UNIVERSITI PUTRA MALAYSIA

NUMERICAL SOLUTIONS OF SINGLE DELAY DIFFERENTIAL EQUATIONS AND SPECIAL SECOND ORDER OSCILLATORY INITIAL VALUE PROBLEMS USING RUNGE-KUTT A AND HYBRID METHODS

SUFIA ZULFA BINTI AHMAD

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By

SUFIA ZULFA BINTI AHMAD

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Chairman : Professor Fudziah Bt Ismail, PhD

Faculty : Science

The first part of the thesis focuses on adapting existing methods for solving first and second order delay differential equations (DDEs). The methods are Improved Runge-Kutta (IRK) and Runge-Kutta (RK) methods which are adapted for solving first order DDEs. The accuracy and stability of the methods when applied to linear first order DDEs are looked into. Next we adapt the existing hybrid methods for solving special second order DDEs. Numerical results are compared in terms of accuracy and computational time with the Runge-Kutta Nyström (RKN) method. Stability of the methods when applied to linear second order DDEs are presented.

The new Semi-Implicit Hybrid methods (SIHMs) are derived for solving system of oscillatory problems. The methods have highest possible order of dissipation and dispersion with small error coefficients. The periodicity intervals of the methods are also given. Numerical results indicate that SIHMs are more efficient compare to the existing methods.

Then the zero-dissipative Phase-Fitted Hybrid methods (PFHMs) are constructed based on the existing explicit hybrid methods. The dispersion relations are developed in order to obtain methods with phase-lag of order infinity. Numerical illustrations indicate that PFHMs are much more efficient than the existing methods.

Finally, we constructed Optimized Hybrid methods (OPHMs) based on the existing non-zero-dissipative hybrid methods. To develop OPHMs; dissipative, dispersive and first derivatives of dispersive relations are required. We found that the non-zero-dissipative hybrid methods are more suitable to be optimized than phase-fitted. Numerical results are also given to prove the claim.

In conclusion, the IRK methods and hybrid methods are more efficient in solving first and second order DDEs respectively. The new methods constructed in this thesis are suitable for solving second-order ODEs and they are more efficient compared to the existing methods.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia
sebagai memenuhi keperluan untuk ijazah Master Sains

PENYELESAIAN BERANGKA BAGI PERSAMAAN PEMBEZAAN TUNDA TUNGGAL DAN PERSAMAAN PEMBEZAAN BIASA PERINGKAT KEDUA BAGI MASALAH NILAI AWAL BENTUK BERAYUN MENGGUNAKAN KAEDAH RUNGE-KUTTA DAN HIBRID

Oleh

SUFIA ZULFA BINTI AHMAD

Jun 2013

Pengerusi : **Professor Fudziah Bt Ismail, PhD**

Fakulti : **Sains**

Bahagian pertama tesis memberi tumpuan terutamanya untuk penyesuaian kaedah sedia ada bagi penyelesaian persamaan pembezaan tunda (PPT) untuk peringkat pertama dan kedua. Kaedah Runge-Kutta tertambah baik (RKT) dan Runge-Kutta (RK) disesuaikan untuk menyelesaikan PPT peringkat pertama. Kejituan dan kestabilan kaedah yang diterapkan kepada PPT peringkat pertama juga diteliti. Seterusnya kaedah sedia ada hibrid disesuaikan bagi menyelesaikan PPT peringkat kedua. Keputusan berangka dibandingkan dari segi ketepatan dan masa pengiraan dengan kaedah Runge-Kutta Nyström (RKN). Kestabilan kaedah apabila diterapkan kepada PPT peringkat kedua juga dipersembahkan.

Kaedah Separa-Tersirat Hibrid (STH) dibina untuk penyelesaian sistem masalah bentuk berayun. Kaedah ini mempunyai peringkat serakan dan lesapan yang tertinggi serta ralat pekali yang kecil. Selang berkala bagi kaedah turut diberikan. Keputusan berangka menunjukkan kaedah STH adalah lebih efektif dari kaedah-kaedah sedia ada.

Kaedah Lesapan Sifar Suai Secara Fasa Hibrid (SSFH) dibina berdasarkan kaedah sedia ada tak-tersirat hibrid. Hubungan secara serakan dibina untuk mendapatkan kaedah yang mempunyai fasa-lag peringkat infiniti. Ilustrasi berangka menunjukkan bahawa kaedah baru adalah lebih berkesan daripada kaedah-kaedah sedia ada.

Akhir sekali, kaedah Pengoptimuman Hibrid (PH) dibina berdasarkan kaedah hibrid lesapan tidak sifar sedia ada. Bagi pembinaan kaedah PH; hubungan antara lesapan, serakan dan pembezaan pertama serakan diperlukan. Didapati bahawa kaedah hibrid lesapan tidak sifar lebih sesuai untuk dioptimumkan berbanding suai secara fasa. Keputusan berangka juga diberikan untuk membuktikan dakwaan itu.

Kesimpulannya, kaedah- kaedah RKT dan hibrid adalah lebih efektif dalam menyelesaikan PPT peringkat pertama dan kedua. Kaedah-kaedah baru yang dibina di dalam tesis ini adalah sesuai untuk menyelesaikan Persamaan

Pembezaan Biasa (PPB) peringkat kedua dan kaedah tersebut lebih cekap berbanding dengan kaedah yang sedia ada.

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LIST OF ABBREVIATIONS

DDE	Delay Differential Equation
DIRKN	Diagonally Implicit Runge-Kutta Nyström
DIRKN(HS)	Three-stage fourth-order DIRKN method by Sommeijer (1987)
DIRKN3(4)	Three-stage fourth-order dispersive order six of DIRKN method derived by Senu et al (2010a)
DIRKN4(4)	Four-stage fourth-order dispersive order six method DIRKN method derived by Senu et al (2010b)
E-HYBRID3(4)	Explicit three-stage fourth-order hybrid methods developed by Franco (2006)
E-HYBRID5(6)	Explicit five-stage sixth-order hybrid method by Franco (2006)
ETSHM5	Explicit four-stage fifth-order hybrid methods by Franco (2006)
ETSHM5(8,5)	Explicit hybrid method of four-stage fifth-order with dispersion of order eight and dissipation of order five develop by Franco (2006)
ETSHM6	Explicit hybrid methods of five-stage sixth-order develop by Franco (2006)
HYBRID4(4)	Explicit four-stage fourth-order hybrid methods developed by Franco (2006)
IRK3	Explicit third order Improved Runge-Kutta methods derived by Rabiei (2011b)
IRK4	Explicit fourth order Improved Runge-Kutta methods derived by Rabiei (2011c)
IRK5	Explicit fifth order Improved Runge-Kutta methods derived by Rabiei (2011a)

IVP	Initial Value Problem
MPAFRKN4(4)	Modified Phase-fitted and Amplification fitted Runge-Kutta Nyström method of four-stage fourth-order by Papadopoulos et al (2010)
NDDI	Newton Divided Difference Interpolation
ODE	Ordinary Differential Equation
OPHM(ETSHM5(8,5))	New optimized hybrid method four-stage fifth order developed based on ETSHM5(8,5) derived in this thesis
OPHM(ETSHM5)	New optimized hybrid method four-stage fifth order developed based on ETSHM5 derived in this thesis
OPHM(ETSHM6)	New optimized hybrid method five-stage sixth order developed based on ETSHM6 derived in this thesis
OPRKN4(5)	New optimized Runge-Kutta Nyström method of four-stage fifth-order develop by Kosti et al (2012)
PFHM(ETSHM5(8,5))	New phase-fitted hybrid method four-stage fifth order developed based on ETSHM5(8,5) derived in this thesis
PFHM(ETSHM5)	New phase-fitted hybrid method four-stage fifth order developed based on ETSHM5 derived in this thesis
PFHM(ETSHM6)	New phase-fitted hybrid method five-stage sixth order developed based on ETSHM6 in this thesis
PFHM3(4)	Phase-fitted hybrid method of three-stage fourth-order and zero dissipation develop in this thesis
PFHM4(4)	Phase-fitted hybrid method of four-stage fourth-order and zero dissipation develop in this thesis

PFHM5(6)	Phase-fitted hybrid method of five-stage sixth-order and zero dissipation develop in this thesis
PFRKN4(4)	Phase-fitted hybrid method of four-stage fourth-order Runge-Kutta Nyström by Papadopoulos et al (2009)
PH	Kaedah Pengoptimuman Hibrid
PPT	Persamaan Pembezaan Tunda
RK3	Explicit third order Runge-Kutta methods in Butcher (2008)
RK4	Explicit fourth order Runge-Kutta methods developed in Butcher (2008)
RK7(6)	Seven-stage sixth-order RK method in Butcher (2008)
RKT	Kaedah Runge-Kutta tertambah baik
RKN3(4)	Explicit three-stage fourth-order Runge-Kutta Nyström method by Hairer (2010)
RKN4(5)	Explicit four-stage fifth-order Runge-Kutta Nyström method by Hairer (2010)
SIHM4(6,∞)	Semi-implicit hybrid method of order 4 with dispersive order 6 and zero dissipation develop in this thesis
SIHM5(6,∞)	Semi-implicit hybrid method of order 5 with dispersive order six and zero-dissipation develop in this thesis
SIHM5(8,5)	Semi-implicit hybrid method of order 5 with dispersive order eight and dissipation order five develop in this thesis
SSFH	Kaedah Suai Secara Fasa Hibrid

CHAPTER 1

INTRODUCTION

1.1 Literature Review

Differential equations have appeared in many practice originated in engineering, physical, social sciences and recently also great approaches in the field of biology and medicine. Some physical processes occur not only depend on the current state of the system but also the past states. Mathematical models of such process commonly result in differential equations with a time delay. This type of equation is called delay differential equations (DDEs) which the derivative at anytime depends on the solution at prior times and also known as model that incorporating past history. A more realistic model must include some of the past history of the system to determine the future behavior. DDEs often appear in connection with fundamental problems to analyze mathematical model in order to determine the future behavior.

There has been a growing interest in the field of DDEs, such as the work of Kuang (1993), Ismail and Suleiman (2000), Bellen and Zennaro (2003), Taiwo and Odetunde (2010) and many others. There are many applications which are well-known related to DDEs such as population dynamics, epidemiology and reforestation. For example, the process of reforestation involved replanting process will take at least 20 years before the tree reaching maturity. Hence, mathematical model of forest harvesting and regeneration must have time delay built into it.

In the first part of this study we are focusing on adapting existing methods for solving first and second order DDEs. The general form of a single first order delay differential equation with constant delay can be written as

$$y'(x) = f\big(x, y(x), y(x-\tau)\big), a \le x \le b, y(x_0) = y_0, x \in [-\tau, a] \qquad (1.1)$$

where τ is the delay term. Many authors have attempted to increase the efficiency of Runge-Kutta (RK) methods that required a lower number of function evaluations to solve first order initial value problems (IVPs). Consequently, Goeken et al (2000) proposed a class of RK method with higher derivatives approximations for the third and fourth-order methods. Phohomsiri and Udwadia (2004) constructed the accelerated Runge-Kutta (ARK) integration schemes for the third-order using two functions evaluations per step. Then, Udwadia and Farahani (2008) developed the ARK method for higher orders. Rabiei et al (2011b) constructed the Improved Runge-Kutta (IRK) method with reduced number of function evaluations which proposed a method of order three with two stages. Rabiei et al (2011c) then derived the order conditions and constructed the IRK method for solving ordinary differential equations (ODEs). The convergence and stability region of the methods were also discussed. Here, we use IRK and

RK methods to solve (1.1) and compare the methods efficiency in terms of accuracy. We use the same approaches as defined in Ismail and Suleiman (2000) to find the stability region of IRK methods when applied to first order DDEs.

The general form of the special second order delay differential equation with constant delay can be written in the form of

$$\left.\begin{array}{l} y''(x) = f\big(x, y(x), y(x-\tau)\big)\,,\ a \le x \le b, \\ y(x_0) = y_0,\ y'(x_0) = y_0',\ x \in [-\tau, a] \end{array}\right\} \quad (1.2)$$

where τ is the delay terms and first derivative does not appear explicitly. Apparently, the most common methods used for solving second order ODEs numerically is Runge-Kutta Nyström (RKN) method and also Runge-Kutta (RK) method after reducing the IVPs to first order ODEs. Franco (1995) proposed that second order ODEs can be solved using particular explicit hybrid algorithms or special multi-step methods. Coleman (2003) developed the algebraic order conditions of hybrid method up to order nine. Later, Franco (2006) constructed explicit two-step hybrid method of order four, five and six using the algebraic order condition developed by Coleman (2003) which have optimized error constant for solving second order IVPs. In this thesis we adapt the hybrid and RKN methods in order to solve (1.2). The RKN and hybrid methods are compared in terms of accuracy and stability regions. We use the same approaches as defined in Ismail and Suleiman

(2000) to find the stability region of hybrid methods. While, the stability region for RKN method is defined using the way proposed by Kuang and Cong (2005).

Many differential equations which appear in practice are system of second order IVP, in which the first derivative does not appear explicitly as

$$y'' = f(x, y),\ y(x_0) = y_0, y'(x_0) = y_0' \tag{1.3}$$

We are focusing on solving (1.3) directly for which it is known in advance that their solution is oscillating. While dealing with oscillatory problems, we need to consider the algebraic order conditions, dispersion (phase-lag) and dissipation (amplification error) properties when construction of a method. Bursa and Nigro (1980) first introduced the phase-lag of a method. Van der Houwen and Sommeijer (1987) proposed explicit RKN methods of order four, five, and six with reduced phase-lag of order six, eight, and 10 respectively. Senu et al (2010a) developed diagonally implicit RKN (DIRKN) method with dispersion of higher order for solving oscillatory problems. There are also some studies such as Samat et al (2012) in which they developed higher order explicit hybrid methods of order seven with phase-lag order eight and dissipation of order nine. In order to solve (1.3), semi implicit hybrid methods (SIHMs) are developed using the necessary algebraic condition, dispersion and dissipation relation. To implement the methods, accuracy and stability are two further factors for judging the efficiency of a method.

Some authors have developed hybrid methods with the purpose of making the phase-lag of the method smaller. For example, Van de Vyver (2007) provided a theoretical framework for a new type of phase-fitted and amplification-fitted of two-step hybrid methods for solving special second ODEs. Papadopoulos et al (2009) constructed phase fitted RKN (PFRKN) method using the dispersion relation in order to get method with phase lag of order infinity. The method is developed based on the Runge–Kutta–Nyström method of algebraic order four with four (three effective) stages by Dormand, El-Mikkawy and Prince (1987). In the literature, zero-dissipative phase-fitted two-step hybrid methods are developed using the same approaches as in Papadopoulos et al (2009) for solving second order ODEs. In this thesis, the phase-fitted hybrid methods (PFHMs) are constructed based on the existing zero-dissipative explicit hybrid methods originally developed by Franco (2006).

Lastly, we investigate effect of optimized and phase-fitted method for the modification of the existing non-zero dissipative hybrid methods. Simos (2012) developed the methodology of optimization of the efficiency of a hybrid two-step method for the numerical solution of the radial Schrödinger equation. The study is to focus on the vanishing of the phase-lag and its derivatives optimize the efficiency of the hybrid two-step method. Kosti et al

(2012) constructed an explicit RKN method with four stages and fifth algebraic order conditions. The variable coefficients of the preserved method result after nullifying the phase lag, the dissipative error and the first derivative of the phase-lag. In this study, both phase-fitted hybrid methods (PFHMs) and optimized hybrid methods (OPHMs) are develop based on the same non-zero-dissipative explicit methods originally by Franco (2006) for solving the second order ODEs. The OPHMs and PFHMs are constructed using the same approaches used by Kosti et al (2012) and Papadopoulos et al (2009) respectively. Therefore, the investigation of whether optimize methods or phase-fitted improve the accuracy of non-zero-dissipative methods are discussed in the research.

1.2 The Objective of the Thesis

The main objectives of this thesis can be summarized as follows:

1. To compare the efficiency of IRK with RK method, hybrid method with RKN method for solving first and second order DDEs respectively.
2. To construct Semi-Implicit Hybrid methods (SIHMs) using phase-lag and dissipative properties for solving oscillating problems.
3. To develop Phase-Fitted Hybrid methods (PFHMs) from existing zero-dissipative methods for solving second order ODEs.

4. To derive Optimize Hybrid methods (OPHMs) and PFHMs from existing non-zero-dissipative methods and investigate the effect of nullifying the properties of phase-lag, amplification error, and first derivative of phase-lag when designing the methods.

1.3 Outline of the Thesis

In Chapter 1, basic theory of numerical method and analysis of dispersion and dissipation of hybrid methods are discussed. In Chapter two, a brief explanation is given on DDEs and how the numerical methods are adapted for solving DDEs. Comparison of efficiency of the methods and their stability are also given. In Chapter three, we derived SIHMs of order four and two methods of order five. The dispersion and dissipation relations are applied in the derivation of the methods. The stability properties of the methods are also determined. Numerical results are presented and comparisons of the methods with some other implicit and explicit existing methods are given.

In Chapter four, we derive zero-dissipative explicit PFHMs of three-and four-stage fourth-order and five-stage sixth-order which based on the hybrid methods which were originally developed by Franco (2006). To get methods of phase-lag of order infinity, the dispersion properties are imposed for each of the hybrid methods using the same way as proposed in Papadopoulos et al (2009). Numerical results and comparison of the methods with the original hybrid methods and other methods in literature for solving special second order ODEs which have oscillating solutions are also included.

For non-zero-dissipative hybrid method, we construct four-stage fifth-order and five-stage sixth-order OPHMs in Chapter five. The methods are developed using the same approach introduced in Kosti et al (2012) which optimized the methods by imposing the dispersion, dissipation and first derivative of dispersion relation. The methods are based on the non-zero-dissipative hybrid method developed by Franco (2006). Numerical results and comparison on accuracy of the methods with the original hybrid method and methods in literature are also discussed. In addition, based on the same hybrid methods by Franco (2006) we also developed the phase-fitted version of the methods. OPHMs and PFHMs performance are compared to investigate which method improves the explicit non-zero-dissipative hybrid methods in terms of accuracy. Finally, the conclusion of the thesis is given in Chapter six.

1.4 Hybrid Method

An s -stage two-step hybrid method for numerical integration IVPs is in the form

$$Y_i = (1 + c_i)y_n - c_i y_{n-1} + h^2 \sum_{j=1}^{s} a_{ij} f\left(x_n + c_j h, Y_j\right), \tag{1.4}$$

$$y_{n+1} = 2y_n - y_{n-1} + h^2 \sum_{i=1}^{s} b_i f(x_n + c_i h, Y_i), \tag{1.5}$$

for $i = 1, \dots, s$, where the coefficients of b_i, c_i , and a_{ij} can be represented in Butcher tableau by the table of coefficients in Table 1.1.

Table 1.1: s-stage Hybrid methods

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array} = \begin{array}{c|ccc} c_1 & a_{1,1} & \cdots & a_{1,s} \\ \vdots & \vdots & \ddots & \vdots \\ c_s & a_{s,1} & \cdots & a_{s,s} \\ \hline & b_1 & \cdots & b_s \end{array}$$

The methods are characterized by two s-dimensional vectors, $\boldsymbol{b}$ and $\boldsymbol{c}$, with elements b_i and c_i, respectively, and $s \times s$ matrix **A** with elements a_{ij}. In vector notation, for an autonomous system of equations $y'' = f(y)$, (1.4) and (1.5) can be written in the form of

$$y_{n+1} = 2y_n - y_{n-1} + h^2(\boldsymbol{b}^T \otimes I) f(\boldsymbol{Y}),$$

$$\boldsymbol{Y} = (\boldsymbol{e} + \boldsymbol{c}) \otimes y_n - \boldsymbol{c} \otimes y_{n-1} + h^2 (A \otimes I) f(\boldsymbol{Y}), \tag{1.6}$$

where $\boldsymbol{e} = (1, \dots, 1)^T$.

The methods of the form (1.4) and (1.5) can be defined as

$$Y_1 = y_{n-1}, Y_2 = y_n, \tag{1.7}$$

$$Y_i = (1 + c_i)y_n - c_i y_{n-1} + h^2 \sum_{j=1}^{i} a_{ij} f\left(x_n + c_j h, Y_j\right), i = 3, \dots, s, \tag{1.8}$$

$$y_{n+1} = 2y_n - y_{n-1} + h^2[b_1 f_{n-1} + b_2 f_n + \sum_{i=3}^{s} b_i f(x_n + c_i h, Y_i)], \tag{1.9}$$

where $f_{n-1} = f(x_{n-1}, y_{n-1})$, $f_n = f(x_n, y_n)$ and the first two nodes are $c_1 = -1$ and $c_2 = 0$. This method is considered as two-step hybrid method because we only require to evaluate $f(t_n, y_n), f(x_n + c_3 h, Y_3), \dots, f(x_n + c_s h, Y_s)$ for each step after starting procedure. The general form of explicit hybrid method can be written in Butcher tableau in Table 1.2.

Table 1.2: *s*-stage Explicit Hybrid methods

-1	0				
0	0	0			
c_3	$a_{3,1}$	$a_{3,2}$	0		
$\vdots$	$\vdots$	$\vdots$	$\ddots$	0	
c_s	$a_{s,1}$	$a_{s,2}$	$\dots$	$a_{s,s-1}$	0
	b_1	b_2	$\dots$	b_{s-1}	b_s

1.5 Local Truncation Error and Algebraic Condition of Hybrid Method

Algebraic order condition of hybrid method was developed by Coleman (2003). The order conditions for two-step hybrid methods are derived by considering them as one-step methods of the form

$$u_n = u_{n-1} + h\phi(u_{n-1}, h), \tag{1.10}$$

where u_n is an appropriately defined numerical solution vector, and some starting procedure is used to generate u_0.

This approach is prompted by the work of Hairer and Warner (2012) for a class of two-step Runge-Kutta methods for differential equations of first order. The first equation in (1.6) can be written as a pair of equations by defining $F_n := (y_{n+1} - y_n)/h$ so that

$$y_n = y_{n-1} + hF_{n-1},$$

$$F_n = F_{n-1} + h(\boldsymbol{b}^T \otimes I) f(\boldsymbol{Y}).$$

These equations can be written as (1.10) with

$$u_n = \begin{pmatrix} y_n \\ F_n \end{pmatrix} \text{ and } \phi(u_{n-1}, h) = \begin{pmatrix} F_{n-1} \\ (\boldsymbol{b}^T \otimes I) f(\boldsymbol{Y}) \end{pmatrix},$$

where $\boldsymbol{Y}$ is defined by

$$\boldsymbol{Y} = (\boldsymbol{e} + \boldsymbol{c}) \otimes y_n - \boldsymbol{c} \otimes y_{n-1} + h^2 (A \otimes I)\, f(\boldsymbol{Y})$$

$$= \boldsymbol{e} \otimes y_{n-1} + h(\boldsymbol{e} + \boldsymbol{c}) \otimes F_{n-1} + h^2 (A \otimes I)\, f(\boldsymbol{Y}). \qquad (1.11)$$

The vector u_n is an approximation for $z_n = z(x_n, h)$, where z is the exact-value function defined by

$$z(x, h) = \begin{pmatrix} y(x) \\ \frac{y(x+h) - y(x)}{h} \end{pmatrix}. \qquad (1.12)$$

The local truncation error of the method at x_n is

$$d_n = z_n - z_{n-1} - h\phi(z_{n-1}, h), \qquad (1.13)$$

with $$\phi(z_{n-1}, h) = \begin{pmatrix} \frac{y(x_n) - y(x_{n-1})}{h} \\ (\boldsymbol{b}^T \otimes I) f(\boldsymbol{Y}) \end{pmatrix}, \qquad (1.14)$$

where $\boldsymbol{Y}$ is now defined implicitly as (1.11).

The order conditions that developed by Coleman (2003) for a s-stage, up to order seven for explicit hybrid methods are in Table 1.3.

Table 1.3: Order condition

Tree t	$\rho(t)$	Order condition
t_{21}	2	$\sum_{i=1}^{s} b_i = 1$
t_{31}	3	$\sum_{i=1}^{s} b_i c_i = 0$
t_{41}	4	$\sum_{i=1}^{s} b_i c_i^2 = \frac{1}{6}$
t_{42}		$\sum_{i=1}^{s} b_i a_{ij} = \frac{1}{12}$
t_{51}	5	$\sum_{i=1}^{s} b_i c_i^3 = 0$
t_{52}		$\sum_{i=1}^{s} b_i c_i a_{ij} = \frac{1}{12}$
t_{53}		$\sum_{i=1}^{s} b_i a_{ij} c_j = 0$
t_{61}	6	$\sum_{i=1}^{s} b_i c_i^4 = \frac{1}{15}$
t_{62}		$\sum_{i=1}^{s} b_i c_i^2 a_{ij} = \frac{1}{30}$
t_{63}		$\sum_{i=1}^{s} b_i c_i a_{ij} c_j = -\frac{1}{60}$
t_{64}		$\sum_{i=1}^{s} b_i a_{ij} a_{ik} = \frac{7}{120}$
t_{65}		$\sum_{i=1}^{s} b_i a_{ij} c_j^2 = \frac{1}{180}$
t_{66}		$\sum_{i=1}^{s} b_i a_{ij} a_{jk} = \frac{1}{360}$
t_{71}	7	$\sum_{i=1}^{s} b_i c_i^5 = 0$
t_{72}		$\sum_{i=1}^{s} b_i c_i^3 a_{ij} = \frac{1}{30}$
t_{73}		$\sum_{i=1}^{s} b_i c_i^2 a_{ij} c_j = 0$
t_{74}		$\sum_{i=1}^{s} b_i c_i a_{ij} a_{ik} = \frac{1}{30}$

t_{75}	$\sum_{i=1}^{s} b_i\, c_i a_{ij}\, c_j^{\,2} = \frac{1}{72}$
t_{76}	$\sum_{i=1}^{s} b_i\, c_i a_{ij}\, a_{jk} = -\frac{1}{720}$
t_{77}	$\sum_{i=1}^{s} b_i a_{ij}\, a_{ik}\, c_k = -\frac{1}{120}$
t_{78}	$\sum_{i=1}^{s} b_i a_{ij}\, c_j^{\,3} = 0$
t_{79}	$\sum_{i=1}^{s} b_i\, a_{ij}\, c_j\, a_{jk} = \frac{1}{360}$
$t_{7,10}$	$\sum_{i=1}^{s} b_i a_{ij}\, a_{jk}\, c_k = 0$

where value of $i > j > k$. The simplifying condition for hybrid method is

$$\sum_i^s a_{ij} = \frac{(c_i^{\,2}+c_i)}{2},\ \text{for } i = 1, \ldots, s\ ,\ j = i - 1 \text{ and } k = j - 1.$$

1.6 Analysis of the Periodicity, Absolute Stability, Dispersion and Dissipation

Stability analysis of explicit hybrid method has been discussed in Franco (2006). We apply the test equation $y''(t) = (i\lambda)^2 y(x) = -\lambda^2 y(x)$, for $\lambda > 0$ by replacing $f(x,y) = -\lambda^2 y(x)$ to the equation (1.4) and (1.5) and gives

$$Y_i = (1 + c_i)y_n - c_i y_{n-1} - h^2 \textstyle\sum_{j=1}^{s} a_{ij}\, \lambda^2 y(x),\ i = 1, \ldots, s, \tag{1.15}$$

$$y_{n+1} = 2y_n - y_{n-1} - h^2 \textstyle\sum_{i=1}^{s} b_i\, \lambda^2 y(x), \tag{1.16}$$

Let $H = h\lambda$, so equation (1.15) and (1.16) can be written as

$$Y_i = (1 + c_i)y_n - c_i y_{n-1} - H^2 \textstyle\sum_{j=1}^{s} a_{ij}\, y(x),\ i = 1, \ldots, s, \tag{1.17}$$

$$y_{n+1} = 2y_n - y_{n-1} - H^2 \textstyle\sum_{i=1}^{s} b_i\, y(x), \tag{1.18}$$

and equation (1.17) will give

$$\begin{aligned} Y_1 &= (1+c_1)y_n - c_1 y_{n-1} - H^2(a_{11}Y_1 + a_{12}Y_2 + \cdots + a_{1s}Y_s) \\ Y_2 &= (1+c_2)y_n - c_2 y_{n-1} - H^2(a_{21}Y_1 + a_{22}Y_2 + \cdots + a_{2s}Y_s) \\ &\vdots \\ Y_s &= (1+c_s)y_n - c_s y_{n-1} - H^2(a_{s1}Y_1 + a_{s2}Y_2 + \cdots + a_{ss}Y_s) \end{aligned} \tag{1.19}$$

Then, (1.19) and (1.18) can be written in vector form respectively as

$$\boldsymbol{Y} = (\boldsymbol{e}+\boldsymbol{c})y_n - \boldsymbol{c}y_{n-1} - H^2\boldsymbol{A}\boldsymbol{Y}, \tag{1.20}$$

$$y_{n+1} = 2y_n - y_{n-1} - H^2\boldsymbol{b}^{\boldsymbol{T}}\boldsymbol{Y}, \tag{1.21}$$

where $\boldsymbol{Y} = \begin{pmatrix} Y_1 \\ \vdots \\ Y_s \end{pmatrix}, \boldsymbol{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_s \end{pmatrix}, \boldsymbol{e} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \boldsymbol{A} = \begin{pmatrix} a_{11} & \cdots & a_{1s} \\ \vdots & \ddots & \vdots \\ a_{s1} & \cdots & a_{ss} \end{pmatrix}$, and $\boldsymbol{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_s \end{pmatrix}$.

By rearranging (1.20) we obtain

$$\boldsymbol{Y} = (\boldsymbol{I} + H^2\boldsymbol{A})^{-1}(\boldsymbol{e}+\boldsymbol{c})y_n - (\boldsymbol{I} + H^2\boldsymbol{A})^{-1}cy_{n-1} \tag{1.22}$$

where $(\boldsymbol{I} + H^2\boldsymbol{A})^{-1} \neq 0$. We substitute (1.22) into (1.21) and we get

$$y_{n+1} = \left(2 - H^2\boldsymbol{b}^{\boldsymbol{T}}(\boldsymbol{I} + H^2\boldsymbol{A})^{-1}(\boldsymbol{e}+\boldsymbol{c})\right)y_n - (1 - H^2\boldsymbol{b}^{\boldsymbol{T}}(\boldsymbol{I} + H^2\boldsymbol{A})^{-1}c)y_{n-1}. \tag{1.23}$$

Then we can rewrite (1.23) as

$$y_{n+1} - S(H^2)y_n + P(H^2)y_{n-1} = 0. \tag{1.24}$$

From (1.24) we obtain the following recursion

$$\mathrm{P}(\xi, H) = \xi^2 - S(H^2)\xi + P(H^2) = 0 \tag{1.25}$$

where $\quad S(H^2) = 2 - H^2\boldsymbol{b}^{\boldsymbol{T}}(\boldsymbol{I} + H^2\boldsymbol{A})^{-1}(\boldsymbol{e}+\boldsymbol{c})$

and $\quad P(H^2) = 1 - H^2\boldsymbol{b}^{\boldsymbol{T}}(\boldsymbol{I} + H^2\boldsymbol{A})^{-1}\boldsymbol{c}. \quad (1.26)$

The characteristic polynomial (1.25) represents the stability polynomial of hybrid method. The numerical solution defined by the difference (1.24) should be periodic therefore the necessary conditions are

$$P(H^2) \equiv 1, \quad \text{and} \quad |S(H^2)| < 2, \qquad \forall H \epsilon (0, H_p) \tag{1.27}$$

and interval $(0, H_p)$ is known as the periodicity interval of the method. The method is called zero dissipative $(d(H) = 0)$ if it satisfied conditions in (1.27). Otherwise, as the method possesses a finite order of dissipation, the integration process is stable if the coefficients of polynomial in (1.27) satisfy the conditions

$$P(H^2) < 1, \quad \text{and} \quad |S(H^2)| < 1 + P(H^2), \qquad \forall H \epsilon (0, H_s) \tag{1.28}$$

And interval $(0, H_s)$ is known as the interval of absolute stability of the method.

The first analysis of phase-lag was carried out by Bursa and Nigro (1980). Phase analysis can be divided into two parts. First is inhomogeneous which phase error is constant in time and second is homogeneous which the phase error are accumulated as n increases. As proposed by Franco (2006), the phase analysis is investigated using the second order homogeneous linear test model, $y''(x) = -\lambda^2 y(x)$. The steps to define phase analysis of hybrid method are the same from equation (1.15) to (1.26). Given that the exact solution for the homogeneous test $y'' = (i\lambda)^2 y(x)$ is

$$y(x_n) = 2|\varpi|\cos(X + nH). \tag{1.29}$$

The numerical solution of (1.5) is in the form of

$$y_n = 2|c||\rho|^n \cos(\omega + n\varphi). \tag{1.30}$$

Definition 1 (Apply the hybrid method (1.29) and (1.30)) We define the phase-lag $\phi(H) = H - \varphi$. If $\phi(H) = O(H^{q+1})$, then the hybrid method is said to be dispersive of order q. While, the quantity $d(H) = 1 - |\rho|$ is called as amplification error and if $d(H) = O(H^{r+1})$, then the hybrid method is said to have dissipation order r. According to Definition 1, if at a point h, $d(H) = 0$, then the hybrid method has zero dissipation at this point and it is dissipative otherwise. The error $\phi(H)$ and $d(H)$ are accumulated in the numerical process and therefore a cause of inaccuracy which leads to many integration steps to be performed. Hence, in this study we will focus on increasing the order of dispersion q (defined by $\phi(H) = O(H^{q+1})$) and the order of dissipation r (defined by $d(H) = O(H^{r+1})$). Dispersion (phase-lag) is the angle between the true and the approximated solution, whereas dissipation is the distance from a standard cyclic solution.

The following nomenclature given by Van der Houwen and Sommeijer (1987)

$$\phi(H) = H - \cos^{-1}\left(\frac{S(H^2)}{2\sqrt{P(H^2)}}\right) \tag{1.31}$$

$$d(H) = 1 - \sqrt{P(H^2)} \tag{1.32}$$

are called the dispersion error and the dissipation error, respectively. The general form of $S(H^2)$ and $P(H^2)$ for explicit hybrid methods in Franco (2006) is in the form of

$$S(H^2) = 2 - \alpha_1 H^2 + \alpha_2 H^4 - \alpha_3 H^6 + \cdots + \alpha_i H^{2i}, \alpha_i = 0 \text{ for } i > s. \tag{1.33}$$

$$P(H^2) = 1 - \beta_1 H^2 + \beta_2 H^4 - \beta_3 H^6 + \cdots + \beta_i H^{2i}, \beta_i = 0 \text{ for } i > s. \tag{1.34}$$

At the beginning of Chapter 2, we will discuss the stability of Improved Runge-Kutta (IRK) method when applied to first order DDE. The comparison of accuracy in terms of absolute error between IRK and Runge-Kutta (RK) method are also presented. Then, comparison between stability regions and accuracy of hybrid method and RKN method when applied to second order DDE are also discussed. The stability properties of the methods are defined using particular linear test model which will be discussed in detail in the next chapter.

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