UNIVERSITI PUTRA MALAYSIA

IMPLICIT BLOCK METHODS WITH EXTRA DERIVATIVES FOR SOLVING GENERAL HIGHER-ORDER ORDINARY DIFFERENTIAL EQUATIONS WITH APPLICATIONS

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By

ALLOGMANY REEM AYED M

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfillment of the Requirements for the Degree of Doctor of Philosophy

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DEDICATIONS

To
My late father
Thank you for inspiring words and calling me a doctor since I was little. I wouldn't have been able to achieve this success without your constant love and support.
My mother
Who shared all my highs and lows, my trials, failures, achievements, and whose love and devotion have been my pillar of strength.
My husband
Whose love, patience, kindness, and support made everything possible.
My children
Who are my inspiration and power throughout the stormy days.
My entire family and friends
Who have supported me throughout this journey.

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfillment of the requirement for the degree of Doctor of Philosophy

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Chairman: Professor Fudziah Ismail, PhD
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Traditionally, higher order ordinary differential equations (ODEs) are solved by reducing them to an equivalent system of first order ODEs. However, it is more cost effective if they can be solved directly by numerical methods. Block methods approximate the solutions of the ODEs at more than one point at one time step, hence faster solutions can be obtained. It is well-known too that a more accurate numerical results can be obtained by incorporating the higher derivatives of the solutions in the method. Based on these arguments, we are focused on developing block methods with extra derivatives for directly solving second, third and fourth order ODEs.

In this study, two-point and three-point implicit block methods with extra derivatives are derived using Hermite Interpolating polynomial as the basis function. The technique of integration is used in the derivation as it is more straight forward and can easily be carried out compared to the existing technique of collocation and interpolation in which the points need to be collocated and interpolated resulting in a huge system of linear equations which need to be solved simultaneously.

The thesis consists of three parts, the first part of the thesis described the derivation of two-point and three-point implicit block methods which incorporated the second, third and fourth derivatives of the solution for directly solving general second order ODEs. Absolute stability for both block methods is presented. The second part of the thesis is focused on the derivation of two-point and three-point implicit block methods which include the third, fourth and fifth derivatives of the solutions for directly solving general third order ODEs. The last part of the thesis concerned with the construction of two-point and three-point implicit block method which involved the fourth and the fifth derivatives of the solution for directly solving general fourth order ODEs.

The basic properties of all the methods, such as algebraic order, zero-stability, and convergence are established. Numerical results clearly show that the new proposed methods are more efficient in terms of accuracy and computational time when compared with well-known existing methods. Applications in several real fields also illustrate the efficiency of the proposed methods.

In conclusion, the new block methods with extra derivatives and codes developed based on the methods are suitable for solving second, third and fourth order ODEs respectively and can be applied to solve physical problems.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

KAEDAH BLOK TERSIRAT DENGAN TERBITAN TAMBAHAN UNTUK MENYELESAIKAN PERSAMAAN PEMBEZAAN BIASA UMUM PERINGKAT TINGGI DENGAN APLIKASI

Oleh

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Secara tradisinya, persamaan pembezaan biasa (PPB) peringkat tinggi diselesaikan dengan menurunkannya kepada sistem PPB peringkat pertama yang setara. Walau bagaimanapun, adalah lebih menjimatkan kos jika ia dapat diselesaikan secara langsung dengan kaedah berangka. Kaedah blok memberi penghampiran kepada penyelesaian PPB pada lebih dari satu titik pada setiap langkah, oleh itu penyelesaian yang lebih cepat dapat diperolehi. Telah diketahui juga bahawa keputusan berangka yang lebih tepat dapat diperolehi dengan melibatkan terbitan penyelesaian yang lebih tinggi dalam kaedah tersebut. Berdasarkan hujah-hujah ini, kami fokus untuk membangunkan kaedah blok dengan terbitan tambahan untuk menyelesaikan secara langsung PPB peringkat kedua, ketiga dan keempat.

Dalam kajian ini, kaedah blok tersirat dua-titik dan tiga-titik dengan terbitan tambahan dibangunkan menggunakan polinomial interpolasi Hermite sebagai fungsi asas. Teknik kamiran digunakan dalam menerbitkan kaedah ini kerana teknik ini lebih mudah dan senang dilaksanakan berbanding dengan teknik kolokasi dan interpolasi sedia ada, di mana titik-titik tersebut perlu dikolokasi dan diinterpolasi sehingga menghasilkan sistem persamaan linear yang besar dan perlu diselesaikan secara serentak.

Tesis ini merangkumi tiga bahagian, bahagian pertama tesis menerangkan pembinaan kaedah blok tersirat dua-titik dan tiga-titik yang menggabungkan

terbitan kedua, ketiga dan keempat penyelesaian PPB tersebut untuk menyelesaikan secara langsung PPB umum peringkat kedua. Kestabilan mutlak untuk kedua-dua kaedah blok dipersembahkan. Bahagian kedua tesis difokuskan kepada menerbitkan kaedah blok tersirat dua-titik dan tiga-titik yang merangkumi terbitan ketiga, keempat dan kelima dari penyelesaian untuk menyelesaikan secara langsung PPB umum peringkat ketiga. Bahagian terakhir tesis adalah tentang menerbitkan kaedah blok tersirat dua-titik dan tiga-titik yang melibatkan terbitan keempat dan kelima penyelesaian untuk menyelesaikan secara langsung PPB umum peringkat keempat.

Sifat asas semua kaedah, seperti peringkat aljabar, kestabilan sifar dan penumpuan diperkukuhkan. Keputusan berangka menunjukkan dengan jelas bahawa kaedah baharu yang dicadangkan adalah lebih cekap dari segi ketepatan dan masa pengiraan jika dibandingkan dengan kaedah sedia ada yang terkenal. Aplikasi dalam beberapa bidang nyata juga menggambarkan kecekapan kaedah yang dicadangkan.

Kesimpulannya, kaedah blok dengan terbitan tambahan yang baharu dan kod yang dibangunkan berdasarkan kaedah tersebut adalah sesuai untuk menyelesaikan PPB peringkat kedua, ketiga dan keempat masing-masingnya dan boleh digunakan untuk menyelesaikan masalah fizikal.

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LIST OF ABBREVIATIONS

ODEs	Ordinary Differential Equations
IVPs	Initial Value Problems
MAXE	Maximum Absolute Error
FCN	Number of Function Evaluations
NS	Total Number of Steps
h	step size
I2PBDO8	Implicit two-point block method for second-order ODEs
I3PBDO8	Implicit three-point block method for second-order ODEs
I2PBffDO9	Direct implicit two-point block method for third-order ODEs
I3PBffDO9	Direct implicit three-point block method for third-order ODEs
I2PBDO6	Two-point direct implicit block method for fourth-order ODEs
I3PBDO6	Three-point direct implicit block method for fourth-order ODEs

CHAPTER 1

INTRODUCTION

Higher-order ordinary differential equations (ODEs) are found in a wide variety of real-life situations. They are used to model problems arising from the field of applied sciences and engineering in terms of unknown function and their derivatives. Many researchers in the literature have presented theoretical and numerical studies for such ODEs. The analytical solutions of ODEs are complicated or impossible for most of realistic systems of ODEs. Therefore, the need to develop numerical methods to achieve more accurate approximations and to be easy to implement is eminent.

1.1 Ordinary Differential Equations

The n^{th} order ordinary differential equation can be written as

$$y^{(n)} = f(x, y, y' ..., y^{(n-1)}), \quad n = 2, 3, 4 \tag{1.1}$$

subject to the initial conditions

$$y(a) = y_0, \qquad y^{(i)}(a) = y_0^{(i)}, \qquad 0 \le i \le n-1, \qquad a \le x \le b.$$

In (1.1), the quantity being differentiated, y is called as the dependent variable, while the quantity with respect to which y is differentiated, x is called as an independent variable.

1.1.1 Initial Value Problems

Definition 1.1 The initial value problem(IVPs) of a system of s first-order ODEs can be defined as:

$$y' = f(x, y) \tag{1.2}$$

subject to the initial conditions

$$y(x_0) = y_0, \quad x \in [a, b]$$

where

$$f : \Re \times \Re^s \to \Re^s,$$
$$y(x) = [y_1(x), y_2(x), \ldots, y_s(x)]^T,$$
$$f(x, y) = [f_1(x, y), f_2(x, y), ..., f_s(x, y)]^T.$$

and y_0 is a given vector of initial conditions

Definition 1.2 The initial value problem(IVPs) of a system of s general second-order ODEs can be defined as:

$$y'' = f(x, y, y') \tag{1.3}$$

subject to the initial conditions

$$y(x_0) = y_0, \quad y'(x_0) = y'_0, \quad x \in [a, b],$$

where

$$\begin{aligned} f : \Re \times \Re^s \times \Re^s &\to \Re^s, \\ y(x) &= [y_1(x), y_2(x), \ldots, y_s(x)]^T, \\ y'(x) &= [y'_1(x), y'_2(x), \ldots, y'_s(x)]^T, \\ f(x, y, y') &= [f_1(x, y, y'), f_2(x, y, y'), ..., f_s(x, y, y')]^T. \end{aligned}$$

and y_0, y'_0 are given vectors of initial conditions.

1.1.2 Existence and Uniqueness of Solution for the general n^{th} order IVPs

We shall suppose that always exists a unique solution of the problems in this thesis. Thus, the hypotheses of the following theorem is fulfilled by each component of the system.

Theorem 1.1 (Wend (1967))
Let D be domain defined by the inequalities $0 \leq x - x_0 < a, \ \mid s_i - y_i \mid < b_i, \ 0 \leq i \leq n-1$, where $y_i \geq 0$ for $i > 0$. Suppose the function $f(x, s_0, s_1, ..., s_{n-1})$ in (1.1) is nonnegative, continuous and nondecreasing in x, and continuous and nondecreasing in s_i for each $0 \leq i \leq n-1$ in the domain D. If in addition $f(x, y_0, ..., y_{n-1}) \neq 0$ in D for $x > x_0$, then the IVP (1.1) has at most one solution in D.

1.1.3 Linear Multistep Method

The linear $k-$step methods use k of the previous points to determine the sequence y_n that takes form of a linear relationship between y_{n+i}, f_{n+i}, where $i = 0, 1, 2, ..., k$ which can be written for the n^{th}-order ODEs as

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i y'_{n+i} + ... + h^n \sum_{i=0}^{k} \gamma_i f_{n+i}. \tag{1.4}$$

Where α_i, β_i, and γ_i are constants with $\alpha_k \neq 0$ and n is the order of the differential equation. The method (1.4) is implicit if $\beta_k \neq 0$, $\gamma_k \neq 0$ and explicit if $\beta_k = 0$, $\gamma_k = 0$.

Definition 1.3 (Gear (1971); Lambert (1991); Fatunla (1988))
The linear operator ℓ associated with Equation (1.4) can be defined as

$$\ell[y(x):h] = \sum_{i=0}^{k}[\alpha_i y(x+ih) - h\beta_i y'(x+ih) - h^2\gamma_i y''(x+ih) - \ldots - h^n\delta_i y^{(n)}(x+ih)]. \tag{1.5}$$

Where $y(x)$ is an arbitrary function that is differentiable on $[a,b]$ and n is the order of the differential equation. By expanding the test function and its derivatives as the Taylor series at x and collecting the terms yields

$$\ell[y(x):h] = C_0 y(x) + C_1 hy'(x) + C_2 h^2 y''(x) + \ldots + C_p h^{(p)} y^{(p)}(x) + \ldots$$

where the coefficients c_p are constants defined as

$$C_0 = \sum_{i=0}^{k} \alpha_i = \bar{0},$$

$$C_1 = \sum_{i=0}^{k}(i\alpha_i - \beta_i) = \bar{0},$$

$$C_2 = \sum_{i=0}^{k}(\frac{i^2}{2!}\alpha_i - i\beta_i - \gamma_i) = \bar{0},$$

$$\vdots$$

$$C_p = \sum_{i=0}^{k}(\frac{i^p}{p!}\alpha_i - \frac{i^{p-1}}{(p-1)!}\beta_i - \ldots - \frac{i^{p-n}}{(p-n)!}\delta_i).$$

Definition 1.4 (Fatunla (1988))
The multistep method (1.4) and the associated linear (1.5) have order p if $C_0 = C_1 = \ldots. = C_{p+(n-1)} = 0$ and $C_{p+n} \neq 0$, where n is the order of the differential equation. Then, the coefficient C_{p+n} is the error constant.

Definition 1.5 (Lambert (1991); Fatunla (1988))
The multistep method (1.4) can be said consistent if it has order $p \geq 1$.
The first and second characteristic polynomials of the linear multistep method are defined as

$$\rho(\zeta) = \sum_{i=0}^{k} \alpha_i \zeta^i$$

$$\sigma(\zeta) = \sum_{i=0}^{k} \beta_i \zeta^i$$

Definition 1.6 (Lambert (1991) and Henrici (1962))
The linear multistep method is said to be zero-stable if the following conditions are fulfilled,

- All roots of the first characteristic polynomial , $| \zeta_i | \leq 1$.
- The multiplicity for those roots with$| \zeta_i | = 1$, must not exceed n.

Theorem 1.2 (Henrici (1962))
The necessary and sufficient conditions for a method to be convergent are that it be consistent and zero stable.

Definition 1.7 (Stoer and Bulirsch (1991))
If f a function of distinct numbers x_i; $i = 0, 1, n$, then there is exists a unique polynomial P of degree n at most with property $f(x_i) = P(x_i)$ for every i. The Polynomial of Hermite interpolation P has the following form

$$P_n(x) = \sum_{i=0}^{n} \sum_{k=0}^{m_i-1} f^{(k)}(x_i)\bar{L}_{i,k}(x). \tag{1.6}$$

Where, $\bar{L}_{(i,k)}(x)$ is the generalized Lagrange polynomial which defined as,

$$\bar{L}_{i,m_{i-1}}(x) = l_{i,m_{i-1}}(x)$$

$$l_{i,k}(x) = \frac{(x-x_i)^k}{k!} \prod_{j=0,j\neq i}^{n} (\frac{x-x_j}{xi-x_j})^m; \quad j,i = 0,1,...,n, \quad k = 0,1,...,m_{i-1}$$

$$\bar{L}_{i,k}(x) = l_{i,k}(x) - \sum_{v=k+1}^{m_i-1} l_{i,k}^{(v)}(x_i)\bar{L}_{i,v}(x)$$

1.2 Problem Statement

Throughout recent literature, the direct numerical methods to approximate higher-order ordinary differential equations ODEs (2.1) are being considerably explored. The reason why these numerical methods were implemented is that some of these higher-order ODEs lack an approximated solution or the current numerical methods are less accurate. However, collocation and interpolation techniques are utilized as direct methods in general. The points need to be collocated and interpolated after which a system of linear equations must be resolved in order to obtain the method's coefficients. More recently, researchers have developed methods with additional derivatives in solving ODEs. Hence, adding extra derivatives to the method leads to obtain a more accurate numerical results. Therefore, we develop direct $r-$point block implicit methods by using interpolation and integration strategy which can be implemented in a straightforward manner for solving both the linear and nonlinear general higher-order ODEs with impressive results.

1.3 Objectives of Study

This study aims to construct block methods for solving higher-order IVPs. The implementation of the block methods is expected to simultaneously obtain the approximation at two and three points. These methods should give better results in terms of accuracy. The objectives of this study can be accomplished by:

- To derive the implicit two-point and three-point block methods with the second, third, and fourth derivatives of the solution based on Hermite interpolation polynomial for solving general second-order initial value problems.

- To derive the implicit two-point and three-point block methods with the third, fourth, and fifth derivatives of the solution based on Hermite interpolation polynomial for solving general third-order initial value problems.
- To drive the implicit two-point and three-point block methods with the fourth and fifth derivatives of the solution based on Hermite interpolation polynomial for solving general fourth-order initial value problems.
- To investigate the stability and convergence of the methods.
- To compare the effectiveness of the proposed methods with its counterparts.
- To apply the proposed methods to solve real-life applications.

1.4 Outline of the Study

A brief description of the thesis's organization will be given as follows:

Chapter 1, discusses the overview of ordinary differential equations. The definitions and the theories which are related to the proposed methods are presented. Chapter 2, reviews some of previous studies on numerical methods for solving higher-order ODEs.

Chapter 3, 4 and 5 deals with the implicit block methods for solving higher-order ODEs. Chapter3, provides the formulation of two-point and three-point implicit block methods based on Hermite interpolation polynomial for solving general second-order ODEs. The methods proposed include the second, third, and fourth derivatives of the solutions. The stability of these methods is also discussed in this chapter. Chapter 4 provides the formulation of two-point and three-point implicit block methods based on Hermite interpolation polynomial for solving general third-order ODEs. The proposed methods include the third, fourth, and fifth derivatives of the solutions. Characteristics of the proposed methods, including algebraic order, zero stability, and convergence, are analysed. Chapter 5 provides the formulation of two-point and three-point implicit block methods based on Hermite interpolation polynomial for solving general fourth-order ODEs. The proposed methods include the fourth and fifth derivatives of the solution. Characteristics of the proposed methods, including algebraic order, zero stability, and convergence, are also analysed. Some applications of the proposed methods for solving conventional problems are provided.

Finally, Chapter 6 summarizes the works of this study and provides suggestions for future work.

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In 2004, she gained admission into Taibah University to pursue a degree of Bachelor in Mathematics. She graduated with Excellent with Honours in 2008. In 2010, she started her Master of Statistics. Before she graduated, she was offered a scholarship from the Ministry of Higher Education of Saudi Arabia to study in the United States of America. Then, she immediately enrolled to study another Master of Applied Mathematics at California State University, East-Bay (CSUEB). During her study, she won the best essay in the American language Program contest at CSUEB in 2012. Then, she graduated with first-class Honours in 2015, and also she was awarded membership in Golden key at California State University, East Bay, the world's largest collegiate honor society since she is a top performer in her graduate class.

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LIST OF PUBLICATIONS

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