UNIVERSITI PUTRA MALAYSIA

SOLVING DELAY DIFFERENTIAL EQUATIONS BY RUNGE-KUTTA METHOD USING DIFFERENT TYPES OF INTERPOLATION

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By

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Thesis Submitted in Fulfillment of the Requirements for the Degree of Master of Science in the Faculty of Science and Environmental Studies Universiti Putra Malaysia

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TO MY PARENTS

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfillment of the requirement for the degree of Master of Science

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Chairman: Fudziah bt Ismail, Ph.D.

Faculty: Science and Environmental Studies

Introduction to delay differential equations (DDEs) and the areas where they arise are given. Analysis of specific numerical methods for solving delay differential equation is considered. A brief discussion on Runge-Kutta method when adapted to delay differential equation is introduced.

Embedded Singly Diagonally Implicit Runge-Kutta (SDIRK) method of third order four-stage in fourth order five-stage which is more attractive from the practical point of view is used to solve delay differential equations. The delay term is approximated using three typess of interpolation that is the divided difference interpolation, Hermite interpolation and In't Hout interpolation. Numerical results based on these three interpolations are tabulated and compared.

Finally, the stability properties of SDIRK method when applied to DDEs using Lagrange interpolation and In't Hout interpolation are investigated and their regions of stability are presented.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia
sebagai memenuhi keperluan untuk ijazah Master Sains

MENYELESAIKAN PERSAMAAN PEMBEZAAN LENGAH DENGAN KAEDAH RUNGE-KUTTA MENGGUNAKAN INTERPOLASI YANG BERBEZA

Oleh

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Pengenalan kepada persamaan pembezaan lengah (PPL) dan bidang di mana ianya kerap muncul diberikan. Analisis bagi beberapa kaedah berangka bagi menyelesaikan PPL dibincangkan. Kaedah Runge-Kutta bila disesuaikan bagi menyelesaikan PPL juga turut dibincangkan.

Kaedah Terbenam Runge-Kutta Pepenjuru Tunggal Tersirat (RKPTT) peringkat tiga tahap empat dalam peringkat empat tahap lima yang lebih cekap dari segi praktikalnya digunakan bagi menyelesaikan persamaan pembezaan lengah. Sebutan lengahnya di perolehi dengan menggunakan tiga jenis interpolasi iaitu interpolasi beza bahagi, interpolasi Hermite dan interpolasi yang diterbitkan oleh In't Hout. Keputusan berangka berdasarkan ketiga-tiga interpolasi ini dibandingkan.

Akhir sekali, ciri ciri ke stabilan kaedah RKPTT bagi menyelesaikan PPL menggunakan interpolasi Lagrange dan interpolasi yang diterbitkan oleh In't Hout dikaji, dan rantau kestabilannya diberikan.

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LIST OF ABBREVIATIONS

IVP	Initial Value Problem
ODE	Ordinary Differential Equation
ODEs	Ordinary Differential Equations
SDIRK	Singly Diagonally Implicit Runge-Kutta
BDF	Backward Differentiation Method
DDE	Delay Differential Equation
DDEs	Delay Differential Equations
RDE	Retarded Delay Differential Equation
NDE	Neutral Delay Differential Equation
CRK	Continuous Runge-Kutta
RKF	Runge-Kutta Felhberg

CHAPTER I

INTRODUCTION TO DELAY DIFFERENTIAL EQUATION AND INTERPOLATION

Delay Differential Equation

Many physical systems posses the feature of having a delayed response to input conditions, so that the rate at which processes occur depends not only on the current state of the system but also the past states. Mathematical models of such process commonly result in differential equations with a time delay. Equations of this type are called delay differential equations (DDEs).

Delay differential equations have their origin in domains of application, such as physics, engineering, biology, medicine and economics. They appear in connection with fundamental problems to analyze mathematical model and to determine the future behavior. Because of that, in recent years, there has been a growing interest in the numerical treatment of DDEs, such as the work of Feldstein and Goodman (1973), Orbele and Pesch (1981), Thompson (1990), Paul (1992), Ismail (1999) and many more. Two examples of DDEs applications are as follows:

1. Population growth (Beliar, 1991) as a model of biological processes can be modeled by state dependent DDE

$$y'(t) = \frac{b(y(t)) - b(y(t - L(y(t))))}{1 - L'(y(t))b(t - L(y(t)))}$$

where $y(t)$ is the population size and $L(.)$ is the lifespan of indivduals in the population.

2. The variation in market price, $p(t)$, of a particular commodity (Karoui, 1992), can be modeled by the following DDE.

$$p'(t) = p(t) f(D(p(t)), S(p(t-\tau))),$$

where $D(.)$ and $S(.)$, denote the demand and supply functions for the commodity.

Assumptions and Definitions

Generally a DDE refers to both a retarded type of DDE (RDE) and a neutral type of DDE (NDE). RDE is an ODE involving solution of the delay term and is given by

$$\begin{aligned} y'(t) &= f(t, y(t), y(t-\tau_1(t, y(t))), y(t-\tau_2(t, y(t))), \ldots, \\ &\qquad y(t-\tau_v(t, y(t))) \qquad \text{for } t \geq 0 \qquad [1.1] \\ y(t) &= \varphi(t) \qquad \text{for } t \leq 0, \end{aligned}$$

Where y, f and φ are N-vector functions and τ_i i=1(1) v are scalar functions. An NDE is an ODE involving both the solution and the derivative of the delay term itself and is given by

$$\begin{aligned} y'(t) &= f(t, y(t), y(t-\tau_1(t, y(t))), \ldots, y(t-\tau_v(t, y(t))), \\ &\qquad y'(t-\tau_{v+1}(t, y(t))), \ldots, y'(t-\tau_{v+w}(t, y(t))) \qquad t \geq t_0 \qquad [1.2] \\ y(t) &= \varphi(t), \; y'(t) = \varphi'(t) \qquad t \leq t_0 \end{aligned}$$

y, f, φ and φ' are N-vector functions and τ_i $i=1(1)v+w$ are scalar functions.

For simplicity assume v=1 for RDE and v=1, w=1 for NDE. So, a simple RDE can be written as

$$\left.\begin{array}{ll} y'(\mathrm{t}) = f(t, y(t), y(t-\tau_1(t, y(t))) & t \geq t_0 \\ y(t) = \varphi(t) & t \leq t_0 \end{array}\right\} \quad [1.3]$$

And a simple NDE can be written as

$$\left.\begin{array}{ll} y'(\mathrm{t}) = f(t, y(\mathrm{t}), y(t-\tau_1(t, y(t))), \; y'(t-\tau_2\;(t, y(t))) & t \geq t_0 \\ y(t) = \varphi(t) \qquad\qquad t \leq t_0 & \end{array}\right\} \quad [1.4]$$

If $v > 1$ in [1.1], then we have RDE with multiple delays, if $v > 1$ and $w > 1$ in [1.2], then it is called NDE with multiple delays.

Generally DDE is referred to both retarded type and neutral type of delay differential equations. However many authors refer to the retarded type of DDE as DDE only. Since in this thesis we are only concerned with DDE of the retarded type, it will therefore be referred to as DDE only.

The function $\tau(t, y(t))$ is called the delay, $t-\tau(t, y(t))$ is called the delay argument, the value of $y(t-\tau(t, y(t))$ is the solution of the delay term or commonly referred to as the delay term only.

If the delay is a function of time t, then it is called time dependent delay. If the delay is a function of time t and $y(t)$, it is called the state dependent delay. A delay argument that passes the current time $(t-\tau(t, y(t)) > t$, it is called an advanced delay. In our work here, we are using DDE with time dependent delay and also state dependent delay.

We will assume the existence, uniqueness of the solution to the problems under consideration. For example, sufficient conditions for the existence and uniqueness of solutions to RDE [1.3] are f continuous with respect to t, $y(t)$, and $y(t-\tau_1(t, y(t)))$, $y(t)$ is continuous, f satisfies a Lipschtiz condition in the last two arguments, φ is continuous, and f is bounded (Driver 1977).

Let $y(t)$ be a solution of a differential equation and $u_i(t)$ be a continuous approximation to $y(t)$ on $[t_i, t_{i+1}]$ associated with a method. The method is said to be convergent if

$$\max_{0 \le i \le N} \max_{t_i \le i \le t_{i+1}} \left\| u_i(t) - y(t) \right\| \to 0 \text{ as } H = \max h_i \to 0 \text{ and } N \to \infty,$$

where $h_i = t_{i+1} - t_i$ and $\sum_{i=0}^{N} h_i = t_F - t_0$.

We define a local solution of the RDE [1.3] as the solution of

$$y'(\mathrm{t}) = f(t, y(t), y(t-\tau_1(t, y(t)))) \qquad t \ge t_0$$

$$y(t) = z(t) \qquad t \le t_0$$

where $z(t)$ is a continuous approximation to $y(t)$ on $(t_0, t_n]$ associated with a method and $z(t) = \varphi(t)$ for $t \le t_0$.

A method is said to be order p if the local error is of order $p+1$,i.e.

$$\left\| y_n(t_n + \tau h_n) - z_n(t_n + \tau h_n) \right\| = O(h_n^{p+1}),$$

for all $\tau \in [0, 1]$. Here $z_n(t)$ is the continuous approximation to $y(t)$ on $[t_n, t_{n+1}]$ associated with the method.

A method for RDE [1.3] is said to be consistent if the method satisfies

$$z_n(t_n + \tau h_n) = z(t_n) + f(t_n, z(t_n), z(t_n - \sigma(t_n, z(t_n)))) + O(h_n^2) \quad \text{for } \tau \in [0, 1].$$

Numerical Methods for Delay differential Equations

It appears in the literature that there is wide interest in the numerical solution of DDE and many approaches have been adopted for solving particular equations. Most of the numerical methods for ordinary differential equations can be adapted to give corresponding techniques for delay differential equations. The range of methods therefore comprises one step methods (including Euler's method and Runge-Kutta type method), multistep methods and block-implicit methods. In each of these methods the standard formula must generally be augmented by an interpolation formula.

If $y'(x)$ is bounded and Rieman -integrable,

$$y(x+h) = y(x) + h f(x, y) + O(1);$$

the $O(1)$ term is $O(h^2)$ if $y'(x)$ is Lipsctiz contiuous.

From this relation we construct Euler's formula for DDE (1.3), namely

$$y(x+h) = y(x) + h f(x, y(x), y(x - \tau(x))) \qquad \text{for } x \geq x_0 \qquad [1.5]$$

Choosing a sequence $x_{r+1} = x_r$ $(r \geq 0)$, and substituting $h = h_r$ in [1.5], we compute values $y_r = y(x_r)$ satisfying

$$y_{r+1} = y_r + h_r f(x_r, y_r, y(x_r - \tau_r)) \tag{1.6}$$

where $\tau_r = \tau(x_r)$. If $x_r - \tau_r \notin \{x_s\}_0^r$, then we require $y(x_r - \tau_r)$, and the use of piecewise-constant interpolation suggests the substitution

$$y(x_r - \tau_r) := y(x_q) \text{ where } x_r - \tau_r \in (x_q, x_{q+1})$$

in the interpolation yield

$$y(x_r - \tau_r) := \frac{\sigma_r'' y(x_q) + \sigma_r' y(x_{q+1})}{h_q} \tag{1.7}$$

where $\qquad h_q = x_{q+1} - x_q$

$$\sigma_r' = (x_r - x_q) - \tau_r, \sigma_r'' = h_q - \sigma_r'.$$

The trapezium rule gives a simple formula with a degree of added sophistication. With minimum condition

$$y(x+h) = y(x) + \frac{1}{2}h[f(x) + f(x+h)]$$

when applied to DDE, will give

$$y(x+h) = y(x) + \frac{1}{2}h[f(x, y(x), y(x-\tau(x))) + f(x+h, y(x+h), y(x+h-\tau(x+h))]$$

Given a mesh x_r with $h_r = x_{r+1} - x_r$ we can obtain values $y_r = y(x_r)$ from the relations

$$y_{r+1} = y_r + \frac{1}{2}h[f(x_r, y_r, y(x_r - \tau_r) + f(x_{r+1}, y_{r+1}, y(x_{r+1}, y_{r+1}, y(x_{r+1} - \tau_{r+1}))]$$

for $r = 0, 1, 2, \ldots$, when the values $y(x_r - \tau_r)$ and $y(x_{r+1} - \tau_{r+1})$ are obtained by interpolation on values of $\{y\}$ and with extrapolation if $\tau_{r+1} < h_r$.

Neves (1975a) developed a numerical method based on the fourth order Merson's formula with a two-point Hermite interpolant defined by the endpoints of each step. The Fortran code DMRODE is presented in the (Neves 1975a).

Oppelstrup (1978) described a numerical method based on the Runge-Kutta Fehlberg 5(4) formula with a three-point fourth-degree Hermite-Birkhoff interpolant.

Oberle and Pesch (1981) developed a numerical method for RDEs with constant delays. The method is based on the Runge-Kutta Fehlberg 4(5) and 7(8) pairs with three- and Five-Point Hermite interpolants. These mesh points are assumed to be chosen such that they satisfy the following three conditions: they do not cross discontinuity; extrapolation is avoided; the delay argument under consideration should be close to the center of the mesh points chosen. Propagated derivative discontinuity points are determined at the beginning of integration.

Arndt (1984) developed a general convergence theory for RDEs by considering that a numerical method is actually trying to solve the initial value problem for ODEs

$$\overline{y}'(t) = f(t, \overline{y}(t), z(t - \sigma(t, z(t)))), \qquad \text{for } t_0 \leq t \leq t_F$$

$$\overline{y}(t_0) = y_0,$$

$$(\overline{y}(t) = (\phi) \text{ for } t \leq t_0)$$

where $z(t)$ is a known continuous approximation to $\overline{y}(t)$. This approximation $z(t)$ is known before the numerical solution is computed in each step unless the delay argument is in the current step.

The latest technique in approximating the delay term, when using Runge-Kutta method to solve DDE is using continuous Runge-Kutta method itself. As we know, Runge-Kutta method produces approximations only at discrete points, sometimes continuous approximation to the solution of ODE is needed to obtain dense output for plotting the solution of the ODE, to find roots of a function associated with the solution or to handle discontinuity. Consequently CRK formulas, which produce continuous approximation to the solution of ODE, have been developed by Dormand and Prince (1986), Enright et al. (1986), Gladwell et al. (1987), Shampine (1986) and Verner (1993). They are of the form

$$z_n(t_n+\theta h_n) = z_{n-1}(t_n) + h_n \sum_{i=1}^{\bar{s}+1} \bar{b}_i(\theta) k_i$$

$$k_i = f(t_n + c_i h_n, Y_i)$$

$$Y_i = z_{n-1}(t_n) + h_n \sum_{j=1}^{i-1} a_{ij} k_j$$

for i=1, 2,..., $\bar{s}+1$, $\theta \in [0,1]$, $\bar{b}_i(\theta), i = 1(1)\bar{s}+1$ are polynomials in θ and $z_{-1}(t_0) = y_0$. The CRK formula is constructed by adding extra stage s+1 to $\bar{s}+1$ with c_{s+1} , $a_{s+1j} = b_j$ $j=1,\dots,s$ to the original Runge-Kutta method.

Define a CRK formula for the DDE by

$$z_n(t_n+\theta h_n) = z_{n-1}(t_n) + h_n \sum_{i=1}^{\bar{s}+1} b_i(\theta) k_i$$

$$k_i = f(t_n + c_i h_n, Y_i, z(t_n + c_i h_n - \tau(t_n + c_i h, Y_i)))$$

$$Y_i = z_{n-1}(t_n) + h_n \sum_{j=1}^{i-1} a_{ij} k_j$$

for $i=1(1)\ \bar{s}+1$ and $z_{n-1}(t_0) = \varphi(t_0)$.

It is proven that, CRK method has can handle some of the difficulties, which DDE inherits, such as the ability to cope with discontinuity.

Runge-Kutta Method

A q-stage Runge-Kutta method can be written as

$$\left.\begin{aligned} k_i &= f(x_n + c_i h, y_n + h \sum_{j=1}^{q} a_{ij} k_j) \\ y_{n+1} &= y_n + \sum_{i=1}^{q} b_i k_i \end{aligned}\right\} \quad [1.8]$$

or it can be written in a table form as

$$\begin{array}{c|c} c_1 & a_{11}, \ldots, a_{1q} \\ c_2 & a_{21}, \ldots, a_{2q} \\ \cdot & \\ \cdot & \\ c_q & a_{q1}, \ldots, a_{qq} \\ \hline & b_1 \ , \ldots, b_q \end{array}$$

A= $\{a_{ij}\}$ is called the generating matrix, b is the vector weights and c is the vector abscissae. The method is said to be explicit if $a_{ij} = 0$ for $i \leq j$, semi implicit if $a_{ij} = 0$ for $i < j$ and fully implicit otherwise.

When we applied Runge-Kutta method [1.8] to DDE [1.3], the following are obtained:

$$k_i = f(\,t_n + c_i h,\, y(t_n + c_i h),\, y(\,t_n + c_i h - \tau\,))$$

or

$$k_i = f(\,t_n + c_i h,\, y(t_n + h\sum_{j=1}^{i} a_{ij} k_j\,),\, y(\,t_n + c_i h - \tau\,))$$

$$y_{n+1} = y_n + \sum_{i=1}^{q} b_i k_i$$

where $(\,t_n + c_i h - \tau\,)$ is the delay argument and interpolation is needed to approximate the value of the delay term $y(\,t_n + c_i h - \tau\,)$.

Using Embedding Method for the Local Error Estimation

An essential part of any integration algorithm for ODE is the procedure to estimate the local error, which provides a basis for choosing the next setpsize. One of the most popular procedure is by using an embedding method, where each integration step is performed twice, using the pth and $(p+1)$th order methods and local truncation error can be obtained by taking

$$LTE = y_{n+1} - y^{*}_{n+1} \qquad [1.9]$$

Here y^{*}_{n+1} is the integration using the pth order method. Usually in Runge-Kutta method, the pth order is obtained as a by product of the $(p+1)$th order method. Thus a significant saving in computational effort can be made which makes such technique very attractive from the practical point of view.

Interpolation

Definition: Interpolation is the process of finding and evaluating a function whose graph goes through a set of given points or the computing of values for a tabulated function at points not in the table, is historically a most important task. Interpolation was the first technique for obtaining an approximation of a function. Polynomial interpolation was then used in quadrature methods and methods for the numerical solution of ordinary differential equation.

Many famous mathematicians have their names associated with procedures for interpolation such as: Gauss, Newton, Bessel, and Stirling. The need to interpolate began with the early studies of astronomy when the motion of heavenly bodies was to be determined from periodic observations. A good interpolation polynomial needs to provide a relatively accurate approximation over an entire interval.

A Review of Previous Work

Neves (1975b) used two-points Hermite interpolation defined by the end points of each step to evaluate the delay term. Orbele and Pesch (1981) used three-points Hermite interpolation for the 4(5) Runge-Kutta Fehlberg method (RKF), and five-points Hermite interpolation for the 7(8) pair. The interpolation order and thereby the number of support points have to be adapted to the order of the method. If RKF method is of order p then the interpolation order must be equal or greater than p. For Hermite interpolation, if i_p denotes the number of support points for the interpolation, then the following inequality must hold