The details of the calculation can be found in textbooks written by the authors such as Griffin et al. (1996) and Pethick and Smith (2008).

Having discussed briefly on the ideal Bose gas theory, it is time now to introduced the real system of Bose-Einstein condensate confined within generic trapping potential $V(\mathbf{r})$ where the bosons interacts repulsively or attractively. The trapping potential can come in various form like three-dimensional (can also be lower dimensional) symmetrical or asymmetrical Harmonic potentials, cigar-shape, pancake-shape which is obtained by squeezing of particular coordinate and so on. An interacting BEC in a trapping potential $V(\mathbf{r})$ can be represented in a grand-canonical Hamiltonian within second quantization language:

$$\hat{H} = \int d\mathbf{r}\hat{\Psi}^{\dagger}(\mathbf{r})\left[-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r}) - \mu\right]\hat{\Psi}(\mathbf{r}) + \frac{1}{2}g\int d\mathbf{r}\hat{\Psi}^{\dagger}(\mathbf{r})\hat{\Psi}^{\dagger}(\mathbf{r})\hat{\Psi}(\mathbf{r})\Psi(\mathbf{r}) \tag{1.2.3}$$

where μ is the chemical potential and g is the inter-particle interaction (or coupling) strength. For a three-dimensional system, inter-particle coupling strength is given by $g = \frac{4\pi\hbar^2 a}{m}$. In this expression, a represent the scattering length (for which $a > 0$ indicates repulsive interaction while $a < 0$ indicates attractive interactions between boson), m represents the atomic (boson) mass, whereas $\hbar$ is the Planck's constant h divide by 2π respectively. For lower dimensional cases, the coupling constant changes accordingly, for further clarification see in the textbook by Pethick and Smith (2008). The annihilation $\Psi(\mathbf{r}')$ and creation $\Psi^{\dagger}(\mathbf{r})$ field operators obeys the Bose-Einstein commutation relation

$$[\hat{\Psi}(\mathbf{r}),\hat{\Psi}^{\dagger}(\mathbf{r}')] = \delta(\mathbf{r}-\mathbf{r}') \quad [\hat{\Psi}(\mathbf{r}),\hat{\Psi}(\mathbf{r}')] = [\hat{\Psi}^{\dagger}(\mathbf{r}),\hat{\Psi}^{\dagger}(\mathbf{r}')] = 0\,. \tag{1.2.4}$$

The particle-number operator is defined by $\hat{N} = \int d\mathbf{r}\,\hat{\Psi}^{\dagger}(\mathbf{r})\hat{\Psi}(\mathbf{r})$, hence it is quite natural to interpret $\hat{\Psi}^{\dagger}\hat{\Psi}$ as the number density operator. Both $\hat{H}$ and $\hat{N}$ are Hermitian and can be diagonalized simultaneously since they commute with each other $[\hat{N},\hat{H}] = 0$. Hence any orthogonal basis state $|\Psi_{NE}\rangle$ are simultaneously eigenstate of $\hat{N}$ and $\hat{H}$ and having properties $\hat{N}|\Psi_{NE}\rangle = N|\Psi_{NE}\rangle$ and $\hat{H}|\Psi_{NE}\rangle = E|\Psi_{NE}\rangle$ where $\langle\Psi_{NE}|\Psi_{NE}\rangle = 1$. Here N and E are the total particle number and energy of the system respectively. Also it can be shown that $[\hat{\Psi}(\mathbf{r}),\hat{N}] = \hat{\Psi}(\mathbf{r})$ and $[\hat{\Psi}^{\dagger}(\mathbf{r}),\hat{N}] = -\hat{\Psi}(\mathbf{r})$, from which one can deduce that $\hat{N}\hat{\Psi}(\mathbf{r})|\Psi_{NE}\rangle = (N-1)\hat{\Psi}(\mathbf{r})|\Psi_{NE}\rangle$ and $\hat{N}\hat{\Psi}^{\dagger}(\mathbf{r})|\Psi_{NE}\rangle = (N+1)\hat{\Psi}^{\dagger}(\mathbf{r})|\Psi_{NE}\rangle$. Obviously $\hat{\Psi}(\mathbf{r})|\Psi_{NE}\rangle$ is also an eigenstate of $\hat{N}$ but with eigenvalue $N-1$, similarly $\hat{\Psi}^{\dagger}(\mathbf{r})|\Psi_{NE}\rangle$ is and eigenstate of $\hat{N}$ with eigenvalue $N+1$. From here we can deduce that $\Psi(\mathbf{r})$ acting on $|\Psi_{NE}\rangle$ annihilate one particle from the field while $\Psi^{\dagger}(\mathbf{r})$ acting on $|\Psi_{NE}\rangle$ creates one particle from the field. Detailed background idea on Bose-Einstein condensation calculations can be referred in the popular statistical mechanic text-books such as Pathria (2001) and Reichl (2017). With that being clarified, we can now move forward to find dynamical equation of the interacting BEC defined by the Hamiltonian (1.2.3). Using the Heisenberg equation of motion used in Pethick and Smith (2008):

$$i\hbar\frac{\partial\hat{\Psi}(\mathbf{r},t)}{\partial t} = [\hat{\Psi}(\mathbf{r},t),\hat{H}]\,, \tag{1.2.5}$$

employing the commutation relation above and performing some algebra using the Hamiltonian Eq. (1.2.3), yields the following non-linear Schrödinger equation

$$i\hbar\frac{\partial\hat{\Psi}(\mathbf{r},t)}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r})\right)\hat{\Psi}(\mathbf{r},t) + g\hat{\Psi}^{\dagger}(\mathbf{r},t)\hat{\Psi}(\mathbf{r},t)\hat{\Psi}(\mathbf{r},t)\,. \tag{1.2.6}$$

The particle field operator can be written as $\hat{\Psi} = \Psi + \tilde{\psi}$ which is the sum of classical wave-function Ψ (please not here we have dropped the hat symbol) describing the ground state BEC and a fluctuation field operator $\tilde{\psi}$ representing excitations. The wave function Ψ is also known as the order parameter (a complex function) and it is defined by the statistical average of particle field operator $<\hat{\Psi}>=\Psi$ and $<\hat{\Psi}^\dagger>=\Psi^*$. This method of splitting the field operator into a complex function plus small perturbation of field operator is known as the Bogoliubov approach (Abrikosov et al. (2012)). Taking the expectation value of Eq. (1.2.6), one obtains the celebrated Gross-Pitaevskii (GP) equation, as used in Parkins and Walls (1998) and also by Leggett (2001):

$$i\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r}) + g|\Psi(\mathbf{r},t)|^2\right)\Psi(\mathbf{r},t) \qquad (1.2.7)$$

Here the third-order correlation function has been decorrelated into products of single-order moments $<\hat{\Psi}^\dagger\hat{\Psi}\hat{\Psi}>\approx<\hat{\Psi}^\dagger><\hat{\Psi}><\hat{\Psi}>=|\Psi|^2\Psi$, which is known as the mean-field approximation (see also the detail section (5.1)). The mean-field approximation is valid for a macroscopic system with large of number of atoms and extremely weak quantum fluctuation (see also the discussion in section 5.1). The wave function is normalized to the total atom number N as $\int d\mathbf{r}|\Psi(\mathbf{r},t)|^2 = N$. By employing the ansatz $\Psi(\mathbf{r},t) = \Psi(\mathbf{r})\exp[-i\mu t/\hbar]$, stationary solution for the BEC can be obtained. In the given ansatz, chemical potential $\mu = \frac{\partial E}{\partial N}$ of the system is introduced. The chemical potential fixes the normalization condition with the atom number. Employing the ansatz on Eq. (1.2.7), the GPE reduces to:

$$(-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r}) + g|\Psi(\mathbf{r})|^2)\Psi(\mathbf{r}) = \mu\Psi(\mathbf{r}). \qquad (1.2.8)$$

1.3 Double-well Bose-Einstein condensate

Double-well Bose-Einstein Condensate (BEC) exhibits fascinating quantum phenomena like tunneling, decoherence and entanglement. Javanainen (1986), have shown that the effect due to the formation of Josephson-junction in a weakly coupled superconductors is mimicked by the two-mode BEC system. In their system, BEC atoms are initially assigned to occupy a particular trap or distributed appropriately among the two traps. Coherence between the two separated traps is preserved by the tunneling process. BEC population was found to oscillate even if there is no disparity between the number of atoms in each well.

A two-mode BEC (symmetric double-well) system were studied using the SU(2) symmetric group by Milburn et al. (1997). They have identified a novel phase called the macroscopic quantum self-trapping (MQST) where atoms tend to localize within their respective wells as the on-site inter-particle interaction is enhanced. The tunneling of atoms between traps were suppressed in this process. Modulated sequence of collapse and revival of the population was shown in their work. A similar model were further researched in the weak and stronger interaction regime by Smerzi et al. (1997) and Raghavan et al. (1999). They found similarity between the latter quantum system with classical pendulum dynamics. Here we detail their method and explain in brief their result. In the literature review and rest of the chapter we focus on the dissipative BEC which is the main theme of this thesis.

Asymmetric double-well trap as in Figure 1.1 can be obtained by placing a laser barrier produces a double-well curvatures with $N_{1,2}$ and $\varepsilon^o_{1,2}$ the number of particles and zero-point energies at each trap respectively, see for example given in Smerzi et al. (1997). Using the latter trapping potential we can study the dynamics of Bose-condensate confined within a double-well setting. Considering the GP Eq. (1.2.7) and taking total wave function in the following ansatz:

$$\Psi(\mathbf{r},t) = \psi_1(t)\Phi_1(\mathbf{r}) + \psi_2(t)\Phi_2(\mathbf{r}) \tag{1.3.1}$$

where $\Phi_j(\mathbf{r})$ are the orthogonal ground state spatial wave-function of each trap, the double-well condensate system is well described by the following set of coupled equations:

$$i\hbar\frac{\partial \psi_1}{\partial t} = (\varepsilon^o_1 + g_1 N_1)\psi_1 - \Omega\psi_1 \tag{1.3.2}$$

$$i\hbar\frac{\partial \psi_2}{\partial t} = (\varepsilon^o_1 + g_2 N_1)\psi_2 - \Omega\psi_2\,. \tag{1.3.3}$$

The time-dependent mode (site) wave-functions are defined by $\psi_j = \sqrt{n_j}\exp[i\theta_j]$ for $j = 1,2$ where n_j and θ_j are the number of particles and phases of the trap 1 and trap 2. The coupling between the spatial wave-function of the individual traps is defined by parameter Ω, that will be defined later. The total number of atom is denoted by $N = n_1 + n_2$ and for the validity of the model small number of atoms (say $\approx 10^3$) so tht very small phases fluctuation is assume to have well defined phase θ_j, see for instance see pioneering references such as Smerzi et al. (1997) and Milburn et al. (1997). The value of ε^o_j, U_j and Ω can be evaluated by:

$$\varepsilon^o_j = \int d\mathbf{r}[\frac{\hbar^2}{2m}|\nabla\Phi_j|^2 + |\Phi_j|^2 V_{ext}(\mathbf{r})], \tag{1.3.4}$$

$$U_j = g\int d\mathbf{r}|\Phi_j|^4, \tag{1.3.5}$$

$$\Omega = -\int d\mathbf{r}[\frac{\hbar^2}{2m}(\nabla\Phi_1)(\nabla\Phi_2) + \Phi_1\Phi_2 V_{ext}(\mathbf{r})]. \tag{1.3.6}$$

For a symmetric double-well trap case, with $\Delta E = \varepsilon^o_1 - \varepsilon^o_2 = 0$, and $U_1 = U_2 = U$, re-writing the phase difference by $\theta = \theta_1 - \theta_2$ and population imbalance in term of $s = n_1 - n_2/N$, Eqs. (1.3.2) and (1.3.3) can be transformed into the following phase-space equations:

$$\frac{ds}{dt} = \sqrt{1-s^2}\,\sin\theta\,, \tag{1.3.7}$$

$$\frac{d\theta}{dt} = -\frac{s\cos\theta}{\sqrt{1-s^2}} + \chi\, s\,. \tag{1.3.8}$$

In the above set of equations, time has been rescaled in $2\Omega\, t$ unit while $\chi = gN/2\Omega$ is the rescaled non-linear interaction strengths. The population imbalance s and phase difference between the modes θ are canonical conjugates with $\dot{s} = -\partial H/\partial\theta$ and $\dot{\theta} = \partial H/\partial s$. The system subscribed to the mean-field approximation can be represented by the classical Hamiltonian reported in (Milburn et al. (1997); Smerzi et al. (1997); Raghavan et al. (1999)):

$$\hat{H} = \Omega\sqrt{1-s^2}\cos\theta + (1/2)\chi\, s^2\,, \tag{1.3.9}$$

Figure 1.2 illustrates population imbalance and it's corresponding phase space diagram (s,θ) for two values of χ representing opposite regime of non-linear linear interaction strength. Many plots we made by varying χ, but we just select these two plots to show the distinct feature of the non-dissipative two-mode BEC phases. Please note that, χ is a ratio between the inter-particle interaction with the tunnelling coupling strength. At weak χ, system's population imbalance shows Josephson's oscillation indicating a Quantum Tunnelling state (QTS). At strong χ, the system exhibits macroscopic quantum self-trapping (MQST), as reported in (Smerzi et al. (1997); Raghavan et al. (1999); Milburn et al. (1997)). The temporal mean population imbalance is non-zero at MQST. These fixed-point are obtained by setting $\dot{s} = 0$ and $\dot{\theta} = 0$. The eigenvalues of the Jacobian matrix (Perko (2013)), gives the following basic characteristic to the fixed-point stability:

1. pair of imaginary eigenvalues correspond to an elliptic fixed point,
2. two real eigenvalues denote a hyperbolic fixed point,
3. two real eigenvalues with opposite sign indicates a saddle fixed point,

The Jabocian matrix using Eqs. (1.3.7)-(1.3.8) for the non-dissipative BEC is the following:

$$J = \begin{pmatrix} \frac{\partial\dot{s}}{\partial s} & \frac{\partial\dot{s}}{\partial\theta} \\ \frac{\partial\dot{\theta}}{\partial s} & \frac{\partial\dot{\theta}}{\partial\theta} \end{pmatrix} = \begin{pmatrix} s\sin\theta/\sqrt{1-s^2} & -\sqrt{1-s^2}\cos\theta \\ \Lambda + \cos\theta/[1-s^2]^{3/2} & -s\sin\theta/\sqrt{1-s^2} \end{pmatrix} \tag{1.3.10}$$

Eigenvalue χ, of the above Jacobian Matrix is calculated for the the trivial fixed point $(s_o = 0, \theta_o = 0)$ at weak $\chi = 0.5$ and stronger $\chi = 2.5$. At weak $\chi = 0.5$ we obtained a pair of complex eigenvalues $\chi = \pm\sqrt{3}\,i$ which indicates elliptic phase trajectory around this fixed point. At stronger $\chi = 2.5$, we obtained a pair of opposite real eigenvalues $\chi = \pm 1$ which indicates hyperbolic trajectory around the same fixed point indicating MQST, see for instance in references (Smerzi et al. (1997); Raghavan et al. (1999); Milburn et al. (1997)). Pioneering experimental work to study the dynamics of weakly coupled Bose-Einstein condensates confined within a symmetric double-well potential were performed and reported by Albiez et al. (2005). They observed time evolution for the formation of the Josephson oscillations which mimics tunneling oscillation in superconductor and also the novel macroscopic self-trapping by sweeping across a measured critical initial population imbalance s_C. The measurement qualitatively confirmed theoretical calculation one may yield controlling parameters such as tunneling coupling and non-linear interaction strength or initial population imbalance applying on Eqs. (1.3.7)-(1.3.8). Later Zibold et al. (2010), reported experimental measurement on the Josephson tunneling effect using rubidium spinor Bose-Einstein condensate. Their experimental measurements correctly identify the locations of different types of fixed points (plasma, π oscillation and hyperbolic), matching the theoretically predicted result of Eqs. (1.3.7)-(1.3.8). Their experimental measurements qualitatively confirms the characterization of the theoretically predicted two-mode BEC phase and also revealing a classical bifurcation.

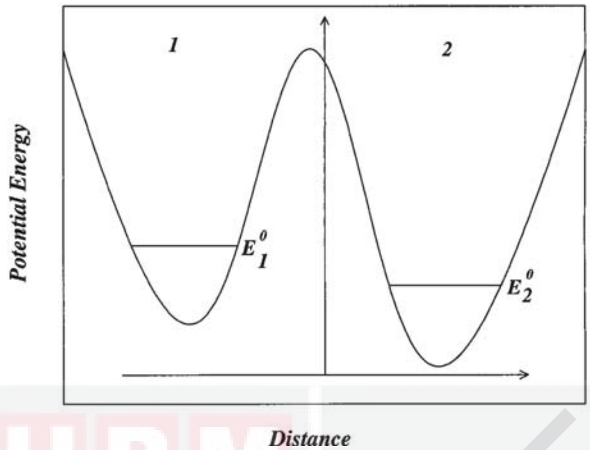

Figure 1.1: The schematic picture of a BEC in a asymmetric double-well trap, adapted from Smerzi et al. (1997)**.**

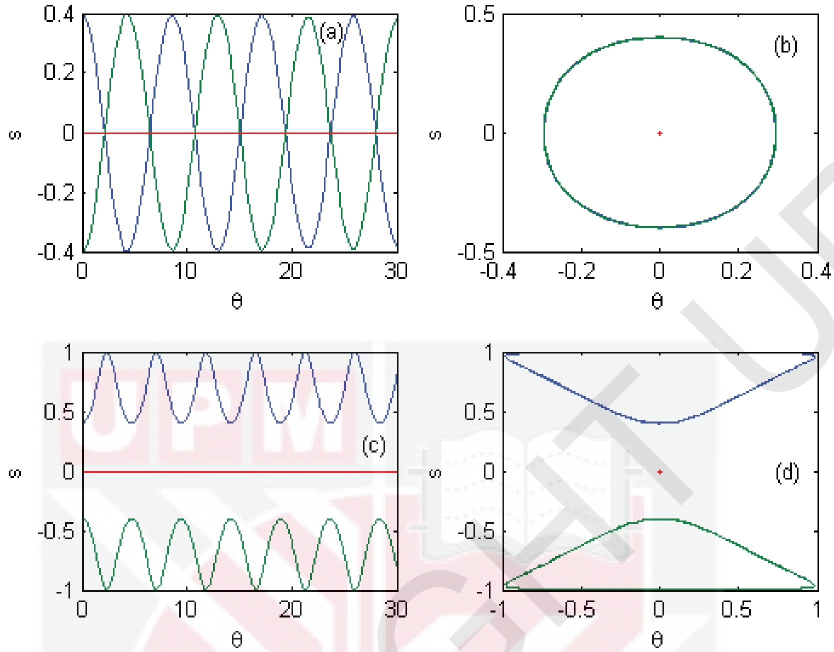

Figure 1.2: The population imbalance and its corresponding phase-space diagram at weak $\chi = 0.5$ (top panels) and strong $\chi = 2.5$ (lower panels) nonlinear interaction strength, solving Eqs. (1.3.7) and (1.3.8). Coloured lines corresponds to initial conditions: blue for ($s(0) = 0.4, \theta(0) = 0$), green for ($s(0) = -0.4, \theta(0) = 0$) and red for $s(0) = 0, \theta(0) = 0$). The top panel (a) indicates Rabi oscillation while panel (b) shows elliptic trajectory around point fixed-point $(0,0)$ is known as plasma oscillation. Lower panel (c) exhibit partial oscillations with time average $< s > \neq 0$ where the fixed-point at $(0,0)$ has been bifurcated into hyperbolic fixed-points, indicating MQST.

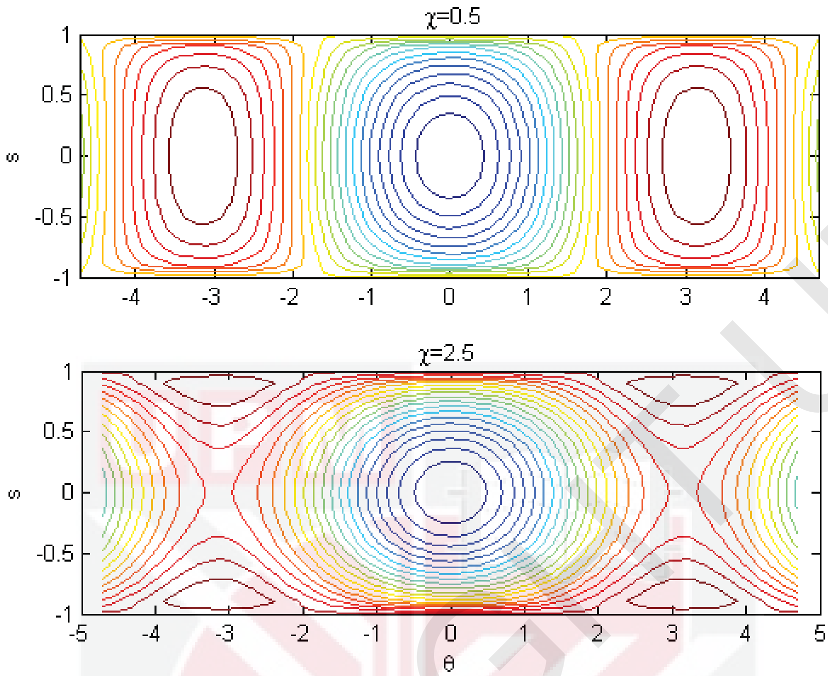

Figure 1.3: Constant energy lines in phase-space (s,θ) using Eq. (1.3.9). Top panel is for weak interaction $\chi = 0.5$, elliptic trajectories around fixed points $(0,0)$, $(0,\pm\pi)$ exhibiting QTS. Lower panel is the contour plot of Eq. (1.3.9) for stronger non-linear interaction $\chi = 2.5$, elliptic fixed point at $(0,0)$ and hyperbolic fixed points at $(0,\pm\pi)$ corresponding to MQST.

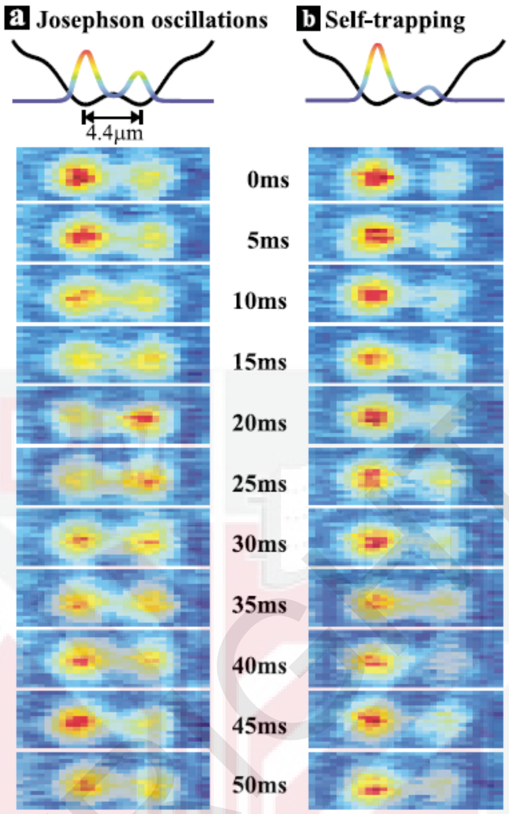

Figure 1.4: Figure reported by Albiez et al. (2005) **showing dynamics on a weakly connected Bose-Einstein condensates confined by a symmetric double-well potential as observed from absorption images. Josephson oscillation is shown in the left panel (a). MQST is exhibited on the right panel (b).**

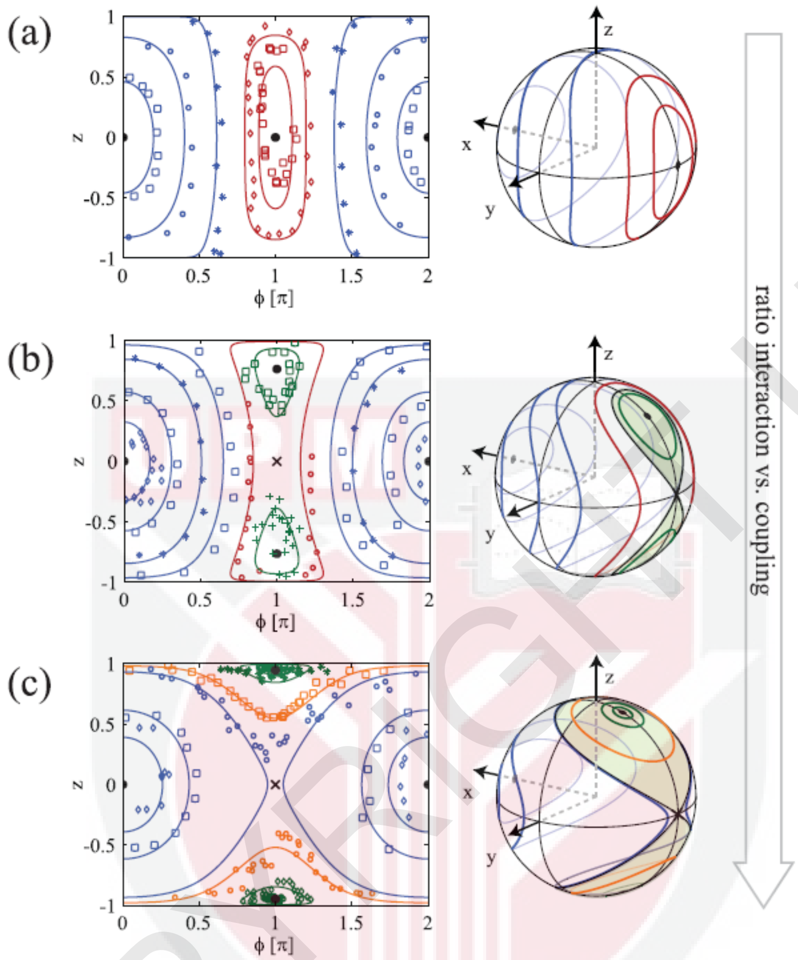

Figure 1.5: Figure reported by Zibold et al. (2010). **Experimentally observed phase portraits data showing the dynamics by increasing the non-linear interaction strength parameter Λ. Comparison is made with theoretical result produced by Eqs. (1.3.7)-(1.3.8) [solid line] for the corresponding parameters. The theoretical trajectories are replicated on the Bloch spheres model for clarity. Results on phase diagram are (a) QTS in the Rabi regime observed at $\Lambda = 0.78$, (b) Josephson regime seen at $\Lambda = 1.55$ while stable macroscopic quantum self-trapping obtained for average π phase. The result are verified by the experimental data indicated by green squares and green crosses.**

1.4 Motivation

There is a resurgent trend in understanding quantum dissipative systems with the development of nano-technology and atom manipulation engineering. However, scientists and engineers are still struggling with matters related to the quantum measurement paradox when the system under consideration is with quantum nature while observable has to be classical. Worst still when the system involves quantum matter like Bose-Einstein condensate that exists for ultra-low temperature. In experimental set-up, the double-well condensate will interact with the reservoir of thermal atoms making the study more complex. Dissipation, atom losses, and decoherence are unavoidable phenomenon that needs to be addressed in order to fully understand the dynamics of such a complex system. Hence there exist still vast room to explore the open or closed quantum system like the double-well Bose-Einstein-reservoir system.

Most of these works treat Born-Markov approximation within the weak-coupling limit. In general, for Markovian dynamics, the kinetics of particles are independent of its past events. When the system is perturbed from equilibrium, it recuperate abruptly. However, rather fast processes involving non-local dissipation having problem using this approach. Due to this limitation, new approaches involving non-Markovian quantum dissipative beyond the weak-coupling approximations were considered. In a Non-Markovian system, the past events of the particles interacting with the environment influences the future motion of the particles and the system recovers slowly to equilibrium when it is perturbed unlike the Markovian case. The working mechanism of the non-Markovian system has not been fully explored. The emergence of new issues and the inadequate knowledge to address them lead to new avenues for research exposition. The resurgent of interest in the field like quantum thermometry and quantum thermal machine as reported by Kosloff (2013) and Hofer et al. (2017), also motivates us to explore similar research field.

1.5 Problem Statement of the Present Research

We are interested to study the dynamics of the Bose-Einstein condensate out-coupled to thermal reservoirs which can be represented by bosonic harmonic oscillators. To be precise, we study the dynamics of two Bose-Einstein condensates confined within a double-well where a bosonic Josephson junction is established between the traps. The atoms of each BEC are interacting repulsively. The presence of atom tunneling between the traps and on-site inter-particle interaction mould a small Bose-Hubbard model (BHM). The double-well BEC sub-system is then exposed to two separate heat baths making the system lossy and drives the system out of equilibrium state. The appropriate equation to describe the system is the generalized quantum Langevin equation where the operational dynamics can be easily characterized. For example in our case, we wanted to study the system subjecting to Markovian or non-Markovian noise and dissipation. For our system to operate within non-Markovian dynamics we employ a memory dissipation kernel (produces coloured noise) whereas the Markovian operational dynamic is obtained by using memory-less dissipation kernel (inducing white noise). The details are given in Chapter 3 and the Appendix A.

1.6 Objective of the work

The work in this thesis focuses on the following objectives:

1. To obtain analytical solution of the interaction free BEC-reservoir system for Markovian and non-Markovian dynamics.
2. To compute and analyse the interacting BEC-reservoir system that operates within Markovian and non-Markovian basis.
3. To compute and analyse the interacting BEC-reservoir system operates within Markovian and non-Markovian basis by employing the noise-dissipation correlated models obeying Fluctuation-dissipation theorem (FDT).

1.7 Structure of the thesis

The thesis covering seven chapters and they are organized in the following manner:

Chapter 1: Basic idea of the Bose-Einstein condensation (BEC) is introduced in this chapter. Theoretical details of a single mode BEC is provided and later extended to cover the double-well BEC (two-mode BEC) system. The theory for two-mode BEC system is illustrated with figures and also supported with experimental results available in the literature. The motivation, statement of the problem and the objective for doing this research are stated. Outline of the thesis is given at the last section.

Chapter 2: Background works leading towards this research is detailed in this chapter. The idea behind open quantum system is mentioned. The two-level atom subjected to environment is given as an example but the approach used is relevant to method used in this thesis. Single-mode dissipative BEC is explained mathematical formalism supported with available experimental result. The master equation within Bloch formalism which is a popular method to study dissipative two-mode BEC system is expounded in this chapter.

Chapter 3: Our system of interest, which is the double-well BEC out-coupled to reservoirs is illustrated with schematic picture. Mathematical formalism is detailed in lucid. The basic assumptions, type and characteristics of the reservoirs, Hamiltonian describing the system and the derivation of the dynamical equation are presented here. Operational dynamics (Markovian or non-Markovian basis) of the system is explained in this chapter.

Chapter 4: We attempted to find exact solution (no approximation) for the interaction free BEC-reservoir system. Rigorous analytical calculation for the system operating within Markovian and non-Markovian basis is derived and reported in this chapter. The results are illustrated with figures.

Chapter 5: The interacting double-well BEC-reservoir system is described within the semi-classical mean-field approach. Physical quantities such as population imbalance, tunnelling current were computed and comparison is made between dynamics generated by memory-less dissipation (Markovian) and by memory dissipation (non-Markovian) on the system. Phase-space analysis were also performed for the Markovian and non-Markovian cases. Characteristic of fixed points and the correspondence with the two-mode BEC phases (Quantum tunneling state [QTS] and Macroscopic quantum self-trapping [MQST]) and how they are effect by dissipation is detail in this chapter.

Chapter 6: The limitation of Chapter 5 is addressed in this chapter by considering the noise-correlation effects on the system. The models obeying Fluctuation-dissipation theorem (FDT) is introduced in this chapter. Physical quantities such as population imbalance, coherence were computed and illustrated in this chapter. The change of state from its initial condition as the system evolved in temporal time is measured by the entanglement-entropy and the results are also depicted in this chapter.

Chapter 7: Finally, we summarize and provide brief analysis on our research findings. A flowchart diagram is given to capture the works covering Chapters 4-6 in the thesis. The summary and result analysis for the main work covered in the thesis are given in separate sequential sections. As a final remark, we noted down the limitation of our models and for possible remedy.

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LIST OF PUBLICATIONS

Rajagopal, K. K., Ibragimov, G., Non-linear dynamics of a double-well Bose-Einstein condensate-reservoirs system. *Brazilian Journal of Physics*, **50**(2): 178-184 (2020).

Rajagopal, K. K., Ibragimov, G., Non-equilibrium dynamics of a double-well Bose-Einstein condensate-dual reservoir system. *Brazilian Journal of Physics*, **51**: 944-953 (2021). [https://doi.org/10.1007/s13538-021-00904-9]

Rajagopal, K. K., Ibragimov, G., Hasim, R. M., and Alias, I. A., Noise-dissipation correlated dynamics of a double-well Bose-Einstein condensate-reservoir system. *Jordanian Journal of Physics*, [accepted for publication]

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