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GENERALIZATION OF HERMITE-HADAMARD TYPE INEQUALITIES AND THEIR APPLICATIONS

ALMUTAIRI, OHUD BULAYHAN

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By

ALMUTAIRI, OHUD BULAYHAN

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfillment of the Requirements for the Doctor of Philosophy

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DEDICATION

This thesis is dedicated to my beloved parents

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Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfillment of the requirement for the degree of Doctor of Philosophy

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By

ALMUTAIRI, OHUD BULAYHAN

December 2020

Chairman: Prof. Adem Kiliçman, PhD
Faculty: Science

This thesis is concerned with the study of generalization, refinement, improvement and extension of Hermite-Hadamard (H-H) type inequalities. These are achieved by using various classes of convex functions and different fractional integrals. We established new integral inequalities of H-H type via *s*-convex functions in the second sense, as well as the new classes of convexities: h-Godunova-Levin and h-Godunova-Levin preinvex functions. We also generalized the inequalities of the H-H type involving Riemann-Liouville via generalized *s*-convex functions in the second sense on fractal sets. We further generalized the H-H type inequalities involving Katugampola fractional integrals via different types of convexities. We also improved several inequalities of H-H type through various classes of convexities by using the conditions $|\mathscr{G}'|^q$ and $|\mathscr{G}''|^q$ for $q \geq 1$. Using the obtained new results, we presented some applications to special means and applications to numerical integration. By comparing the error bounds estimation of numerical integrations, report shows that the present results obtained using generalization of mid-point and trapezoid type inequalities are more efficient. Several quadrature rules were reported to be examined through this approach. The findings of this study are new, more general and to some extend better than many other research results.

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Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

GENERALISASI KETIDAKSAMAAN JENIS HERMITE-HADAMARD DAN APLIKASINYA

Oleh

ALMUTAIRI, OHUD BULAYHAN

December 2020

Pengerusi: Prof. Adem Kiliçcman, PhD
Fakulti: Sains

Tesis ini berkaitan kajian generalisasi, pembaikan, peningkatan dan pengembangan ketaksamaan jenis Hermite-Hadamard (H-H). Ini dicapai dengan menggunakan pelbagai kelas fungsi cembung dan kamiran pecahan yang berbeza. Kami membentuk ketaksamaan kamiran baharu jenis H-H melalui fungsi s-cembung dalam pengertian kedua, serta kelas cembung baharu: h-Godunova-Levin dan fungsi h-Godunova-Levin preinveks. Kami juga mengeneralisasi ketaksamaan jenis H-H yang melibatkan Riemann-Liouville melalui generalisasi fungsi s-cembung dalam pengertian kedua pada set fraktal. Kami seterusnya mengeneralisasikan ketaksamaan jenis H-H yang melibatkan kamiran pecahan Katugampola melalui pelbagai jenis kecembungan. Kami juga memperbaiki beberapa ketaksamaan jenis H-H melalui pelbagai kelas kecembungan menggunakan syarat $|\mathscr{G}'|^q$ dan $|\mathscr{G}''|^q$ untuk $q \geq 1$. Menggunakan dapatan kajian yang baharu, kami mengemukakan beberapa aplikasi dengan cara khusus dan aplikasi untuk pengamiran berangka. Dengan membandingkan anggaran batas ralat terhadap pengamiran berangka, kami laporan menunjukkan bahawa keputusan yang peroleh menggunakan generalisasi titik tengah dan ketaksamaan jenis trapezoid adalah lebih efisyen. Beberapa peraturan kuadratur dilaporkan untuk diperiksa melalui pendekatan ini. Hasil dapatan kajian ini adalah baharu, lebih umum dan dari sudut tertentu lebih baik daripada dapatan kajian yang lain.

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This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Doctor of Philosophy. The members of the Supervisory Committee were as follows:

Adem kiliçman, PhD
Professor
Faculty of Science
Universiti Putra Malaysia
(Chairperson)

Siti Hasana Sapar, PhD
Associate Professor
Faculty of Science
Universiti Putra Malaysia
(Member)

Fadzilah Md Ali, PhD
Assistant Professor
Faculty of Science
Universiti Putra Malaysia
(Member)

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LIST OF ABBREVIATIONS

$\mathbb{N}$	The natural numbers
$\mathbb{R}$	The real number
$\mathbb{R}_+$	The set of positive real number
$\mathbb{R}^\alpha$	Real line numbers on a fractal space
$[m_1, m_2]$	Real interval
$\Gamma(m)$	Euler's Gamma (Gamma) function
$\beta(m_1, m_2)$	Beta function
$L_1[m_1, m_2]$	The space of integrable functions on $[m_1, m_2]$
K_s^2	s-Convex function in the second sense
GK_s^2	Generalized s-Convex function in the second sense
$SGX(\frac{1}{h}, \zeta)$	h-Godunova-Levin function
$SGXP(\frac{1}{h}, \zeta)$	h-Godunova-Levin preinvex function
$\max(.,.)$	Maximum
$\min(.,.)$	Minimum
$A(.,.)$	The arithmetic mean
$G(.,.)$	The geometric mean
$L(.,.)$	The logarithmic mean
$L_\vartheta(.,.)$	The generalized log-mean

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CHAPTER 1

INTRODUCTION

1.1 Background

Inequalities are simply referred to as the disparity between two quantities. They are generally considered among the powerful tools in mathematical analysis, as well as other branches of mathematics, including approximation theory and numerical analysis. One example of these is the arithmetic- geometric mean inequality used by Erdos and Grunwald (1939) when estimating integrals through triangles (Aigner et al., 2010). The development of theory of inequalities begins dating back to the time of P.L. Chebyshev, A.L. Cauchy and C.V. Gauss, whose contributions include the discovery of theoretical foundations that approximate iterative methods. Others are Hardy et al. (1952) whose classical work was the transformation of inequalities from a collection of individual formulas to a systematic discipline. This development was widely celebrated by researchers as since then different types of inequalities have been reported in the literature due to their broad areas of applications. The utility of mathematical inequalities is developing along with other branches of mathematics. This development is not only for theoretical mathematicians, but also for those working in applied areas, such as mathematical modelling. In the previous millennium, researchers have witnessed the strength of inequalities through which enormous new results were obtained.

The emergence of Hermite-Hadamard (H-H) inequality is considered as the most important discovery in the study of inequalities since they attract the interests of many scientists. As mentioned by Mitrinović and Lacković (1985), this inequality was first appeared in the literature through the effort of Hadamard (1893); however, the result was first discovered by Hermite (1883). Following this fact, many researchers referred the result as the H-H inequality. This inequality was stated in the monograph of Dragomir and Pearce (2004) as the first fundamental result for convex functions defined in the interval of real numbers with a natural geometrical interpretation that can be applied to investigate a variety of problems. Inequalities play important roles in understanding many mathematical concepts, such as probability theory, numerical integration and integral operator theory. Throughout the last century, the H-H type inequalities have been considered among the fast growing fields in mathematical analysis, through which vast problems in engineering, economics and physics have been studied (Dragomir and Pearce, 2004; Bainov and Simeonov, 2013; Wang and Feckan, 2018). Due to the enormous importance of these inequalities, many extensions, refinements and generalizations of their related types have been equally investigated (Ion, 2007; Özdemir et al., 2011; Latif et al., 2012; Zabandan et al., 2012; Ali et al., 2017; Hwang and Dragomir, 2017; Prabseang et al., 2019; Bin-Mohsin et al., 2019; Kunt et al., 2019; Duc et al., 2020).

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Therefore, the H-H type inequalities, by which many results are studied, play important roles in the theory of convex functions. The convexities, along with many types of their generalizations, can be applied in different fields of sciences (Demuynck, 2009; Pennanen, 2012; Liu et al., 2020), through which many generalizations of H-H inequality for a variant types of convexities have been studied. Other extensions of H-H inequality include the formulation of problems related to fractional calculus, a branch of calculus dealing with derivatives and integrals of non-integer order (see Gorenflo and Mainardi, 1997; Dragomir, 2019; Dahmani and Belhamiti, 2020).

Nowadays, the real-life applications of fractional calculus exist in most areas of studies (Baleanu et al., 2010; de Oliveira et al., 2019). To make the application of fractional calculus easier, mathematicians have defined its derivatives and integrals in many different ways. In this thesis, we are interested in formulating fractional integrals, along with their generalizations, to obtain some new fractional H-H type inequalities involving different types of convexities.

1.2 Convex functions

The concept of a convex function was first introduced to elementary calculus when discussing necessary conditions for a minimum or maximum value of a differentiable function. The convex function was later recognized as an active area of study by Jensen (1905). In modern studies, a convex function is considered as a link between analysis and geometry, which makes it a powerful tool for solving many practical problems.

Definition 1.1 *(Niculescu and Persson, 2006) Let V be an interval in $\mathbb{R}$. A function $\mathscr{G}: V \to \mathbb{R}$ is said to be convex if*

$$\mathscr{G}(\zeta m_1 + (1-\zeta)m_2) \leq \zeta \mathscr{G}(m_1) + (1-\zeta)\mathscr{G}(m_2) \tag{1.2.1}$$

holds for all $m_1, m_2 \in V$ and $\zeta \in [0,1]$.

If inequality (1.2.1) strictly holds for any distinct points m_1 and m_2, where $\zeta \in (0,1)$, then the function is said to be a strictly convex. Meanwhile, a function $-\mathscr{G}$ is convex (strictly convex), then $\mathscr{G}$ is concave (strictly concave).

Since results in convex functions can be reproduced in terms of their concave analogues, this study, therefore, concentrates on convexity.

Geometrically, a function $\mathscr{G}$ is convex given that the line segment joining any two points on the graph lies above (or on) the graph (see Figure 1.1). Meanwhile, if the line segment connecting the two points is below (or on) the graph, the function is concave.

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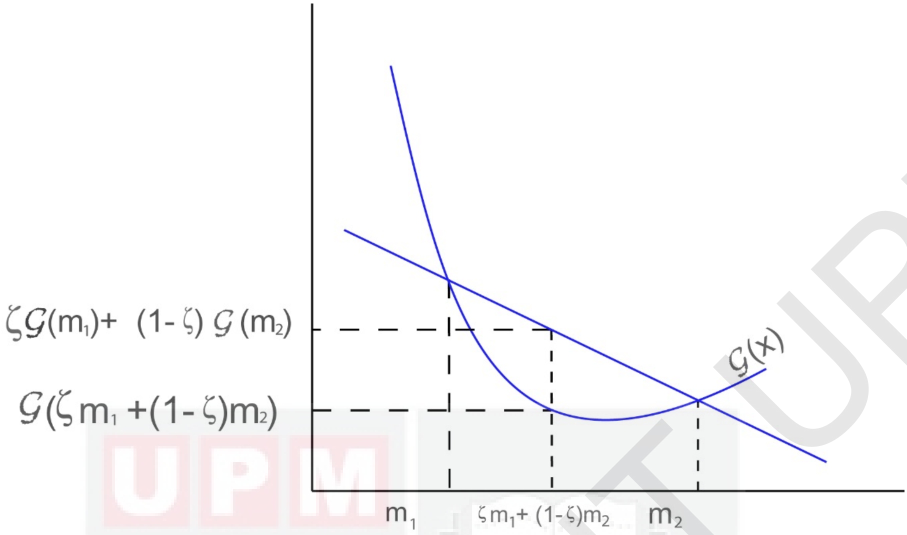

Figure 1.1: The geometrical representation of inequality (1.2.1)

Example 1.1 *Given a function $\mathscr{G}: V \subseteq \mathbb{R} \to \mathbb{R}$ for any $m \in \mathbb{R}$, we have the following examples.*

i. *$\mathscr{G}(m) = c_1 m + c_2$, where $c_1, c_2 \in \mathbb{R}$. The function $\mathscr{G}(m)$ is both concave and convex on $(-\infty, \infty)$. Thus, it is referred to as an affine.*

ii. *The functions $\mathscr{G}(m) = m^2$ and $\mathscr{G}(m) = e^m$ are both convex functions on $\mathbb{R}$.*

iii. *$\mathscr{G}(m) = \ln m$ is a concave function on $\mathbb{R}_+ = [0, \infty)$.*

In order to clearly describe a convex function, different definitions are given as follows.

Definition 1.2 *(Peajcariaac and Tong, 1992) Let $c_1, c_2, c_3 \in V$ such that $c_1 < c_2 < c_3$, the function $\mathscr{G}$ is convex if and only if (iff)*

$$\frac{\mathscr{G}(c_1) - \mathscr{G}(c_2)}{c_1 - c_2} \leq \frac{\mathscr{G}(c_2) - \mathscr{G}(c_3)}{c_2 - c_3}. \tag{1.2.2}$$

Inequality (1.2.2) can be interpreted geometrically as follows. The slope of the line segment joining $(c_1, \mathscr{G}(c_1))$ and $(c_2, \mathscr{G}(c_2))$ is less than that of line segment joining $(c_2, \mathscr{G}(c_2))$ and $(c_3, \mathscr{G}(c_3))$.

Definition 1.3 *(Niculescu and Persson, 2006) A function $\mathscr{G}: V \subseteq \mathbb{R} \to \mathbb{R}$ is called J-convex or convex in Jensen sense if*

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$$\mathscr{G}\left(\frac{m_1+m_2}{2}\right) \leq \frac{\mathscr{G}(m_1)+\mathscr{G}(m_2)}{2} \tag{1.2.3}$$

holds for all $m_1, m_2 \in V$.

Considering the importance of inequality (1.3), Jansen was the first to relate it with a convex function. If J-convex is continuous, the inequality (1.3) is equivalent to (1.2.1), a convexity.

Theorem 1.1 *(Niculescu and Persson, 2006) Suppose that* $\mathscr{G} : V \subseteq \mathbb{R} \to \mathbb{R}$ *is a continuous function. Then* $\mathscr{G}$ *is convex iff* $\mathscr{G}$ *is J-convex, that is,*

$$\mathscr{G}\left(\frac{m_1+m_2}{2}\right) \leq \frac{\mathscr{G}(m_1)+\mathscr{G}(m_2)}{2},$$

for all $m_1, m_2 \in V$.

For other results of convex function, we refer the reader to Phelps (2009), Borwein et al. (2010), Udriste (2013) and Ullah et al. (2019).

Convex functions can be generalized in different ways. Thus, in the following subheading, we present the definition of generalized convex function on fractal sets, along with their examples and a related theorem.

1.2.1 Generalized convex functions on fractal sets

The definition of generalized convex functions on fractal sets $\mathbb{R}^\alpha (0 < \alpha \leq 1)$ is given by Mo et al. (2014) as follows.

Definition 1.4 *Let* $\mathscr{G} : V \subseteq \mathbb{R} \to \mathbb{R}^\alpha$. *For any* $m_1, m_2 \in V$ *and* $\zeta \in [0,1]$, *if the following inequality*

$$\mathscr{G}(\zeta m_1 + (1-\zeta)m_2) \leq \zeta^\alpha \mathscr{G}(m_1) + (1-\zeta)^\alpha \mathscr{G}(m_2)$$

holds, then $\mathscr{G}$ *is called a generalized convex function on* V.

A linear function $\mathscr{G}(m) = c_1^\alpha m^\alpha + c_2^\alpha, c_1, c_2, m \in \mathbb{R}$ is both generalized convex and concave. Meanwhile, the following functions serve as examples of strictly convex.

i. $\mathscr{G}(m) = m^{\alpha p}, p > 1$.

ii. $\mathscr{G}(m) = E_\alpha(m^\alpha), m \in \mathbb{R}$, where $E_\alpha(m^\alpha) = \sum_{k=0}^{\infty} \frac{m^{\alpha k}}{\Gamma(1+k\alpha)}$ is the Mittag-Leffer function (Gorenflo et al., 2014).

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Now, the following theorem gives an important characteristic of generalized convex function.

Theorem 1.2 *(Mo et al., 2014) Let $\mathscr{G} : V \subseteq \mathbb{R} \to \mathbb{R}^{\alpha}$, then $\mathscr{G}$ is a generalized convex function iff the inequality*

$$\frac{\mathscr{G}(c_1) - \mathscr{G}(c_2)}{(c_1 - c_2)^{\alpha}} \leq \frac{\mathscr{G}(c_2) - \mathscr{G}(c_3)}{(c_2 - c_3)^{\alpha}}$$

holds, for any $c_1, c_2, c_3 \in V$ with $c_1 < c_2 < c_3$.

For further results of generalized convex function on fractal sets, we refer the reader to Set and Tomar (2016), Sun (2017) and Sarıkaya et al. (2019).

In the following subheading, we discuss the relation between monotonicity and differentiability of a convex function.

1.2.2 Monotonicity and differentiability of convex functions

An important property of a function, through which its graph either increases or decreases, is called monotonicity. A function $\mathscr{G}$, preserving the order, can be monotonically increasing (decreasing) if for any two points $m_1, m_2 \in V$ such that $m_1 \leq m_2$, and $V \subseteq \mathbb{R}$ then $\mathscr{G}(m_1) \leq (\geq) \mathscr{G}(m_2)$.

The following theorems describe the relation between the monotonicity and the derivatives of convex functions.

Theorem 1.3 *(Peajcariaac and Tong, 1992) Suppose that $\mathscr{G} : V \subseteq \mathbb{R} \to \mathbb{R}$ is a convex (strictly convex) function. Then $\mathscr{G}'_{-}(m)$ and $\mathscr{G}'_{+}(m)$ exist, which are increasing (strictly increasing) on V° (the interior of V).*

Theorem 1.4 *(Peajcariaac and Tong, 1992) Let $\mathscr{G}$ be a differentiable function on (m_1, m_2). Therefore, $\mathscr{G}$ is convex (strictly convex) iff $\mathscr{G}'$ is increasing (strictly increasing).*

The second derivative is important since it can be used to determine the convexity of a twice differentiable function.

Theorem 1.5 *(Robert and Varberg, 1973) Let $\mathscr{G} : V \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function. If $\mathscr{G}''$ exists on V°, then $\mathscr{G}$ is convex (strictly convex) on V iff $\mathscr{G}''(m) \geq (>)$ 0 for every $m \in V$.*

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1.3 Different types of convex functions

The theory of convexity deals with large classes, such as Godunova-Levin, s-convex and preinvex functions. These, termed as the generalization of convexity, play important roles in optimization theory and mathematical programming. In this subheading, therefore, we give basic definitions of different classes of convex functions. Important properties and examples are also discussed herein.

1.3.1 Godunova-Levin and P-functions

The space Godunova-Levin function, denoted by $Q(V)$, was introduced by Godunova and Levin (1985). They noted that both the positive monotone and positive convex functions are belonged to $Q(V)$. Due to the importance of this function, we present it as follows.

Definition 1.5 *(Mitrinović et al., 2013) A non-negative function $\mathscr{G} : V \subseteq \mathbb{R} \to \mathbb{R}$ is called Godunova-Levin function (denoted by $\mathscr{G} \in Q(V)$) if*

$$\mathscr{G}(\zeta m_1 + (1-\zeta)m_2) \leq \frac{1}{\zeta}\mathscr{G}(m_1) + \frac{1}{1-\zeta}\mathscr{G}(m_2) \tag{1.3.1}$$

holds, for all $m_1, m_2 \in V$ and $\zeta \in (0,1)$.

Example 1.2 *(Dragomir et al., 1995) For $x \in [m_1, m_2]$, the function*

$$\mathscr{G}(x) = \begin{cases} 1, & m_1 \leq x < \frac{m_1+m_2}{2} \\ 4, & x = \frac{m_1+m_2}{2} \\ 1, & \frac{m_1+m_2}{2} < x \leq m_2 \end{cases}$$

is in the class $Q(V)$.

Godunova-Levin function was restricted to a space called $P(V)$ contained in $Q(V)$. This class is defined by Dragomir et al. (1995) as follows.

Definition 1.6 *A non-negative function $\mathscr{G} : V \subseteq \mathbb{R} \to \mathbb{R}$ is called P-function (denoted by $\mathscr{G} \in P(V)$) if*

$$\mathscr{G}(\zeta m_1 + (1-\zeta)m_2) \leq \mathscr{G}(m_1) + \mathscr{G}(m_2)$$

holds, for all $m_1, m_2 \in V$ and $\zeta \in [0,1]$.

Therefore, all non-negative monotone and convex functions are contained in $P(V)$.

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For other results of Godunova-Levin and P-functions, see Radulescu et al. (2009), Fujii et al. (2011), Fang and Shi (2014), Kadakal et al. (2017) and Bekar (2019).

1.3.2 s-convex function in the second sense

The definition of s-convex function in the second sense or s-Breckner convex is given as follows.

Definition 1.7 *(Breckner, 1978) A function $\mathscr{G} : [0,\infty) \to \mathbb{R}$ is said to be s-convex in second sense (denoted by $\mathscr{G} \in K_s^2$), if*

$$\mathscr{G}(\zeta_1 m_1 + \zeta_2 m_2) \leq \zeta_1^s \mathscr{G}(m_1) + \zeta_2^s \mathscr{G}(m_2) \tag{1.3.2}$$

holds, for all $m_1, m_2 \in [0,\infty)$, $\zeta_1, \zeta_2 \geq 0$, $\zeta_1 + \zeta_2 = 1$ and $0 < s \leq 1$.

Choosing $s = 1$ reduces s-convexity in second sense to the classical convex function on $[0,\infty)$.

The following property that is connected to s-convex function in the second sense is given bellow.

Theorem 1.6 *(Hudzik and Maligranda, 1994) If $\mathscr{G} \in K_s^2$, then $\mathscr{G}$ is non-negative on $[0,\infty)$.*

For some properties of s-convexity in second sense, see the references (Dragomir and Fitzpatrick, 1999; Du et al., 2017; Usta et al., 2018; Gozpinar et al., 2019).

Hudzik and Maligranda (1994) present the example of s-convex function in the second sense as follows.

Example 1.3 *Let $0 < s < 1$ and $c_1, c_2, c_3 \in \mathbb{R}$. Defining*

$$\mathscr{G}(m) = \begin{cases} c_1, & m = 0 \\ c_2 m^s + c_3, & m > 0 \end{cases}$$

for $m \in \mathbb{R}_+$, we have

i. If $c_2 \geq 0$ and $0 \leq c_3 \leq c_1$, then $\mathscr{G} \in K_s^2$,

ii. If $c_2 > 0$ and $c_3 < 0$, then $\mathscr{G} \notin K_s^2$.

As Hudzik and Maligranda mentioned that the condition $\zeta_1 + \zeta_2 = 1$ in definition 1.7 can be replaced by $\zeta_1 + \zeta_2 \leq 1$, equivalently.

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Theorem 1.7 *(Hudzik and Maligranda, 1994) Suppose that $\mathscr{G} \in K_s^2$. The inequality (1.3.2) holds for all $c_1, c_2 \in \mathbb{R}_+$ and $\zeta_1, \zeta_2 \geq 0$ with $\zeta_1 + \zeta_2 \leq 1$ iff $\mathscr{G}(0) = 0$.*

The geometric description of s-convex curve, given in the definition below, was clearly explained in Pinheiro (2007).

Definition 1.8 *A function $\mathscr{G} : V \subseteq \mathbb{R} \to \mathbb{R}$ is called an s-convex in the second sense for $0 < s < 1$, if the graph of the function is below a bent chord L that is between any two points. This means that, for every compact interval $W \subset V$, the following inequality*

$$\sup_{W}(L - \mathscr{G}) \geq \sup_{\partial W}(L - \mathscr{G})$$

holds, with boundary ∂W.

The s-convex function of second sense can be referred as the limiting curve. This differentiates the curves of s-convex in second sense from others which are not. Following this, Pinheiro determines the affects of the choice of s on the limiting curve.

For further results on s-convex function in the second sense, we refer the reader to Suneja et al. (1993), Dragomir and Fitzpatrick (2000), Alomari and Darus (2008), Dragomir (2016), Shuang et al. (2013) and Li and Du (2017).

1.3.3 Generalized s-convex in the second sense on fractal sets

The definition of the generalized s-convex function on fractal sets is given as follows.

Definition 1.9 *(Mo et al., 2014) A function $\mathscr{G} : V \subseteq \mathbb{R}_+ \to \mathbb{R}^\alpha$ is a generalized s-convex in the second sense on fractal sets if*

$$\mathscr{G}(\zeta_1 m_1 + \zeta_2 m_2) \leq \zeta_1^{\alpha s}\mathscr{G}(m_1) + \zeta_2^{\alpha s}\mathscr{G}(m_2) \tag{1.3.3}$$

holds, for all $m_1, m_2 \in V$, $0 < s \leq 1$, $\zeta_1, \zeta_2 \geq 0$ and $\zeta_1 + \zeta_2 = 1$. This class of function is denoted by GK_s^2.

The generalized s-convex function in the second sense becomes s-convex function when $\alpha = 1$.

One should note that the following theorems along with example can be found in Mo et al. (2014).

Theorem 1.8 *Let $\mathscr{G} \in GK_s^2$. Inequality (1.3.3) holds for all $m_1, m_2 \in \mathbb{R}_+$ and $\zeta_1, \zeta_2 \geq 0$ with $\zeta_1 + \zeta_2 < 1$ iff $\mathscr{G}(0) = 0^\alpha$.*

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Theorem 1.9 *Let* $0 < s < 1$. *If* $\mathscr{G} \in GK_s^2$, *then* $\mathscr{G}$ *is non-negative on* $[0, +\infty)$.

Theorem 1.10 *Let* $0 < s_1 \leq s_2 \leq 1$. *If* $\mathscr{G} \in GK_{s_2}^2$ *and* $\mathscr{G}(0) = 0^\alpha$, *then* $\mathscr{G} \in GK_{s_1}^2$.

Considering the properties of the generalized s-convex in the second sense, we present the following example.

Example 1.4 *Let* $0 < s < 1$, *and* $a_1^\alpha, a_2^\alpha, a_3^\alpha \in \mathbb{R}^\alpha$. *For* $m \in \mathbb{R}_+$, *we define*

$$\mathscr{G}(m) = \begin{cases} a_1^\alpha, & m = 0 \\ a_2^\alpha m^{s\alpha} + a_3^\alpha, & m > 0. \end{cases}$$

Thus, we have the following:

i. If $a_2^\alpha \geq 0^\alpha$ *and* $0^\alpha \leq a_3^\alpha \leq a_1^\alpha$, *then* $\mathscr{G} \in GK_s^2$,

ii. If $a_2^\alpha > 0^\alpha$ *and* $a_3^\alpha < 0^\alpha$, *then* $\mathscr{G} \notin GK_s^2$.

For more results related to the generalized s-convex in the second sense on fractal sets , the interested reader is directed to Kılıçman and Saleh (2015a), Budak et al. (2016) and Partap et al. (2019).

1.3.4 s-Godunova-Levin function

In order to unify the concepts of Godunova-Levin and P-functions, Dragomir (2016) introduced s-Godunova-Levin as follows.

Definition 1.10 *A function* $\mathscr{G} : V \subseteq \mathbb{R} \to [0, \infty)$ *is said to be s-Godunova-Levin, (denoted by* $\mathscr{G} \in Q_{s_2}(V)$*), if*

$$\mathscr{G}(\zeta m_1 + (1-\zeta)m_2) \leq \frac{1}{\zeta^s}\mathscr{G}(m_1) + \frac{1}{(1-\zeta)^s}\mathscr{G}(m_2) \tag{1.3.4}$$

holds, for all $m_1, m_2 \in V$, $\zeta \in (0,1)$ *and* $0 \leq s \leq 1$.

Choosing $s = 1$ reduces s-Godunova-Levin function to the the class of Godunova-Levin. Also, taking $s = 0$ we have the class of P-function. Thus, we have the following, $P(V) = Q_0(V) \subseteq Q_{s_2}(V) \subseteq Q_1(V) = Q(V)$. For more results on s-Godunova-Levin of convexity, we refer the reader to Noor et al. (2014), Park (2015) and Kashuri and Liko (2018).

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1.3.5 Preinvex function

Preinvex functions are among the most important classes of generalized convex functions. This concept, playing important roles in many disciplines, was proposed by Ben-Israel and Mond (1986). Since then, a preinvex function has become an active area of study.

Definition 1.11 *(Hanson, 1981) A set $V \subseteq \mathbb{R}$ is called an invex if there exists a function $\eta : V \times V \to \mathbb{R}$ such that*

$$m_1 + \zeta \eta(m_2, m_1) \in V$$

holds, for all $m_1, m_2 \in V$ and $\zeta \in [0,1]$.

The invex set V can also referred to as an η-connected set.

Definition 1.12 *(Ben-Israel and Mond, 1986) Suppose that $V \subseteq \mathbb{R}$ is an invex set with respect to $\eta : V \times V \to \mathbb{R}$. A function $\mathscr{G} : V \to \mathbb{R}$ is called preinvex with respect to η, if*

$$\mathscr{G}(m_1 + \zeta \eta(m_2, m_1)) \leq (1-\zeta)\mathscr{G}(m_1) + \zeta \mathscr{G}(m_2) \tag{1.3.5}$$

holds, for all $m_1, m_2 \in V$ and $\zeta \in [0,1]$.

Further generalizations can be found in Weir and Mond (1988); Matloka (2014); Jue-You (2010); Meftah et al. (2017); Meftah and Souahi (2019).

Thus far, this section is concerned with the descriptions of different types of convexities: Godunova-Levin, P-function, s-convex in the second sense, s-Godunova-Levin, generalized convex function on fractal sets, generalized s-convex function in the second sense on fractal sets and preinvex function. In order to observe the relations between these functions, we combine them in Figure 1.2.

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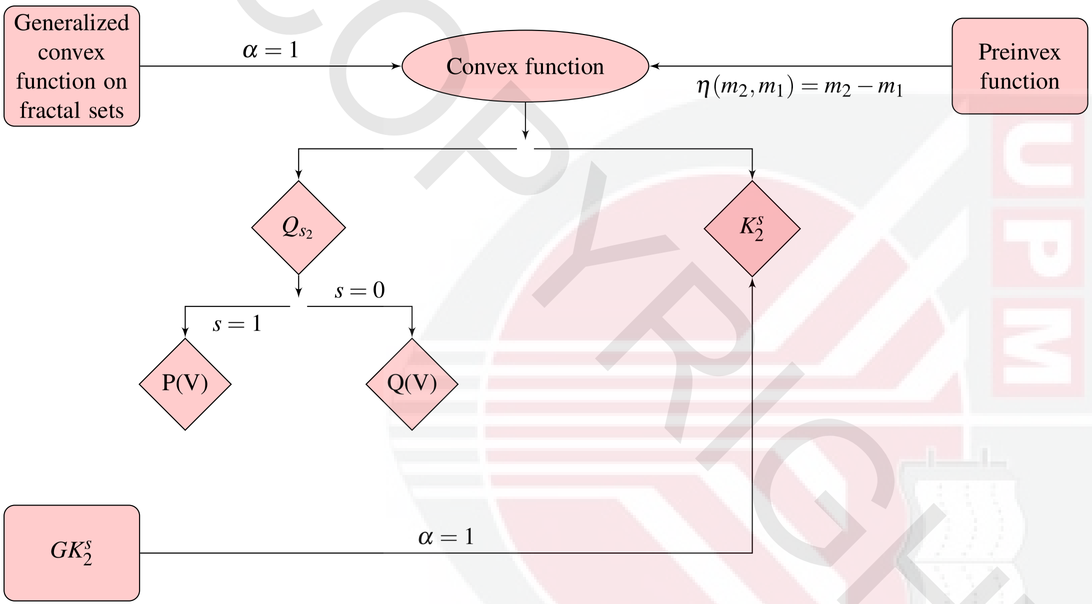

Figure 1.2: The relations between different types of convexities

1.4 Hermite-Hadamard inequality

H-H inequality plays a vital role in the theory of convexity. This inequality estimates the integral average of any convex functions through the midpoint and trapezoidal formula of a given domain. While the midpoint formula estimates the integral from the left, the trapezoidal formula estimates it from the right. More precisely, the classical H-H inequality is considered as follows.

Theorem 1.11 *(Dragomir and Pearce, 2004) Let $\mathscr{G} : [m_1, m_2] \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on $[m_1, m_2]$ with $m_1 < m_2$, then*

$$(m_2 - m_1)\mathscr{G}\left(\frac{m_1 + m_2}{2}\right) \leq \int_{m_1}^{m_2} \mathscr{G}(x)dx \leq (m_2 - m_1)\frac{\mathscr{G}(m_1) + \mathscr{G}(m_2)}{2} \tag{1.4.1}$$

holds.

The proof of inequality (1.4.1) is provided here for simplicity. Though the proof of the theorem exists, this is the first time (1.4.1) is proved using a similar technique reported in Sarikaya et al. (2012).

Proof:
Let $\mathscr{G}$ be a convex function on the interval $[m_1, m_2]$. Taking $\zeta = \frac{1}{2}$ in inequality (1.2.1) for $x, y \in [m_1, m_2]$, we have

$$\mathscr{G}\left(\frac{x + y}{2}\right) \leq \frac{\mathscr{G}(x) + \mathscr{G}(y)}{2}. \tag{1.4.2}$$

Substituting $x = \zeta m_1 + (1 - \zeta)m_2$ and $y = (1 - \zeta)m_1 + \zeta m_2$ in (1.4.2), we get

$$2\mathscr{G}\left(\frac{m_1 + m_2}{2}\right) \leq \mathscr{G}(\zeta m_1 + (1 - \zeta)m_2) + \mathscr{G}((1 - \zeta)m_1 + \zeta m_2). \tag{1.4.3}$$

Integrating inequality (1.4.3) with respect to ζ over $[0, 1]$, we have

$$\begin{aligned} 2\mathscr{G}\left(\frac{m_1 + m_2}{2}\right) &\leq \int_0^1 \mathscr{G}(\zeta m_1 + (1 - \zeta)m_2)d\zeta + \int_0^1 \mathscr{G}((1 - \zeta)m_1 + \zeta m_2)d\zeta \\ &= \frac{2}{m_2 - m_1}\int_{m_1}^{m_2} \mathscr{G}(x)dx. \end{aligned} \tag{1.4.4}$$

In order to prove the second part of inequality (1.4.1), we used Definition 1.1, for $\zeta \in [0, 1]$ to arrive at

$$\mathscr{G}(\zeta m_1 + (1 - \zeta)m_2) \leq \zeta\mathscr{G}(m_1) + (1 - \zeta)\mathscr{G}(m_2)$$

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and

$$\mathscr{G}((1-\zeta)m_1+\zeta m_2) \leq (1-\zeta)\mathscr{G}(m_1)+\zeta\mathscr{G}(m_2).$$

When the above inequalities are added, we obtain the following

$$\begin{aligned}\mathscr{G}(\zeta m_1+&(1-\zeta)m_2)+\mathscr{G}((1-\zeta)m_1+\zeta m_2)\\&\leq \zeta\mathscr{G}(m_1)+(1-\zeta)\mathscr{G}(m_2)+(1-\zeta)\mathscr{G}(m_1)+\zeta\mathscr{G}(m_2).\end{aligned} \tag{1.4.5}$$

Integrating inequality (1.4.5) with respect to ζ over $[0,1]$, we have

$$\begin{aligned}\int_0^1 \mathscr{G}(\zeta m_1+(1-\zeta)m_2)d\zeta+\int_0^1 \mathscr{G}((1-\zeta)m_1+\zeta m_2)d\zeta\\ \leq [\mathscr{G}(m_1)+\mathscr{G}(m_2)]\int_0^1 d\zeta.\end{aligned}$$

Thus,

$$\frac{2}{m_2-m_1}\int_{m_1}^{m_2}\mathscr{G}(x)dx \leq \mathscr{G}(m_1)+\mathscr{G}(m_2)$$

completes the proof. □

The H-H inequality is geometrically described in Niculescu and Persson (2006), and we have summarized it as follows:

The area under the graph of $\mathscr{G}$ on $[m_1,m_2]$ is between the areas of two trapeziums. While the area of the first trapezium is formed by the points of coordinates $(m_1,\mathscr{G}(m_1)),(m_2,\mathscr{G}(m_2))$ with the $x-$axis, that of the second trapezium is formed by the tangent to the graph of $\mathscr{G}$ at $\left(\frac{m_1+m_2}{2},\mathscr{G}\left(\frac{m_1+m_2}{2}\right)\right)$ with the $x-$axis (see Figure 1.3).

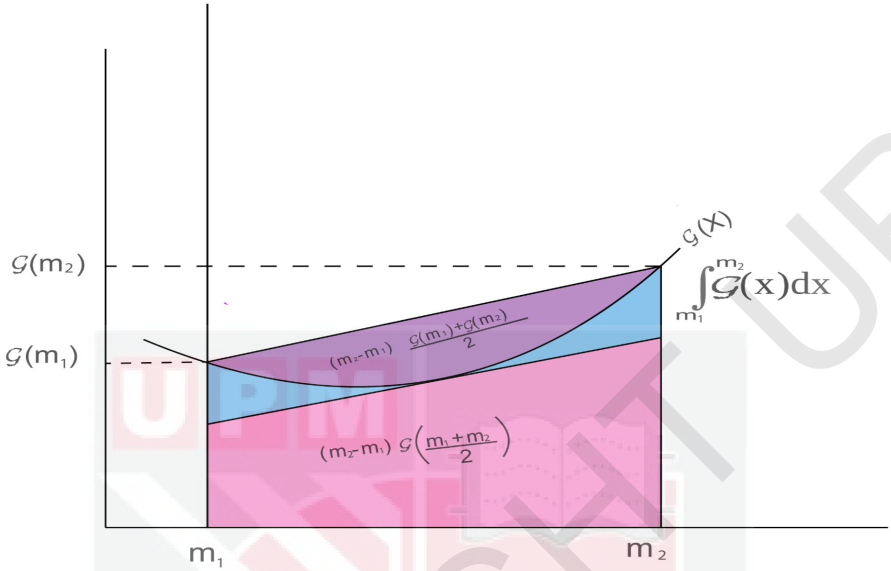

Figure 1.3: The geometrical representation of inequality (1.4.1)

The example of H-H inequality is given as follows.

Example 1.5 *(Niculescu and Persson, 2004) If we choose $\mathcal{G} = e^x$ with $x \in \mathbb{R}$, the H-H inequality yields*

$$e^{(m_1+m_2)/2} < \frac{e^{m_2} - e^{m_1}}{m_2 - m_1} < \frac{e^{m_1} + e^{m_2}}{2},$$

for $m_1 < m_2$ in $\mathbb{R}$.

For more examples of H-H inequality, see Khattri (2010) and Dragomir and Pearce (2004).

1.5 Characterisations of Convexity via H-H inequality

The importance of the H-H inequality is that each of its two sides is characterized a convex function. The necessary and sufficient condition for a continuous function $\mathcal{G}$ to be convex on (m_1, m_2) is given in the following theorem.

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Theorem 1.12 *(Hardy et al., 1952) Let $\mathscr{G}$ be a continuous function on (m_1, m_2). Then $\mathscr{G}$ is convex iff*

$$\mathscr{G}(x) \leq \frac{1}{2z} \int_{x-z}^{x+z} \mathscr{G}(\zeta) d\zeta, \tag{1.5.1}$$

for $m_1 \leq x - z \leq x \leq z + k \leq m_2$.

It can be shown that inequality (1.5.1) is equivalent to the first part of (1.4.1) when $\mathscr{G}$ is continuous on $[m_1, m_2]$ (see Dragomir and Pearce, 2004).

The second part of inequality (1.4.1) can be applied as a convexity criterion in the following theorem.

Theorem 1.13 *(Robert and Varberg, 1973) Let $\mathscr{G}$ be continuous function on $[m_1, m_2]$. Then $\mathscr{G}$ is convex iff*

$$\frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \mathscr{G}(x) dx \leq \frac{\mathscr{G}(a_1) + \mathscr{G}(a_2)}{2},$$

for all $m_1 < a_1 < a_2 < m_2$.

1.6 Fractional calculus

Recently, integral inequalities have been studied, by many researchers, using fractional calculus, which is concerned with derivatives and integrals of non-integer order. The derivatives of fractional calculus are defined via fractional integrals. Fractional calculus have been receiving more attention due to their applications in different fields of science and technology (see Cafagna, 2007; Machado et al., 2011; Yang, 2019; Hilfer, 2019).

In order to understand the fractional calculus, three types of special functions, Euler Gamma, Beta and generalized Beta functions, are introduced here due to their importance.

Definition 1.13 *The Euler Gamma function, which is a generalization of factorial function, is defined by*

$$\Gamma(m) = \int_0^\infty r^{m-1} \mathrm{e}^{-r} dr, \quad \text{for } m > 0.$$

For instance, $\Gamma(1) = 1, \Gamma(2) = 1$ and $\Gamma(3/2) = \frac{\sqrt{\pi}}{2}$.

Corollary 1.1 *For $m \in \mathbb{N}$, we have $\Gamma(m+1) = m!$.*

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One should note that this is not the only definition of a gamma function. Other definitions also exist except for a non-negative integer. For example, a gamma function for a complex number can also be defined (see Andrews et al., 1999; Sebah and Gourdon, 2002; Thukral, 2014). Therefore, the gamma function, appearing in most fractional integrals, is studied along with its properties by many researchers.

A basic property of Γ that is frequently used in this study can be easily shown through the integration by parts,

$$\Gamma(m+1) = m\Gamma(m).$$

Meanwhile, the Beta function is defined as follows:

Definition 1.14 *The Beta function β is given as*

$$\beta(m_1, m_2) = \int_0^1 r^{m_1 - 1}(1-r)^{m_2 - 1} dr, \quad m_1, m_2 > 0.$$

Mostly, the notation $\beta(m_1, m_2)$ is conveniently used to replace the combination of Gamma function. The relation between the two functions is given by Gradshteyn and Ryzhik (1980). One should note that the property $\beta(m_1, m_2) = \beta(m_2, m_1)$ guarantees that the Beta function is symmetric.

The generalization of the Beta function can be written as

$$\beta_\rho(m_1, m_2) = \int_0^1 \rho\left(1 - r^\rho\right)^{m_2 - 1}\left(r^\rho\right)^{m_1 - 1} r^{\rho - 1} dr, \quad m_1, m_2, \rho > 0.$$

Note that, as $\rho \to 1, \beta_\rho(m_1, m_2) \to \beta(m_1, m_2)$.

Now, we recall some basic definitions of fractional integrals that can be further used when developing some new results in this thesis. Important references include the monographs of Baleanu et al. (2011), Nigmatullin (1992), Kilbas et al. (2006), Diethelm (2010), Tarasov (2011), Malinowska et al. (2015), Salati et al. (2019), Garrappa et al. (2019), Hilfer and Luchko (2019), Das (2020) and Soradi-Zeid et al. (2020) are available for further reading. Also, the results reported in these references are considered as important discoveries on fractional calculus.

The next part of this section introduces the Riemann-Liouville fractional integrals, which can be severally used in other parts of this work.

Definition 1.15 *Let $\mathcal{G} \in L_1[m_1, m_2]$. The left and right sides Riemann-Liouville in-*

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tegrals denoted by $J^{\lambda}_{m_1^+}\mathscr{G}$ *and* $J^{\lambda}_{m_2^-}\mathscr{G}$ *of order* $\lambda \in \mathbb{R}_+$ *are defined by*

$$J^{\lambda}_{m_1^+}\mathscr{G}(x) = \frac{1}{\Gamma(\lambda)}\int_{m_1}^{x}(x-\gamma)^{\lambda-1}\mathscr{G}(\gamma)d\gamma, \quad x > m_1$$

and

$$J^{\lambda}_{m_2^-}\mathscr{G}(x) = \frac{1}{\Gamma(\lambda)}\int_{x}^{m_2}(\gamma-x)^{\lambda-1}\mathscr{G}(\gamma)d\gamma, \quad x < m_2,$$

respectively.

If $\lambda = 1$ in the above equalities, we get the classical integral.

One should note that the Hadamard fractional integrals differ from those of the Riemann-Liouville since in the former the logarithmic functions of arbitrary exponents are included in the kernels of the integrals. Therefore, the Hadamard fractional integrals are defined as follows.

Definition 1.16 *(Samko et al., 1993) Let* $\lambda > 0$ *with* $m-1 < \lambda \leq m, m \in \mathbb{N}$, *and* $m_1 < x < m_2$. *The left and right sides Hadamard fractional integrals denoted by* $H^{\lambda}_{m_1^+}\mathscr{G}(x)$ *and* $H^{\lambda}_{m_2^-}\mathscr{G}(x)$ *of order* λ *of a function* $\mathscr{G}$ *are given as*

$$H^{\lambda}_{m_1^+}\mathscr{G}(x) = \frac{1}{\Gamma(\lambda)}\int_{m_1}^{x}\left(\ln\frac{x}{\gamma}\right)^{\lambda-1}\frac{\mathscr{G}(\gamma)}{\gamma}d\gamma$$

and

$$H^{\lambda}_{m_2^-}\mathscr{G}(x) = \frac{1}{\Gamma(\lambda)}\int_{x}^{m_2}\left(\ln\frac{\gamma}{x}\right)^{\lambda-1}\frac{\mathscr{G}(\gamma)}{\gamma}d\gamma,$$

respectively.

Anatoly (2001), Butzer et al. (2002a,b) and Kilbas et al. (2006) provide useful background and properties of Hadamard fractional integrals.

The following proposition is related to the Hadamard integrals.

Proposition 1.1 *(Kilbas et al., 2006) If* $\lambda > 0$ *and* $0 < m_1 < m_2 < \infty$, *the following relations hold:*

$$\left(H^{\lambda}_{m_1^+}\left(\log\frac{\gamma}{m_1}\right)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\lambda)}\left(\log\frac{x}{m_1}\right)^{\beta+\lambda-1}$$

and

$$\left(H^{\lambda}_{m_2^-}\left(\log\frac{m_2}{\gamma}\right)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\lambda)}\left(\log\frac{m_2}{x}\right)^{\beta+\lambda-1}.$$

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The Riemann-Liouville fractional integrals, along with the Hadamard's fractional integrals, are generalized through the recent work of Katugampola (2015). These two integrals were combined and given in a single form. The following definition Katugampola (2015) modifies the old version Katugampola (2011) for Katugampola fractional integrals.

Definition 1.17 *Let $[m_1, m_2] \subset \mathbb{R}$ be a finite interval. The left and right-sided Katugampola fractional integrals of order $\lambda > 0$ for $\mathscr{G} \in X_c^p(m_1, m_2)$ are defined by*

$$^{\rho}I^{\lambda}_{m_1^+}\mathscr{G}(x) = \frac{\rho^{1-\lambda}}{\Gamma(\lambda)} \int_{\mathrm{m}_1}^{x} \frac{\gamma^{\rho-1}}{(x^{\rho} - \gamma^{\rho})^{1-\lambda}} \mathscr{G}(\gamma) d\gamma$$

and

$$^{\rho}I^{\lambda}_{m_2^-}\mathscr{G}(x) = \frac{\rho^{1-\lambda}}{\Gamma(\lambda)} \int_{x}^{m_2} \frac{\gamma^{\rho-1}}{(\gamma^{\rho} - x^{\rho})^{1-\lambda}} \mathscr{G}(\gamma) d\gamma,$$

with $m_1 < x < m_2$ and $\rho > 0$.

Following this, the space $X_c^p(m_1, m_2)(c \in \mathbb{R}, 1 \leq p \leq \infty)$ is introduced as follows.

Definition 1.18 *(Anatoly, 2001) Let the space $X_c^p(m_1, m_2)(c \in \mathbb{R}, 1 \leq p \leq \infty)$ of those complex-valued Lebesgue measurable functions $\mathscr{G}$ on $[m_1, m_2]$ for which $|\mathscr{G}|_{X_c^p} < \infty$, where the norm is defined by*

$$|\mathscr{G}|_{X_c^p} = \left(\int_{m_1}^{m_2} |\zeta^c \mathscr{G}(\zeta)|^p \frac{d\zeta}{\zeta} \right)^{1/p} < \infty \quad (1 \leq p < \infty, c \in \mathbb{R})$$

and for the case $p = \infty$

$$|\mathscr{G}|_{X_c^{\infty}} = \operatorname{ess} \sup_{m_1 \leq \zeta \leq m_2} \left(\zeta^c |\mathscr{G}(\zeta)| \right) \quad (c \in \mathbb{R}),$$

where ess sup $|\mathscr{G}(\zeta)|$ stands for the essential maximum of $|\mathscr{G}(\zeta)|$.

If $c = 1/p$, $(X_c^p(m_1, m_2))$ reduces to $(L_p(m_1, m_2))$, the p-integrable function.

Important references on Katugampola fractional integrals and their applications are suggested for further reading (Butkovskii et al., 2013; Gaboury et al., 2013; Richard, 2014; Katugampola, 2014).

The relations among Katugampola fractional integrals, Riemann-Liouville integrals and Hadamard integrals are given in the next theorem. The left-sided version of the relation is considered here for its simplicity since similar results also exist for the right-sided operators.

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Theorem 1.14 *(Katugampola, 2014) Let $\lambda > 0$ and $\rho > 0$. Then for $x > m_1$, we have*

i. $\lim_{\rho\to 1} {}^{\rho}I^{\lambda}_{m_1^+}\mathscr{G}(x) = J^{\lambda}_{m_1^+}\mathscr{G}(x)$,

ii. $\lim_{\rho\to 0^+} {}^{\rho}I^{\lambda}_{m_1^+}\mathscr{G}(x) = H^{\lambda}_{m_1^+}\mathscr{G}(x)$.

Remark 1.1 *One should note that, while (i) is concerned with the Riemann-Liouville operators, (ii) is related to the Hadamard operators.*

1.7 Hölder integral inequality

The Hölder integral inequality plays an important role in both pure and applied sciences. Other areas of applying this inequality include the theory of convexity, which can be considered as one of the active and fast growing fields of studies in mathematical sciences. Thus, the Hölder's integral inequality is described in the following theorem.

Theorem 1.15 *(Mitrinović and Vasic, 1970) Suppose that $p > 1$ and $1/p + 1/q = 1$. If $\mathscr{G}$ and $\mathscr{K}$ are real functions on $[m_1, m_2]$ such that $|\mathscr{G}|^p$ and $|\mathscr{K}|^q$ are integrable functions on $[m_1, m_2]$, then*

$$\int_{m_1}^{m_2} |\mathscr{G}(x)\mathscr{K}(x)|dx \leq \left(\int_{m_1}^{m_2} |\mathscr{G}(x)|^p dx\right)^{\frac{1}{p}} \left(\int_{m_1}^{m_2} |\mathscr{K}(x)|^q dx\right)^{\frac{1}{q}}$$

holds.

The other version of Hölder integral inequality is called the power-mean integral, which is given in the following theorem.

Theorem 1.16 *(Mitrinović et al., 2013) Suppose that $q \geq 1$. Let $\mathscr{G}$ and $\mathscr{K}$ be real mappings on $[m_1, m_2]$. If $|\mathscr{G}|$ and $|\mathscr{G}||\mathscr{K}|^q$ are integrable functions in the given interval, then*

$$\int_{m_1}^{m_2} |\mathscr{G}(x)\mathscr{K}(x)|dx \leq \left(\int_{m_1}^{m_2} |\mathscr{G}(x)|dx\right)^{1-\frac{1}{q}} \left(\int_{m_1}^{m_2} |\mathscr{G}(x)||\mathscr{K}(x)|^q dx\right)^{\frac{1}{q}}$$

holds.

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1.8 Motivation

As mentioned earlier, H-H inequality plays fundamental roles in both pure and applied sciences. This cannot be overemphasized since the inequality frequently serves as a basis of many modelling problems. Extensions and generalizations of H-H inequality remain an active area of studies since several journal articles, books and monographs are devoted to reporting new findings connected to such inequalities. For example, the new fractional inequalities were produced via H-H type involving Riemann-Liouville fractional integrals (Sarikaya et al., 2012; Dragomir, 2017b). However, new results of H-H type inequalities involving Katugampola fractional integrals are lacking (Chen and Katugampola, 2017). Therefore, this study is concerned with the generalization of H-H type inequalities via different fractional integrals including Katugampola type.

1.9 Problem Statement

The problem, in this study, is to formulate generalized H-H inequality for several forms of convex functions via different types of fractional integrals, such as Riemann-Liouville and Katugampola. Other problems include the estimation of mid-point type and trapezoid type inequalities connected with inequality (1.4.1). Furthermore, H-H type inequalities play a vital role in estimating error bounds of quadrature formula for numerical integrations (Dragomir and Agarwal, 1998; Wang and Feckan, 2018; Mehrez and Agarwal, 2019). Since the error bound estimate using Taylor's expansion involves the second derivative, H-H type inequalities can improve the estimate by considering only the first derivative. This would involve less partitioning points when compared to Taylor's methods (Dragomir and Wang, 1998; Agarwal et al., 2018).

1.10 Objectives

The main objective of this study is to generalize H-H type inequalities including their applications. These can be achieved through the following objectives.

1. To generalize integral inequalities of H-H type via s-convex function in the second sense.
2. To establish new integral inequalities of H-H type via new classes of convexities.
3. To improve the inequalities of H-H type via Riemann-Liouville integrals for generalized s-convex function on fractal sets.

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4. To generalize H-H type inequalities for classical convex and generalized convex functions on fractal sets involving Katugampola fractional integrals.

5. To extend H-H type inequalities via Katugampola fractional integrals for generalized s-convex function on fractal sets.

6. To construct inequalities for special means of real numbers and estimate error bounds of quadrature formula for the numerical integration.

The contributions of this thesis are given in Figures 1.4 and 1.5, respectively.

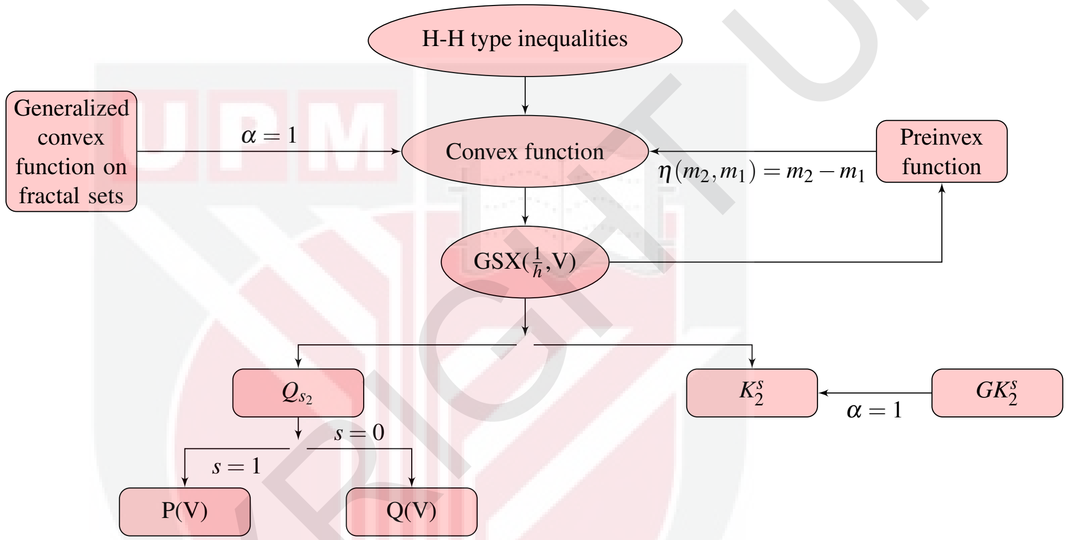

Figure 1.4: The generalization of H-H type inequalities for different types of convexities

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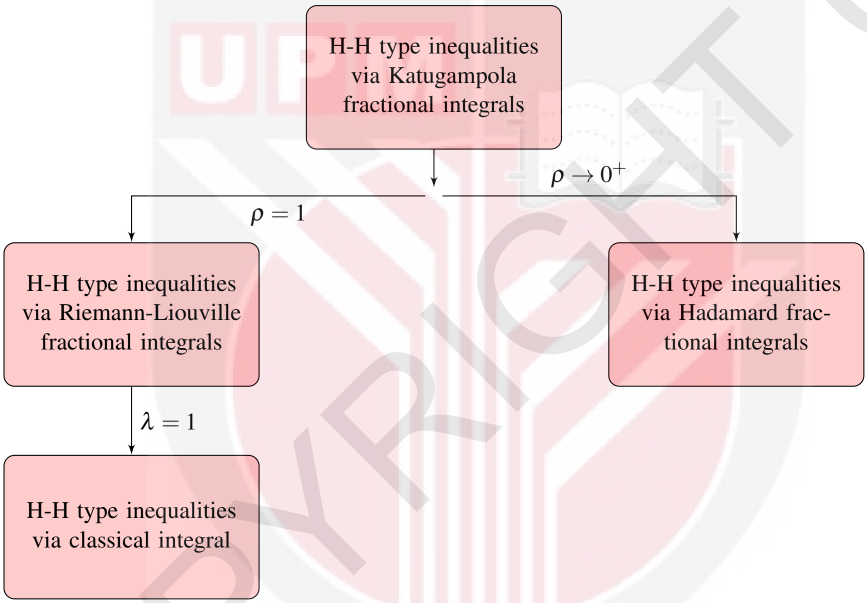

Figure 1.5: The generalization of H-H type inequalities via different fractional integrals

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1.11 Research Methodology

Different types of fractional integrals are used to establish variant inequalities of H-H type involving different types of convexities. The new H-H type inequalities derived though different convexities, as well as fractional integrals, can be generalized by applying Hölder's and power-mean inequalities to the corresponding powers. Furthermore, some special means of real numbers are derived, and error bounds to some quadrature rules for numerical integration are also estimated.

1.12 Outline of the thesis

This thesis is devoted to the generalization and improvement of H-H type inequalities for different types of convexities and different fractional integrals, together with their applications. In order to achieve the goal of this thesis, the following outlines are given:

In Chapter 1, we present a general introduction of inequalities and theory of convex functions. This serves as the basis for understanding the subsequent chapters of this thesis. Basic concepts presented in this chapter include results of generalized convex function on fractal sets, s-convex in the second sense, generalized s-convex in the second sense on fractal sets, P-function, Godunova-Levin and preinvex functions including some of their properties. We equally investigate some properties that deal with geometrical interpretation of convex and s-convex functions. The developments of concepts that can lead to the generalization of different type of convex functions are outlined. We also give some background on fractional calculus. Some known inequalities of Hölder and power-mean are introduced. This chapter also presents the motivation and objectives of the whole study.

Chapter 2 focuses on the review of previous studies conducted by other researchers. Some of these previously published works are extended and generalized by most of our results. Thus, literature on some H-H type inequalities, along with their related generalizations and refinements, are rigorously conducted. Works on the special means of real numbers and the quadrature formula for the numerical integration are also presented.

In Chapter 3, new integral inequalities for s-convexity linked with H-H inequality are established. Some of the results we obtained in this chapter are the generalizations of inequalities in Mehrez and Agarwal (2019). Also, when numerically comparing the findings of the studies, our results performed better than those reported in Mehrez and Agarwal (2019). As an application, the inequalities for special means are derived. Error estimates for the midpoint formula are also studied in this chapter.

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Chapter 4 deals with the study of new classes of convexity called h-Godunova-Levin and h-Godunova-Levin preinvexity. Using these new classes, a number of new inequalities of H-H type are established. This chapter is organized through the following steps:

-Firstly, the new classes of h-Godunova-Levin, denoted by $SGX\left(\frac{1}{h},\zeta\right)$, are introduced, along with their properties. This class of function unifies different classes of convexities: s-Godunova-Levin, P-function, s-convexity and Godunova-Levin. We further prove new H-H type inequalities via h-Godunova-Levin.

-Secondly, we introduce a new definition of h-Godunova-Levin preinvexity, denoted by $SGXP\left(\frac{1}{h},\zeta\right)$, which can be the generalization of preinvexity. Also, we present new H-H type inequalities for h-Godunova-Levin preinvexity.

Finally, some applications to special means and application to numerical integration are given.

In Chapter 5, we present new H-H type inequalities via Riemann-Liouville integrals of a function $\mathscr{G}$ taking its value in a fractal subset of $\mathbb{R}$. This function also possesses an appropriate generalized s-convexity on fractal sets. It is shown that these fractal inequalities give rise to a generalized s-convexity, a property of $\mathscr{G}$. We also prove certain inequalities involving Riemann-Liouville integrals of a function $\mathscr{G}$ provided that the absolute value of the first or second order derivative of $\mathscr{G}$ possesses an appropriate fractal s-convexity. We show that the newly established inequalities are the generalizations of those in Dragomir and Fitzpatrick (1999) and Set et al. (2014).

Chapter 6 defines a new identity for the Katugampola fractional integrals. Using this identity, we studied a new integral inequality for a function whose first derivative in absolute value is convex. The new generalized H-H inequality for generalized convex function on fractal sets involving Katugampola type fractional integral is established. This can be the generalization of the work of Chen and Katugampola (2017). The trapezoid and mid-point type inequalities are also proposed for the generalized convex function involving Katugampola fractional integrals. This, in a single form, would generalize the Riemann-Liouville and the Hadamard integrals.

Chapter 7 considers some new integral inequalities for generalized s-convexity via Katugampola fractional integrals on fractal sets linked with the H-H inequality. We present some inequalities for the type of mappings whose derivatives in absolute value are the generalized s-convexity. In addition, we obtain some new inequalities linked with convexity and generalized s-convexity via classical integrals as well as Riemann-Liouville fractional integrals in form of a corollary. As applications, the inequalities for special means are derived.

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In Chapter 8, we summarize the entire work by recalling some important results obtained in this study. We equally recommend further studies for some open-ended problems.

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REFERENCES

Abbas, G. and Farid, G. (2016). Some integral inequalities for m-convex functions via generalized fractional integral operator containing generalized mittag-leffler function. *Cogent Mathematics*, 3(1):1269589.

Agarwal, P. (2017). Some inequalities involving hadamard-type k-fractional integral operators. *Mathematical Methods in the Applied Sciences*, 40(11):3882–3891.

Agarwal, P., Dragomir, S. S., Jleli, M., and Samet, B. (2018). *Advances in mathematical inequalities and applications*. Springer.

Aigner, M., Ziegler, G. M., Hofmann, K. H., and Erdos, P. (2010). *Proofs from the Book*, volume 274. Springer.

Ali, T., Khan, M. A., and Khurshidi, Y. (2017). Hermite-hadamard inequality for fractional integrals via eta-convex functions. *Acta Mathematica Universitatis Comenianae*, 86(1):153–164.

Alomari, M. and Darus, M. (2008). The hadamard's inequality for s-convex function of 2-variables on the co-ordinates. *International Journal of Math. Analysis*, 2(13):629–638.

Alomari, M. W., Darus, M., and Kirmaci, U. S. (2011). Some inequalities of hermite-hadamard type for s-convex functions. *Acta Mathematica Scientia*, 31(4):1643–1652.

Anatoly, A. K. (2001). Hadamard-type fractional calculus. *Journal of the Korean Mathematical Society*, 38(6):1191–1204.

Andrews, G. E., Askey, R., and Roy, R. (1999). *Special functions*, volume 71. Cambridge University Press.

Bainov, D. D. and Simeonov, P. S. (2013). *Integral inequalities and applications*, volume 57. Springer Science and Business Media.

Baleanu, D., Güvenç, Z. B., Machado, J. T., et al. (2010). *New trends in nanotechnology and fractional calculus applications*. Springer.

Baleanu, D., Machado, J. A. T., and Luo, A. C. (2011). *Fractional dynamics and control*. Springer Science and Business Media.

Banic, S. (2009). Mappings connected with hermite-hadamard inequalities for superquadratic functions. *Journal of Mathematical Inequalities*, 3(4):577–589.

Barani, A. and Barani, S. (2012). Hermite–hadamard type inequalities for functions when a power of the absolute value of the first derivative is p-convex. *Bulletin of the Australian Mathematical Society*, 86(1):126–134.

Barani, A., Ghazanfari, A. G., and Dragomir, S. S. (2012). Hermite-hadamard inequality for functions whose derivatives absolute values are preinvex. *Journal of Inequalities and Applications*, 2012(1):247.

© COPYRIGHT UPM

Bekar, K. (2019). Hermite-hadamard type inequalities for trigonometrically p–functions. *Comptes rendus de l'Academie bulgare des Sciences*, 72(11).

Ben-Israel, A. and Mond, B. (1986). What is invexity? *The ANZIAM Journal*, 28(1):1–9.

Bin-Mohsin, B., Awan, M. U., Noor, M. A., Riahi, L., Noor, K. I., and Almutairi, B. (2019). New quantum hermite-hadamard inequalities utilizing harmonic convexity of the functions. *IEEE Access*, 7:20479–20483.

Borwein, J. M., Vanderwerff, J. D., et al. (2010). *Convex functions: constructions, characterizations and counterexamples*, volume 109. Cambridge University Press Cambridge.

Breckner, W. W. (1978). Stetigkeitsaussagen für eine klasse verallgemeinerter konvexer funktionen in topologischen linearen räumen. *Pupl. Inst. Math*, 23:13–20.

Budak, H. and Sarikaya, M. Z. (2016). On hermite-hadamard type inequalities for s-convex mappings via fractional integrals of a function with respect to another function. *Fasciculi Mathematici*, 57(1):25–36.

Budak, H., Sarikaya, M. Z., and Set, E. (2016). Generalized ostrowski type inequalities for functions whose local fractional derivatives are generalized s-convex in the second sense. *Journal of Applied Mathematics and Computational Mechanics*, 15(4):11–21.

Bullen, P. (2003). Handbook of means and their inequalities, vol. 260 kluwer academic publisher. *Dordrecht, The Netherlands*.

Bullen, P. S., Mitrinović, D. S., and Vasic, M. (2013). *Means and their inequalities*, volume 31. Springer Science and Business Media.

Butkovskii, A., Postnov, S. S., and Postnova, E. (2013). Fractional integro-differential calculus and its control-theoretical applications. i. mathematical fundamentals and the problem of interpretation. *Automation and Remote Control*, 74(4):543–574.

Butzer, P. L., Kilbas, A. A., and Trujillo, J. J. (2002a). Compositions of hadamard-type fractional integration operators and the semigroup property. *Journal of Mathematical Analysis and Applications*, 269(2):387–400.

Butzer, P. L., Kilbas, A. A., and Trujillo, J. J. (2002b). Fractional calculus in the mellin setting and hadamard-type fractional integrals. *Journal of Mathematical Analysis and Applications*, 269(1):1–27.

Cafagna, D. (2007). Fractional calculus: A mathematical tool from the past for present engineers [past and present]. *IEEE Industrial Electronics Magazine*, 1(2):35–40.

Chen, H. and Katugampola, U. N. (2017). Hermite-hadamard and hermite-hadamard–fejér type inequalities for generalized fractional integrals. *Journal of Mathematical Analysis and Applications*, 446(2):1274–1291.

© COPYRIGHT UPM

Chudziak, M. and Zoldak, M. (2020). Hermite–hadamard and fejér inequalities for co-ordinated (f, g)-convex functions on a rectangle. *Symmetry*, 12(1):13.

Dahmani, Z. and Belhamiti, M. M. (2020). *Integral inequalities and differential equations via fractional calculus*. IntechOpen.

Das, S. (2020). *Kindergarten of fractional calculus*. Cambridge Scholars Publishing.

de Oliveira, E. C., Jarosz, S., and Vaz Jr, J. (2019). Fractional calculus via laplace transform and its application in relaxation processes. *Communications in Nonlinear Science and Numerical Simulation*, 69:58–72.

Demuynck, T. (2009). A general extension result with applications to convexity, homotheticity and monotonicity. *Mathematical Social Sciences*, 57(1):96–109.

Deng, J. and Wang, J. (2013). Fractional hermite-hadamard inequalities for (α, m)-logarithmically convex functions. *Journal of Inequalities and Applications*, 2013(1):364.

Diethelm, K. (2010). *The analysis of fractional differential equations: An application-oriented exposition using differential operators of Caputo type*. Springer Science and Business Media.

Dragomir, S. (1991). A mapping in connection to hadamard's inequalities. *Akademie der Wissenschaften. Mathematisch-Naturwissenschaftliche Klasse*, 128:17–20.

Dragomir, S. (2002). New refinements of the hermite-hadamard integral inequality for convex functions and applications. *Soochow Journal of Mathematics*, 28(4):357–374.

Dragomir, S. and Agarwal, R. (1998). Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula. *Applied Mathematics Letters*, 11(5):91–95.

Dragomir, S. and Ionescu, N. (1992). Some integral inequalities for differentiable convex functions. *Contributions, Macedonian Acad. of Sci. and Arts*, 13(1):13–17.

Dragomir, S. and Pearce, C. (2004). Selected topics on hermite–hadamard inequalities and applications. *RGMIA Monographs*.

Dragomir, S., Pecaric, J., and Persson, L.-E. (1995). Some inequalities of hadamard type. *Soochow J. Math*, 21(3):335–341.

Dragomir, S. and Wang, S. (1998). Applications of ostrowski's inequality to the estimation of error bounds for some special means and for some numerical quadrature rules. *Applied Mathematics Letters*, 11(1):105–109.

Dragomir, S. S. (1992). Two mappings in connection to hadamard's inequalities. *Journal of Mathematical Analysis and Applications*, 167(1):49–56.

Dragomir, S. S. (2003). Further properties of some mappings associated with hermite-hadamard inequalities. *Tamkang Journal of Mathematics*, 34(1):45–58.

© COPYRIGHT UPM

Dragomir, S. S. (2016). Integral inequalities of jensen type for λ-convex functions. *Matematicki Vesnik*, 68(1):45–57.

Dragomir, S. S. (2017a). Hermite-hadamard type inequalities of first kind for generalized riemann-liouville fractional integrals. *RGMIA Res Rep Coll*, 20:Art–76.

Dragomir, S. S. (2017b). Some inequalities of hermite-hadamard type for symmetrized convex functions and riemann-liouville fractional integrals. *RGMIA Res. Rep. Coll*, 20:15.

Dragomir, S. S. (2019). Hermite-hadamard type inequalities for generalized riemann-liouville fractional integrals of h-convex functions. *Mathematical Methods in the Applied Sciences*.

Dragomir, S. S. and Fitzpatrick, S. (1999). The hadamard inequalities for s-convex functions in the second sense. *Demonstratio Mathematica*, 32(4):687–696.

Dragomir, S. S. and Fitzpatrick, S. (2000). The jensen inequality for s-breckner convex functions in linear spaces. *Demonstratio Mathematica*, 33(1):43–50.

Dragomir, S. S., Milosević, D. M., and Sándor, J. (1993). On some refinements of hadamard's inequalities and applications. *Publikacije Elektrotehnickog Fakulteta. Serija Matematika*, pages 3–10.

Dragomir, S. S., Qaisar, S., Ahmad, F., and Iqbal, M. J. (2018). New hermite-hadamard inequalities via fractional integrals, whose absolute values of second derivatives is p-convex. *Journal of Mathematical Inequalities*, 12(3):655–664.

Du, T., Li, Y., and Yang, Z. (2017). A generalization of simpson's inequality via differentiable mapping using extended (s, m)-convex functions. *Applied Mathematics and Computation*, 293:358–369.

Dubey, R. S. and Goswami, P. (2019). Some fractional integral inequalities for the katugampola integral operator. *AIMS Math*, 4:193–198.

Duc, D. T., Hue, N. N., Nhan, N. D. V., and Tuan, V. K. (2020). Convexity according to a pair of quasi-arithmetic means and inequalities. *Journal of Mathematical Analysis and Applications*, page 124059.

Erdos, P. and Grunwald, T. (1939). On polynomials with only real roots. *Annals of Mathematics*, pages 537–548.

Fang, Z. B. and Shi, R. (2014). On the (p, h)-convex function and some integral inequalities. *Journal of Inequalities and Applications*, 2014(1):1–16.

Fujii, J. I., Kian, M., and Moslehian, M. S. (2011). Operator q-class functions. *Scientiae Mathematicae Japonicae*, 73(1):75–80.

Gaboury, S., Tremblay, R., and Fugère, B.-J. (2013). Some relations involving a generalized fractional derivative operator. *Journal of Inequalities and Applications*, 2013(1):167.

© COPYRIGHT UPM

Garrappa, R., Kaslik, E., and Popolizio, M. (2019). Evaluation of fractional integrals and derivatives of elementary functions: Overview and tutorial. *Mathematics*, 7(5):407.

Godunova, E. and Levin, V. (1985). Neravenstva dlja funkcii sirokogo klassa, soderzascego vypuklye, monotonnye i nekotorye drugie vidy funkii. vycislitel. mat. i. *Fiz. Mezvuzov. Sb. Nauc. Trudov, MGPI, Moskva*, 166:138–142.

Gorenflo, R., Kilbas, A. A., Mainardi, F., and Rogosin, S. V. (2014). *Mittag-Leffler functions, related topics and applications*. Springer.

Gorenflo, R. and Mainardi, F. (1997). Fractional calculus. In *Fractals and fractional calculus in continuum mechanics*, pages 223–276. Springer.

Gozpinar, A., Set, E., and Dragomir, S. S. (2019). Some generalized hermite-hadamard type inequalities involving fractional integral operator for functions whose second derivatives in absolute value are s-convex. *Acta Mathematica Universitatis Comenianae*, 88(1):87–100.

Gradshteyn, I. and Ryzhik, I. (1980). Table of integrals, series and products. *New York: Academic Press, 1980, 5th corr. and enl. ed.*

Hadamard, J. (1893). Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par riemann. *Journal de mathématiques pures et appliquées*, pages 171–216.

Hakiki, M. and Wibowo, A. (2020). Hermite–hadamard–fejèr type inequalities for s-convex functions in the second sense via riemann–liouville fractional integral. In *Journal of Physics(Conference Series)*, volume 1442, page 012039. IOP Publishing.

Hanson, M. A. (1981). On sufficiency of the kuhn-tucker conditions. *Journal of Mathematical Analysis and Applications*, 80(2):545–550.

Hardy, G., Littlewood, J., and Polya, G. (1952). Inequalities. *Cambridge Univ. Press, London and New York*, 2:151–218.

Hermite, C. (1883). Sur deux limites d'une intégrale définie. *Mathesis*, 3(1):1–82.

Hilfer, R. (2019). Mathematical and physical interpretations of fractional derivatives and integrals. *Handbook of Fractional Calculus with Applications*, 1:47–85.

Hilfer, R. and Luchko, Y. (2019). Desiderata for fractional derivatives and integrals. *Mathematics*, 7(2):149.

Hudzik, H. and Maligranda, L. (1994). Some remarks on s-convex functions. *Aequationes Mathematicae*, 48(1):100–111.

Hwang, D.-Y. and Dragomir, S. S. (2017). Extensions of the hermite-hadamard inequality for r-preinvex functions on an invex set. *Bulletin of the Australian Mathematical Society*, 95(3):412–423.

© COPYRIGHT UPM

Ion, D. A. (2007). Some estimates on the hermite-hadamard inequality through quasi-convex functions. *Annals of the University of Craiova-Mathematics and Computer Science Series*, 34:82–87.

İşcan, İ., Set, E., and Özdemir, M. E. (2014). On new general integral inequalities for s-convex functions. *Applied Mathematics and Computation*, 246:306–315.

Jensen, J. L. W. V. (1905). Om konvekse funktioner og uligheder imellem middelvaerdier. *Nyt Tidsskrift for Matematik*, 16:49–68.

Jue-You, L. (2010). On hadamard-type inequalities for s-preinvex functions. *Journal of Chongqing Normal University (Natural Science)*, 27.

Kadakal, H., Kadakal, M., and Iscan, İ. (2017). Some new integral inequalities for n-times differentiable godunova-levin functions. *Cumhuriyet Science Journal*, 38(4):1–5.

Kadakal, M. (2020). Some hermite-hadamard type inequalities for (p; m)-function and quasi m-convex functions. *An International Journal of Optimization and Control: Theories and Applications (IJOCTA)*, 10(1):78–84.

Kang, S. M., Farid, G., Nazeer, W., and Mehmood, S. (2019). $(h-m)$-convex functions and associated fractional hadamard and fejér–hadamard inequalities via an extended generalized mittag-leffler function. *Journal of Inequalities and Applications*, 2019(1):1–10.

Kashuri, A. and Liko, R. (2017). Hermite-hadamard type fractional integral inequalities for generalized (r; s, m)-preinvex functions. *European Journal of Pure and Applied Mathematics*, 10(3):495–505.

Kashuri, A. and Liko, R. (2018). Hermite-hadamard type inequalities for generalized (s,m)-preinvex godunova-levin functions. *Matematicke znanosti*, 534:63–75.

Katugampola, U. N. (2011). New approach to a generalized fractional integral. *Applied Mathematics and Computation*, 218(3):860–865.

Katugampola, U. N. (2014). A new approach to generalized fractional derivatives. *Bull. Math. Anal. Appl*, 6(4):1–15.

Katugampola, U. N. (2015). Mellin transforms of generalized fractional integrals and derivatives. *Applied Mathematics and Computation*, 257:566–580.

Khan, M. A., Chu, Y.-m., Kashurt, A., and Ltko, R. (2019). Hermite-hadamard type fractional integral. *Journal of Computational Analysis and Applications*, page 1487.

Khattri, S. K. (2010). Three proofs of the inequality. *The American Mathematical Monthly*, 117(3):273–277.

Kilbas, A. A. A., Srivastava, H. M., and Trujillo, J. J. (2006). *Theory and applications of fractional differential equations*, volume 204. Elsevier Science Limited.

© COPYRIGHT UPM

Kılıçman, A. and Saleh, W. (2015a). Notions of generalized s-convex functions on fractal sets. *Journal of Inequalities and Applications*, 2015(1):312.

Kılıçman, A. and Saleh, W. (2015b). Some generalized hermite-hadamard type integral inequalities for generalized s-convex functions on fractal sets. *Advances in Difference Equations*, 2015:1.

Kiliçman, A. and Saleh, W. (2018). Generalized preinvex functions and their applications. *Symmetry*, 10(10):493.

Kirmaci, U. S. (2004). Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula. *Applied Mathematics and Computation*, 147(1):137–146.

Kirmaci, U. S., Bakula, M. K., Özdemir, M. E., and Pecarić, J. (2007). Hadamard-type inequalities for s-convex functions. *Applied Mathematics and Computation*, 193(1):26–35.

Kunt, M., Karapinar, D., Turhan, S., and İşcan, İ. (2019). The left riemann-liouville fractional hermite-hadamard type inequalities for convex functions. *Mathematica Slovaca*, 69(4):773–784.

Latif, M. A., Dragomir, S., and Matouk, A. (2012). New inequalities of ostrowski type for co-ordinated convex functions via fractional integrals. *J. Fract. Calc. Appl*, 2(1).

Li, Y. and Du, T. (2017). A generalization of simpson type inequality via differentiable functions using extended (s, m) ϕ-preinvex functions. *J. Comput. Anal. Appl*, 22(4):613–632.

Liu, F., Cai, J., Lemieux, C., and Wang, R. (2020). Convex risk functionals: representation and applications. *Insurance- Mathematics and Economics*, 90:66–79.

Luo, C., Wang, H., and Du, T. (2020). Fejér–hermite–hadamard type inequalities involving generalized h-convexity on fractal sets and their applications. *Chaos, Solitons and Fractals*, 131:109547.

Łupińska, B. and Odzijewicz, T. (2018). A lyapunov-type inequality with the katugampola fractional derivative. *Mathematical Methods in the Applied Sciences*, 41(18):8985–8996.

Machado, J. T., Kiryakova, V., and Mainardi, F. (2011). Recent history of fractional calculus. *Communications in Nonlinear Science and Numerical Simulation*, 16(3):1140–1153.

Malinowska, A. B., Odzijewicz, T., and Torres, D. F. (2015). *Advanced methods in the fractional calculus of variations*. Springer.

Matloka, M. (2014). Inequalities for h-preinvex functions. *Applied Mathematics and Computation*, 234:52–57.

© COPYRIGHT UPM

Meftah, B., Boukerrioua, K., and Chiheb, T. (2017). Hadamard type inequalities for (s, r) preinvex functions in the first sense. *Electronic Journal of Mathematical Analysis and Applications*, 5(2):170–190.

Meftah, B. and Souahi, A. (2019). Fractional hermite-hadamard type inequalities for functions whose derivatives are extended s-(α, m)-preinvex. *An International Journal of Optimization and Control: Theories and Applications (IJOCTA)*, 9(1):73–81.

Mehreen, N. and Anwar, M. (2018). Integral inequalities for some convex functions via generalized fractional integrals. *Journal of Inequalities and Applications*, 2018(1):208.

Mehrez, K. and Agarwal, P. (2019). New hermite-hadamard type integral inequalities for convex functions and their applications. *Journal of Computational and Applied Mathematics*, 350:274–285.

Mercer, A. M. (2005). An improvement of the grüss inequality. *J. Inequal. Pure Appl. Math*, 6(4):93.

Mitrinović, D. and Lacković, I. (1985). Hermite and convexity. *Aequationes Mathematicae*, 28(1):229–232.

Mitrinović, D. S., Pecarić, J., and Fink, A. M. (2013). *Classical and new inequalities in analysis*, volume 61. Springer Science & Business Media.

Mitrinović, D. S. and Vasic, P. M. (1970). *Analytic inequalities*, volume 61. Springer.

Mo, H. and Sui, X. (2017). Hermite-hadamard-type inequalities for generalized s-convex functions on real linear fractal set $R^{\alpha}(0 < \alpha < 1)$. *Mathematical Sciences*, 11(3):241–246.

Mo, H., Sui, X., and Yu, D. (2014). Generalized convex functions and some inequalities on fractal sets. *arXiv preprint arXiv:1404.3964*.

Niculescu, C. and Persson, L.-E. (2006). *Convex functions and their applications*. Springer.

Niculescu, C. P. and Persson, L.-E. (2004). Old and new on the hermite-hadamard inequality. *Real Analysis Exchange*, 29(2):663–686.

Nigmatullin, R. (1992). Fractional integral and its physical interpretation. *Theoretical and Mathematical Physics*, 90(3):242–251.

Noor, M. A. (2007). Hermite-hadamard integral inequalities for log-preinvex functions. *J. Math. Anal. Approx. Theory*, 2(2):126–131.

Noor, M. A., Noor, K. I., and Awan, M. U. (2014). Fractional ostrowski inequalities for s-godunova-levin functions. *International Journal of Analysis and Applications*, 5(2):167–173.

© COPYRIGHT UPM

Okur, N., Işcan, I., and Dizdar, E. Y. (2019). Hermite-hadamard type inequalities for p-convex stochastic processes. *An International Journal of Optimization and Control: Theories and Applications (IJOCTA)*, 9(2):148–153.

Omotoyinbo, O. and Mogbodemu, A. (2014). Some new hermite–hadamard integral inequalities for convex functions. *Int. J. Sci. Innovation Tech*, 1(1):1–12.

Özcan, S. and İşcan, İ. (2019). Some new hermite–hadamard type inequalities for s-convex functions and their applications. *Journal of Inequalities and Applications*, 2019(1):201.

Özdemir, M. E., Latif, M. A., and Akdemir, A. O. (2012). On some hadamard-type inequalities for product of two s-convex functions on the co-ordinates. *Journal of Inequalities and Applications*, 2012(1):21.

Özdemir, M. E., Set, E., and Sarıkaya, M. Z. (2011). Some new hadamard type inequalities for co-ordinated. *Hacettepe Journal of Mathematics and Statistics*, 40(2):219–229.

Park, J. (2015). Hermite-hadamard-like type inequalities for s-convex functions and s-godunova-levin functions of two kinds. *Applied Mathematical Sciences*, 9(69):3431–3447.

Partap, N., Jain, S., and Chugh, R. (2019). Supremacy of s-convexity scheme and generation of advanced fractal using faster iteration. *Nonlinear Studies*, 26(2).

Peajcariaac, J. E. and Tong, Y. L. (1992). *Convex functions, partial orderings, and statistical applications*. Academic Press.

Pearce, C. E. and Pecarić, J. (2000). Inequalities for differentiable mappings with application to special means and quadrature formulae. *Applied Mathematics Letters*, 13(2):51–55.

Pearce, C. E. and Rubinov, A. (1999). P-functions, quasi-convex functions, and hadamard-type inequalities. *Journal of Mathematical Analysis and Applications*, 240(1):92–104.

Peng, S., Wei, W., and Wang, J. (2014). On the hermite-hadamard inequalities for convex functions via hadamard fractional integrals. *Facta Universitatis, Series: Mathematics and Informatics*, 29(1):55–75.

Pennanen, T. (2012). Introduction to convex optimization in financial markets. *Mathematical programming*, 134(1):157–186.

Phelps, R. R. (2009). *Convex functions, monotone operators and differentiability*, volume 1364. Springer.

Pinheiro, M. R. (2007). Exploring the concept of s-convexity. *Aequationes mathematicae*, 74(3):201–209.

Prabseang, J., Nonlaopon, K., and Tariboon, J. (2019). Quantum hermite–hadamard inequalities for double integral and q-differentiable convex functions. *J. Math. Inequal*, 13:675–686.

© COPYRIGHT UPM

Radulescu, M., Radulescu, S., and Alexandrescu, P. (2009). On the godunova-levin-schur class of functions. *Math. Inequal. Appl*, 12(4):853–862.

Rashid, S., Noor, M. A., Noor, K. I., and Akdemir, A. O. (2019). Some new generalizations for exponentially s-convex functions and inequalities via fractional operators. *Fractal and Fractional*, 3(2):24.

Richard, H. (2014). *Fractional calculus: an introduction for physicists*. World Scientific.

Robert, A. W. and Varberg, D. E. (1973). *Convex functions*. Academic Press, New York.

Salati, A. B., Shamsi, M., and Torres, D. F. (2019). Direct transcription methods based on fractional integral approximation formulas for solving nonlinear fractional optimal control problems. *Communications in Nonlinear Science and Numerical Simulation*, 67:334–350.

Samko, S. G., Kilbas, A. A., Marichev, O. I., et al. (1993). *Fractional integrals and derivatives*, volume 1. Gordon and Breach Science Publishers, Yverdon-les-Bains, Switzerland.

Sarıkaya, M. Z., Budak, H., and Erden, S. (2019). On new inequalities of simpson's type for generalized convex functions. *The Korean Journal of Mathematics*, 27(2):279–295.

Sarikaya, M. Z. and Kiris, M. E. (2015). Some new inequalities of hermite-hadamard type for s-convex functions. *Miskolc Mathematical Notes*, 16(1):491–501.

Sarikaya, M. Z., Saglam, A., and Yildirim, H. (2008). On some hadamard-type inequalities for h-convex functions. *J. Math. Inequal*, 2(3):335–341.

Sarikaya, M. Z., Set, E., Yaldiz, H., and Başak, N. (2012). Hermite-hadamard's inequalities for fractional integrals and related fractional inequalities. *Mathematical and Computer Modelling*, 57(9-10):2403–2407.

Sebah, P. and Gourdon, X. (2002). Introduction to the gamma function. *American Journal of Scientific Research*, pages 2–18.

Set, E., Choi, J., and Çelik, B. (2018). Certain hermite-hadamard type inequalities involving generalized fractional integral operators. *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, 112(4):1539–1547.

Set, E. and Karaoğlan, A. (2017). Hermite-hadamard and hermite-hadamard-fejer type inequalities for (k, h)-convex function via katugampola fractional integrals. *KJM,(5)*, pages 181–191.

Set, E. and Mumcu, I. (2018). Hermite-hadamard type inequalities for quasi-convex functions via katugampola fractional integrals. *International Journal of Analysis and Applications*, 16(4):605–613.

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Set, E., Sarikaya, M. Z., Özdemir, M. E., and Yıldırım, H. (2014). The hermite-hadamard's inequality for some convex functions via fractional integrals and related results. *Journal of Applied Mathematics, Statistics and Informatics*, 10(2):69–83.

Set, E. and Tomar, M. (2016). New inequalities of hermite-hadamard type for generalized convex functions with applications. *Facta Universitatis, Series: Mathematics and Informatics*, 31(2):383–397.

Shuang, Y., Yin, H.-P., and Qi, F. (2013). Hermite–hadamard type integral inequalities for geometric-arithmetically s-convex functions. *Analysis International mathematical journal of analysis and its applications*, 33(2):197–208.

Soradi-Zeid, S., Jahanshahi, H., Yousefpour, A., and Bekiros, S. (2020). King algorithm-a novel optimization approach based on variable order fractional calculus with application in chaotic financial systems. *Chaos, Solitons and Fractals*, 132:109569.

Sun, W. (2017). Generalized harmonically convex functions on fractal sets and related hermite-hadamard type inequalities. *J. Nonlinear Sci. Appl*, 10(11):5869–5880.

Sun, W. (2019). On generalization of some inequalities for generalized harmonically convex functions via local fractional integrals. *Quaestiones Mathematicae*, 42(9):1159–1183.

Suneja, S., Singh, C., and Bector, C. (1993). Generalization of preinvex and b-vex functions. *Journal of Optimization Theory and Applications*, 76(3):577–587.

Tarasov, V. E. (2011). *Fractional dynamics: applications of fractional calculus to dynamics of particles, fields and media*. Springer Science and Business Media.

Thukral, A. K. (2014). Factorials of real negative and imaginary numbers-a new perspective. *SpringerPlus*, 3(1):658.

Tomar, M., Agarwal, P., and Choi, J. (2020). Hermite-hadamard type inequalities for generalized convex functions on fractal sets style. *Boletim da Sociedade Paranaense de Matemática*, 38(1):101–116.

Toplu, T., Set, E., Iscan, I., and Maden, S. (2019). Hermite-hadamard type inequalities for p-convex functions via katugampola fractional integrals. *Facta Univ. Ser. Math. Inform*, 34:149–164.

Udriste, C. (2013). *Convex functions and optimization methods on Riemannian manifolds*, volume 297. Springer Science and Business Media.

Ullah, S. Z., Khan, M. A., and Chu, Y.-M. (2019). A note on generalized convex functions. *Journal of Inequalities and Applications*, 2019(1):1–10.

Usta, F., Budak, H., Sarıkaya, M. Z., and Set, E. (2018). On generalization of trapezoid type inequalities for s-convex functions with generalized fractional integral operators. *Filomat*, 32(6).

© COPYRIGHT UPM

Varosanec, S. (2007). On h-convexity. *Journal of Mathematical Analysis and Applications*, 326(1):303–311.

Vivas, M., Hernández, J., and Merentes, N. (2016). New hermite-hadamard and jensen type inequalities for h-convex functions on fractal sets. *Revista Colombiana de Matemáticas*, 50(2):145–164.

Wang, J. and Feckan, M. (2018). *Fractional Hermite-Hadamard Inequalities*, volume 5. Walter de Gruyter GmbH and Co KG.

Wang, J., Li, X., Feckan, M., and Zhou, Y. (2013a). Hermite-hadamard-type inequalities for riemann-liouville fractional integrals via two kinds of convexity. *Applicable Analysis*, 92(11):2241–2253.

Wang, J., Li, X., and Zhou, Y. (2016). Hermite-hadamard inequalities involving riemann-liouville fractional integrals via s-convex functions and applications to special means. *Filomat*, 30(5):1143–1150.

Wang, J., Li, X., Zhu, C., et al. (2013b). Refinements of hermite-hadamard type inequalities involving fractional integrals. *Bulletin of the Belgian Mathematical Society-Simon Stevin*, 20(4):655–666.

Weir, T. and Mond, B. (1988). Pre-invex functions in multiple objective optimization. *Journal of Mathematical Analysis and Applications*, 136(1):29–38.

Yang, X.-J. (2019). *General fractional derivatives: theory, methods and applications*. CRC Press.

Zabandan, G., Bodaghi, A., and Kılıçman, A. (2012). The hermite-hadamard inequality for r-convex functions. *Journal of Inequalities and Applications*, 2012(1):215.

Zhao, D., Ali, M. A., Kashuri, A., and Budak, H. (2020). Generalized fractional integral inequalities of hermite–hadamard type for harmonically convex functions. *Advances in Difference Equations*, 2020(1):1–14.

Zhou, S.-S., Rashid, S., Noor, M. A., Noor, K. I., Safdar, F., and Chu, Y.-M. (2020). New hermite–hadamard type inequalities for exponentially convex functions and applications. *AIMS Math*, 5:6874–6901.

Zhu, C., Feckan, M., and Wang, J. (2012). Fractional integral inequalities for differentiable convex mappings and applications to special means and a midpoint formula. *Journal of Applied Mathematics, Statistics and Informatics*, 8(2):21–28.

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BIODATA OF STUDENT

I am Ohud Bulayhan Almutairi, a lecturer with the Department of Mathematics, University of Hafr Al Batin, Sudia Arabia. I have joined the University as a demonstrator. Following this, I obtained my master's degree in mathematics from Dammam University, Saudi Arabia. In addition to teaching undergraduate courses and organizing academic workshops, I served the university as a Director of Development Unity in 2017.

I have attended both the national and international conferences and research workshops, some of which were organized by the Saudi Ministry of Higher Education for the Saudi faculty members.

My area of interest includes fractional calculus, integral inequalities and geometry.

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LIST OF PUBLICATIONS

O. Almutairi., A. Kiliçman., (2019). New fractional inequalities of midpoint type via s-convexity and their application. *Journal of Inequalities and Applications*, (1-19), *DOI:10.1186/s13660-019-2215-3*.

O. Almutairi., A. Kiliçman., (2019). Generalized Integral Inequalities for Hermite–Hadamard-Type Inequalities via s-Convexity on Fractal Sets. *Mathematics, MDPI*, (7(11), 1065), *DOI:10.3390/math7111065*.

O. Almutairi., A. Kiliçman., (2019). Some Integral Inequalities for h-Godunova-Levin Preinvexity. *Symmetry, MDPI*, (11(12), 1500), *DOI:10.3390/sym11121500*.

O. Almutairi., A. Kiliçman., (2019). New refinements of the Hadamard inequality on coordinated convex function. *Journal of Inequalities and Applications*, (1-9), *DOI:10.1186/s13660-019-2143-2*.

O. Almutairi., A. Kiliçman., (2020). New Generalized Hermite-Hadamard Inequality and Related Integral Inequalities Involving Katugampola Type Fractional Integrals. *Symmetry, MDPI*, (12(4), 568), *DOI:10.3390/sym12040568*.

O. Almutairi., A. Kiliçman., (2020). Integral inequalities for s-convexity via generalized fractional integrals on fractal sets. *Mathematics, MDPI*, (8(1), 53), *DOI:10.3390/math8010053*.

O. Almutairi., A. Kiliçman., (2020). A comprehensive Review on fractional generalized Hermite-Hadamard type inequalities and fractional form . *International Fundamental Science Congress (IFSC2018)*, Kajang, Malaysia.

O. Almutairi., A. Kiliçman., (2020). A comprehensive Review on fractional generalized Hermite-Hadamard type inequalities and fractional form . *International Conference on Recent Advances in Applied Mathematics (ICRAAM2020)*, 4-6 February, Kuala Lumpur, Malaysia.

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