UNIVERSITI PUTRA MALAYSIA

SOLVING DIRECTLY SPECIAL THIRD AND FOURTH ORDER ORDINARY DIFFERENTIAL EQUATIONS USING LINEAR MULTISTEP AND EXPLICIT OBRECHKOFF METHODS

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By

MARZIEH RAJABI

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Master of Science

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DEDICATIONS

This thesis is dedicated to:
To all of my love;
My mother & father
and my husband

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Master of Science

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November 2018

Chairman:Fudziah binti Ismail, PhD
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This study focuses mainly on developing linear multistep methods which can directly solve special third and fourth order ordinary differential equations (ODEs). We constructed and derived the new explicit and implicit linear multistep methods for different stepnumbers based on Taylor's series expansion. The study in the thesis consists of three parts. The first part of the thesis described the derivation of explicit and implicit multistep methods with step number k equals to three, five and six for directly solving special third order ODEs. The stability of the methods is also investigated. The numerical results revealed that the new methods are more efficient as compared to the existing methods. The second part of the thesis focused on the derivation of explicit and implicit multistep methods with step number k equals to four and five for directly solving special fourth order ODEs. The zero-stability and absolute stability of the new methods are also given. Numerical results clearly show that the new proposed methods are more efficient in terms of accuracy and computational time when compared with well-known existing methods. Finally, the last part of the thesis concerned with the construction of explicit multistep method with extra derivative known as Obrechkoff methods for directly solving special third order ODEs. Stability properties of the methods are also presented. Numerical results show that new methods are more efficient than the existing methods. As a whole, the new proposed methods for directly solving special third and fourth order ordinary differential equations have been presented. The illustrative examples demonstrate the superiority of the new linear multistep and Obrechkoff methods over existing numerical methods in the scientific literature.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Master Sains

MENYELESAIKAN SECARA LANGSUNG PERSAMAAN PEMBEZAAN BIASA KHAS PERINGKAT KETIGA DAN KEEMPAT MENGGUNAKAN KAEDAH MULTILANGKAH LINEAR DAN KAEDAH OBRECHKOFF TAK TERSIRAT

Oleh

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Kajian di dalam tesis ini fokus kepada membangunkan kaedah multilangkah yang boleh menyelesaikan persamaan pembezaan biasa (PBB) khas peringkat ketiga dan keempat secara terus. Kaedah tak tersirat dan tersirat multilangkah yang baharu ini dengan bilangan langkah yang berbeza dibangunkan berdasarkan kembangan siri Taylor. Kajian dalam tesis ini merangkumi tiga bahagian. Bahagian pertama tesis menerangkan pembinaan kaedah multilangkah tak tersirat dan tersirat dengan bilangan langkah k sama dengan tiga, lima dan enam untuk menyelesaikan PPB khas peringkat ketiga secara terus. Kestabilan kaedah juga dikaji. Hasil berangka mendedahkan bahawa kaedah yang baharu adalah lebih cekap berbanding kaedah sedia ada. Bahagian kedua tesis ini memberi tumpuan kepada pembinaan kaedah multilangkah tak tersirat dan tersirat dengan bilangan langkah k sama dengan empat dan lima yang secara langsung dapat menyelesaikan PPB khas peringkat keempat secara langsung. Kestabilan sifar dan kestabilan mutlak kaedah yang baharu juga turut diberikan. Keputusan berangka menunjukkan dengan jelas bahawa kaedah yang baharu dicadangkan lebih cekap dari segi ketepatan dan masa pengiraan apabila dibandingkan dengan kaedah sedia ada yang terkenal. Akhirnya, bahagian akhir tesis berkisar kepada pembinaan kaedah multilangkah tak tersirat dengan pembezaan tambahan yang dikenali sebagai kaedah Obrechkoff untuk menyelesaikan secara langsung PPB khas peringkat ketiga. Ciri-ciri kestabilan kaedah juga dibentangkan. Keputusan berangka menunjukkan bahawa kaedah baharu lebih berkesan daripada kaedah sedia ada. Secara keseluruhannya kaedah baharu yang dicadangkan untuk menyelesaikan secara langsung persamaan pembezaan biasa khas peringkat ketiga dan keempat telah dipersembahkan. Contoh ilustrasi yang diberikan menunjukkan keunggulan kaedah multilangkah dan kaedah Obrechkoff yang baharu berbanding kaedah berangka sedia ada dalam kesusasteraan saintifik.

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This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Master of Science. The members of the Supervisory Committee were as follows:

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Declaration by Members of Supervisory Committee

This is to confirm that:

- the research conducted and the writing of this thesis was under our supervision;
- supervision responsibilities as stated in the Universiti Putra Malaysia (Graduate Studies) Rules 2003 (Revision 2012-2013) are adhered to.

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LIST OF ABBREVIATIONS

ODEs	Ordinary Differential Equations
IVPs	Initial Value Problems
h	Constant stepsize
LMM	Linear Multistep Method
EXLMM3	All explicit linear multistep methods for the special third order ODEs
EXLMM4	All explicit linear multistep methods for the special fourth order ODEs
EXL3M	Explicit linear 3-step method
EXL4 M	Explicit linear 4-step method
EXL5 M	Explicit linear 5-step method
EXL6 M	Explicit linear 6-step method
PCLMM3	All predictor-Corrector linear multistep methods for the 3rd order ODEs
PCLMM4	All predictor-Corrector linear multistep methods for the 4th order ODEs
PCL3 M	Predictor-Corrector method for $k = 3$
PCL4 M	Predictor-Corrector method for $k = 4$
PCL5 M	Predictor-Corrector method for $k = 5$
PCL6M	Predictor-Corrector method for $k = 6$
EXObr3	All explicit Obrechkoff method for the special third order ODEs
EX3Obr	Explicit Obrechkoff method for $k = 3$
EX4Obr	Explicit Obrechkoff method for $k = 4$
EX5Obr	Explicit Obrechkoff method for $k = 5$
ADBM	Adams-Bashforth-Moulton method
RKM	Fourth order Runge-Kutta method
MaxEr	Maximum error of the computed method
T	CPU time
FnEv	Function evaluation

CHAPTER 1

INTRODUCTION TO NUMERICAL ORDINARY DIFFERENTIAL EQUATIONS

Differential equations basically fall into two classes, ordinary and partial, depending on the number of independent variables present in the differential equations. Numerical solution for differential equation has always been an important problem for scientists. One type of the differential equation is Ordinary Differential Equations (ODEs) which involve an unknown function with independent variable and one or more of its derivatives. The study of differential equations is a wide field in pure and applied mathematics, physics and engineering (Radzi et al., 2011).

ODEs have always played an important role in modeling virtually every algebraic, geometric, physical, technical, or biological process from celestial motion to bridge design, to interaction between neurons (Majid et al., 2003). Many contexts of engineering and science such as fluid dynamics, radioactive decay and population growth have been widely used ODEs. The numerical method for ODEs is very important compare to the analytical way of solving because, the anti-derivatives for most realistic systems of ODEs are difficult or impossible to find.

1.1 Introduction to Ordinary Differential Equations

In the following discussion, we consider the initial value problems classified into two categories as follows. First order ODEs and higher order ODEs.

The first order ordinary differential equations are defined by

$$y' = f(x,y), y(a) = \eta, \tag{1.1}$$

for $x \in [a,b]$ and η is any given number.

And the higher order ODEs

$$y^{(v)} = f(x,y,\ldots,y^{(v-1)}), \tag{1.2}$$

where $v = 2,3,4,\ldots,n$ with initial conditions

$$y(a) = y(0) \quad \text{and } y^i(a) = \eta_i, \quad 0 < i < v-1, \quad x \in [a,b].$$

In equation (1.1) , the quantity being differentiated, y is named as the dependent variable, while the quantity with respect to which y is differentiated, x is named as independent variable.

Theorem 1.1 *(Existence and Uniqueness)(Lambert, 1991)*
Let f(x,y) be defined and continuous for all points (x,y) in the region D defined by $a \leq x \leq b$, $-\infty < y < \infty$, a and b finite, and let there exist a constant L such that, for every x, y, y such that (x,y) and (x,y*) are both in D,*

$$|f(x,y) - f(x,y^*)| \leq L|y - y^*|$$

Then, there exists a unique solution y(x) of the initial value problem (1), where y(x) is continuous and differentiable for all (x,y) in D.
The condition is known as Lipschitz condition.Then there exists exactly one function y(x) with the following three properties: i. y(x) is continuous and differentiable for

$x \in [a,b]$,
ii.$y' = f(x,y(x))$, $x \in [a,b]$,
iii.$y(a) = \eta$
the proof is given by Henrici (1962).

Basically, the numerical methods for ODEs are classified as one-step method and multistep method. One-step method requires the information form only one previous point, x_n to find the approximation at the mesh point, x_{n+1}. In the other point of view, multistep method requires the usage of information from more than one previous point to find the next approximation. In general, the linear multistep method or the linear k-step method for (1.1) can be written as:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f\left(x_{n+j}, y_{n+j}\right) \tag{1.3}$$

where α_j and β_j are constants with the conditions $\alpha_k \neq 0$, moreover, α_0 and α_j are not both zero. Since (1.3) can be multiplied by the same constant without altering the relationship, the coefficient α_j and β_j, are arbitrary constant. This arbitrariness has been removed by assuming that $\alpha_k = 1$. Method (1.3) is explicit if $\beta_k = 0$ and implicit if $\beta_k \neq 0$.

Definition 1.1 *(Lambert, 1991)*
The linear difference operator ℓ associated with the linear multistep methods (1.3) is defined by

$$\ell\left[y(x);h\right] = \sum_{j=0}^{k} \alpha_j y(x+jh) - h \sum_{j=0}^{k} \beta_j f(x+jh) \tag{1.4}$$

where $y(x)$ is an arbitrary function that is sufficiency differentiable on $[a,b]$. Expanding the test function and its first derivative as Taylor series about x and collecting the terms to obtain

$$\ell[y(x);h] = C_0 y(x) + C_1 hy(x) + C_2 h^2 y^2(x) + \cdots + C_q h^q y^q(x) + \cdots, \qquad (1.5)$$

where the coefficients C_q are constants independent of $y(x)$. In particular,

$$\begin{aligned}
C_0 &= \sum_{j=0}^{k} \alpha_j, \\
C_1 &= \sum_{j=0}^{k} j\alpha_j - \sum_{j=0}^{k} \beta_j, \\
C_2 &= \sum_{j=0}^{k} \frac{1}{2!} j^2 \alpha_j - \sum_{j=0}^{k} j\beta_j, \\
C_3 &= \sum_{j=0}^{k} \frac{1}{3!} j^3 \alpha_j - \sum_{j=0}^{k} \frac{1}{2!} j^2 \beta_j, \\
&\cdot \\
&\cdot \\
&\cdot \\
C_q &= \sum_{j=0}^{k} \frac{j^q}{q!} \alpha_j - \frac{j^{(q-1)}}{(q-1)!} \beta_j
\end{aligned}$$

for $q = 2, 3, 4, \cdots$

Definition 1.2 *(Lambert, 1991)*
The linear multisptep method (1.11) and the associated linear difference operator defined by (1.4) are said to be of order p if
$C_0 = C_1 = C_2 = \cdots = C_{(p)} = 0$ *and* $C_{p+1} \neq 0$ *The first non-vanishing coefficient,* C_{p+1}*, is called the error constant.*

Definition 1.3 *(Lambert, 1991)*
The linear multistep method (1.11) is said to be consistent if it has order at least one. It follows that the method (1.4) is consistent if and only if

$$\sum_{j=0}^{k} \alpha_j = 0$$

and

$$\sum_{j=0}^{k} j\alpha_j = \sum_{j=0}^{k} \beta_j$$

first and second characteristic polynomials of the linear multistep method (1.4) can be defined as $\rho\,(\zeta)$ *and* $\sigma\,(\zeta)$ *respectively where*

$$\rho\,(\zeta) = \sum_{j=0}^{k} \alpha_j \zeta^j \tag{1.6}$$

$$\sigma\,(\zeta) = \sum_{j=0}^{k} \beta_j \zeta^j \tag{1.7}$$

The linear k-step method (1.4) will be consistent if it has order $p \geqslant 1$ *or it satisfies the following conditions*

$$\rho(1) = 0, \quad \rho'(1) = \sigma(1).$$

In addition, the following definition gives the condition for zero stability of a method.

Definition 1.4 *(Lambert, 1991) The linear multistep is said to be zero-stable if no root of the first characteristic polynomial* $\rho\,(\zeta)$ *has modulus greater than one, and if every root with modulus one is simple.*

Theorem 1.2 *(Lambert, 1991) The necessary and sufficient conditions for a linear multistep method to be convergent are that it be consistent and zero-atable.*

$$\pi\,(\zeta,\hbar) = \rho\,(\zeta) - \hbar\,\sigma\,(\zeta) \tag{1.8}$$

where

$$\hbar = h\lambda.$$

The polynomial $\pi\,(\zeta,\hbar)$ is frequently referred to as characteristic polynomial of the method.

Definition 1.5 *(Lambert, 1991)*
The linear multistep method is said to be absolutely stable for a given $\hbar$ *if, for that* $\hbar$*, all the roots* ζ_s *of stability polynomial,*

$$\pi\,(\zeta,\hbar) = \rho(\zeta) - \hbar\sigma(\zeta) = 0,$$

where $\hbar = h\lambda$ *satisfy* $|\zeta_s| < 1$*,* $s = 1,2,3,\cdots,k$*, and to be absolutely unstable for that* $\hbar$ *otherwise. The region of absolute stability consists of all* $\hbar$ *in the complex plane for which the method is absolutely stable.*

Introduction to Obrechkoff Methods

Explicit differentiation for many problems can be intolerably complicated, but when it is feasible to evaluate the first few total derivatives of y, then generalizations of linear multistep methods which employ such derivatives can be very efficient (Shokri et al, 2011). Such methods are called Obrechkoff methods, although the original work of Obrechkoff in 1940 was concerned only with numerical quadrature, and it would appear that Milne in 1949 was the first to advocate the use of Obrechkoff formulae for the numerical solution of differential equations (Shokri and Shmatikov, 2015).

The k-step Obrechkoff method using the first l derivatives of y may be written as

$$\sum_{j=0}^{k} \alpha_j y\left(n+j\right) = \sum_{i=1}^{l} \left(h^i \sum_{j=0}^{k} \beta_{ij} {y_{n+j}}^{i} \right) \tag{1.9}$$

while $\alpha_k = 1$. In this method the following derivative equations frequently are used

$$\begin{aligned} y^{(1)} &= f(x,y) \\ y^{(2)} &= f_x + f f_y \\ y^{(3)} &= f_{xx} + 2 f f_{xy} + f^2 f yy + f_y (f_x + f f_y), \cdots . \end{aligned} \tag{1.10}$$

Order, error constant, and local truncation error are defined exactly as for linear multistep methods, through the operator ℓ ,

$$\ell\left[y\left(x\right);h\right] = \sum_{j=0}^{k} \alpha_j y(n+j) - \sum_{i=1}^{l} \left(h^i \sum_{j=0}^{k} \beta_{ij} {y_{n+j}}^{i} \right) \tag{1.11}$$

the Obrechkoff method called explicit if $\beta_{ik} = 0$ and is implicit otherwise.

1.2 Problem Statement

Higher order ordinary differential equations are usually solved by reducing the equations into a system of first order ODEs and solved by numerical methods for first order ODEs, however this requires a lot of computational time and effort. A lot of researches have also been done on numerical methods for directly solving general third order ODEs of the form $y''' = f(x,y,y',y'')$ and general fourth order ODEs of the form $y^{(iv)} = f(x,y,y',y'',y''')$, the method requires the computation of the derivatives of y , which is redundant if the equations are of the special types that is $y''' = f(x,y)$ and $y^{(iv)} = f(x,y)$. This triggers the quest to come up with numerical methods which are tailored specifically for directly solving special third and fourth order ODEs.

1.3 The Objectives of the Thesis

The objectives of the thesis are as follows:

- To construct the explicit and implicit multistep methods of step number $k = 3,5$ and 6 for solving special third order ordinary differential equations.
- To derive explicit and implicit multistep method of stepnumber $k = 4$ and $k = 5$ for directly solving special fourth order ordinary differential equations.
- To construct Obrechkoff methods of step number $k = 3,4$ and 5 for directly solving special third order ordinary differential equations.
- To study the stability of all the methods that have been constructed.

1.4 Outline of the Study

Chapter 1 presents the introduction of ODEs and the intitial value problems. It gives definitions and basic theory which are related to the proposed method.

Chapter 2 reviews on the numerical solution of ODEs which has been done by other researchers. Related studies on explicit and implicit linear multistep method are also provided.

Chapter 3 gives the derivation of linear $k = 3,5$ and 6 step methods for directly solving special third order initial value problem. The region of stability of the method is also investigated.

Chapter 4 focused on the linear multistep method with stepnumber $k = 4$ and 5, for directly solving special fourth order ordinary differential equations. The stability of the these methods is also discussed in this chapter.

Chapter 5 deals with the Obrechkoff methods for directly solving the special third order ordinary differential equations. The deviation involves the Taylor's series expansion. The analysis of the stability properties are also presented.

Chapter 6 summarizes the conclusion of the thesis and future work is also recommended.

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