UNIVERSITI PUTRA MALAYSIA

SPECTRAL HOMOTOPY ANALYSIS METHOD AND COMPOSITE CHEBYSHEV FINITE DIFFERENCE METHOD FOR SOLVING INTEGRO-DIFFERENTIAL EQUATIONS

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By

ZOHREH PASHAZADEH ATABAKAN

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Doctor of Philosophy

March 2015

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DEDICATIONS

To all my family members specially my husband, my daughter and my parents
for their love and continuous support
also
to all my teachers

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Doctor of Philosophy

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March 2015

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In this thesis, spectral homotopy analysis method (SHAM) is proposed for solving different type of second order integro-differential equations such as linear and nonlinear Volterra, Fredholm and Volterra-Fredholm integrodifferential equations. Linear and nonlinear systems of second order Fredholm integro-differential equations are solved using SHAM. In this method, the Chebyshev pseudo spectral method is used to solve the linear high-order deformation equations. The convergence analysis of the proposed method is proved, the error estimation of the method is done and the rate of convergence is obtained. Many different examples are solved using spectral homotopy analysis method to confirm the accuracy and the efficiency of the introduced method.

An efficient and accurate method based on hybrid of block-pulse functions and Chebyshev polynomials using Chebyshev-Gauss-Lobatto points is introduced for solving linear and nonlinear Fredholm and system of Fredholm integro-differential equations. The useful properties of Chebyshev polynomials and finite difference method make it a computationally efficient method to approximate the solution of Fredholm integro-differential equations. In this method, the given problem is converted into a system of algebraic equations using collocation points. The error bound of the method is estimated. Several numerical examples have been provided and compared with well-known approaches and exact solutions to confirm that the introduced method is more accurate and efficient. For future studies, some problems are proposed at the end of this thesis.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

KAEDAH ANALISIS SPEKTRUM HOMOTOPI DAN KAEDAH KOMPOSIT CHEBYSHEV BEZA TERHINGGA UNTUK MENYELESAIKAN PERSAMAAN PEMBEZAAN INTEGRO PERINGKAT KEDUA

Oleh

ZOHREH PASHAZADEH ATABAKAN

March 2015

Pengerusi: Prof. Adem Kılıçman, PhD
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Dalam tesis ini, kaedah analisis homotopi spektrum (SHAM) dicadangkan bagi menyelesaikan beberapa jenis berbeza persamaan-integro peringkat kedua seperti persamaan linear dan tak linear Volterra, Fredholm dan persamaan-integro Volterra-Fredholm. Sistem linear dan tak linear persamaan-integro Fredholm peringkat kedua, diselesaikan menggunakan SHAM. Dalam kaedah ini, kaedah Chebyshev pseudo spektrum telah digunakan untuk menyelesaikan persamaan linear berubah bentuk peringkat tinggi. Analisis penumpuan kaedah yang dicadangkan telah dibuktikan, anggaran ralat bagi kaedah ini telah dilakukan dan kadar penumpuan-nya juga diperolehi. Banyak contoh yang berlainan telah diselesaikan dengan menggunakan kaedah homotopi spektrum analisis untuk memastikan kejituan dan kecekapan kaedah yang telah diperkenalkan.

Satu keadah yang cekap dan tepat berdasarkan fungsi hibrid bagi blok-nadi dan polinomial Chebyshev yang menggunakan titik Chebyshev-Gauss-Lobato telah diperkenalkan untuk menyelesaikan persamaan pembezaan-integro linear dan tak linear dan sistem Fredholm. Ciri-ciri berguna polinomial Chebyshev dan kaedah beza terhingga membuatkan kaedah ini lebih cekap dari segi komputasi untuk menganggarkan penyelesaian persamamaan pembezaan-integro Fredholm. Dalam kaedah ini, masalah yang diberi ditukarkan kepada suatu sistem persamaan al-jabar menggunakan titik kolokasi. Batas ralat bagi kaedah ini dianggarkan. Beberapa contoh berangka telah diberikan dan dibandingkan dengan pendekatan yang terkenal dan penyelesaian sebenar untuk memastikan kaedah yang diperke-nalkan adalah lebih tepat dan cekap. Untuk kajian masa hadapan, beberapa masalah telah dicadangkan di akhir tesis ini.

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LIST OF ABBREVIATIONS

IEs	Integral Equations
FIEs	Fredholm Integral Equations
ASPM	Artificial parameter method
ADM	Adomian decomposition method
DM	Decomposition method
HAM	Homotopy Analysis Method
SHAM	Spectral Homotopy Analysis Method
CChFD	Composite Chebyshev finite difference method
ChFD	Chebyshev finite difference method
IDEs	Integro differential equations
HPM	Homotopy Perturbation Method
LCMM	Legendre collocation matrix method
RHFM	Rationalized Haar functions method
LIM	Lagrange interpolation method
WG	Wavelet Galerkin method
WC	Wavelet collocation method
Tm	Tau method
DTM	Differential transfer method
Sa	Sequential approach
IHPM	Improved homotopy perturbation method
LPs	Legendre polynomial solutions
BPF	Block-pulse functions
Hbf	Hat basis function
BPb	Bessel polynomial bases
IRKM	Iterative Reproducing Kernel Method
TP	Taylor polynomial solution
TC	Tau-Chebyshev

CHAPTER 1

INTRODUCTION

1.1 Background

In the past two decades there was a strong interest among physicists, engineers and mathematicians for the theory and numerical modeling of integral equations (IEs). Integral equations can be generally classified into two types as follows:

1. Fredholm integral equations (FIEs)

$$s(x)y(x) = f(x) + \lambda \int_a^b k(x,t)y(t)dt \tag{1.1}$$

where the kernel $k(x,t)$ and the functions $f(x)$ and $s(x)$ are given and λ is a parameter. In terms of value of $s(x)$ the following kinds of Fredholm integral equations can be defined. In particular

(i) If $s(x) = 0$, then integral equations (1.1) becomes

$$f(x) + \lambda \int_a^b k(x,t)y(t)dt = 0, \tag{1.2}$$

and is called Fredholm integral equations of the first kind.

(ii) If $s(x) = 1$, then integral equations (1.1) becomes

$$y(x) = f(x) + \lambda \int_a^b k(x,t)y(t)dt, \tag{1.3}$$

and is called Fredholm integral equations of the second kind.

(iii) If $s(x) \neq \{0, \text{constant}\}$ then integral Eqs. (1.1) is called Fredholm integral equations of the third kind.

2.Volterra integral equations (VIEs)

$$s(x)y(x) = f(x) + \lambda \int_a^x k(x,t)y(t)dt \tag{1.4}$$

where the kernel $k(x,t)$ and the function $f(x)$ are given and λ is a parameter and x is in the domain of integration $[a,b]$. Such as Fredholm integral equations, in terms of value of $s(x)$ Volterra integral equations fall under two types as follows:

(i) If $s(x) = 0$, then integral equations (1.4) becomes

$$f(x) + \lambda \int_a^x k(x,t)y(t)dt = 0, \tag{1.5}$$

and is called Volterra integral equations of the first kind.

(ii) If $s(x) = 1$, then integral equations (1.4) becomes

$$y(x) = f(x) + \lambda \int_a^x k(x,t)y(t)dt, \tag{1.6}$$

and is called Volterra integral equations of the second kind.

(iii) If $s(x) \neq \{0, \text{constant}\}$ then integral equations (1.4) is called Volterra integral equations of the third kind.

1.1.1 Degenerate kernel methods

In Fredholm Alternative Theorem (Jerri, 1999), integral equation with a degenerate kernel function was proposed. The method can be used very easily for solving Fredholm integral equations of the second kind (1.3).

Theorem 1.1 *(Atkinson, 2008) Consider*

$$\eta y(x) - \int_A k(x,t)y(t)dt = f(x), \quad t \in A \tag{1.7}$$

with $\eta \neq 0$ and $A \subseteq \mathbb{R}^m$, for some $m \geq 1$. Also A is considered as a closed bounded set. The integral operator P of (1.7) is defined by

$$\begin{aligned} P &: X \to X, \\ P(y(x)) &= \int_A k(x,t)y(t)dt, \end{aligned}$$

where P is a compact operator and $X = C(A)$ with $\|.\|_\infty$ or $X = L^2(A)$. The kernel function k can be approximated by a sequence of degenerate kernel

$$k(x,t) \approx k_m(x,t) = \sum_{j=1}^{m} a_{j,m}(x)b_{j,m}(t), \quad m \geq 1. \tag{1.8}$$

If associated integral operators P_m satisfy the following condition

$$\lim_{m\to\infty} \|P - P_m\| = 0 \tag{1.9}$$

and suppose x_m is the solution of the following approximating equation

$$f(x) + \int_A k_m(x,t)x_m(t)dt = \eta x_m(x), \quad t \in A, \tag{1.10}$$

and let the convergence (1.9) *be rapid then the solution y_m converge to y rapidly.*

Theorem 1.2 *(Atkinson, 2008) Let $\eta - P$ is an one to one and onto operator and X is a Banach space and P is bounded. In addition, if P_m is a sequence of* *bounded linear operators and satisfies the following property:*

$$\lim_{m\to\infty} \|P - P_m\| = 0,$$

then the operators

$$(\eta - P_m)^{-1} : X \xrightarrow[onto]{1-1} X$$

exist for $m \geq N$, *and*

$$\left\|(\eta - P_m)^{-1}\right\| \leq \frac{\|(\eta - P)^{-1}\|}{1 - \left\|(\eta - P)^{-1}\right\| \|P - P_m\|}.$$

Assume

$$(\eta - P)x = f$$

and

$$(\eta - P_m)x_m = f,$$

for $m \geq N$ *then*

$$\|x - x_m\| \leq \left\|(\eta - P_m)^{-1}\right\| \|Px - P_m x\|, \quad m \geq N. \tag{1.11}$$

An important consequence of the above convergence theorem is that the speed of convergence does not depend on the differentiability of the unknown x, since (1.11) implies,

$$\|x - x_m\| \leq \left\|(\eta - P_m)^{-1}\right\| \|P - P_m\| \|x\|, m \geq N. \tag{1.12}$$

Independent of the differentiability of x , $\|x - x_m\|$ converges to 0 if $\|P - P_m\|$ converges rapidly to 0. But other types of numerical methods that used for solving (1.1) do not have this properties.
If $X = C(A)$ the degenerate kernel (1.8) can be chosen such that the functions $a_i(t)$ are all continuous and the functions $b_j(s)$ are all absolutely integrable.
To use the above convergence theorem notice that:

$$\|P - P_m\| = \max_{x\in A} \int_A |k(x,t) - k_m(x,t)|\, dt.$$

If $X = L^2(A)$, it is necessary that all a_i and b_j belong to $L^2(A)$. To apply the convergence theorem, we use:

$$\|P - P_m\| \leq \left[\int_A \int_A |k(x,t) - k_m(x,t)|^2\, dxdt\right]^{\frac{1}{2}}.$$

If this is not sufficient, then other bounds are often possible. The kernels $k_m(x,t)$ should be chosen to make $\|P - P_m\|$ convergent to zero as rapidly as practicable.

Solution of degenerate kernel integral equations:

If the Eq.(1.8) is substituted into Eq. (1.7), this equation can be rewritten as follows:

$$\eta y_m(x) - \sum_{j=1}^{n} a_j(x) \int_A b_j(t) y_m(t) dt = f(x), t \in A. \tag{1.13}$$

Then the solution y_m can be represented by

$$y_m(x) = \frac{1}{\eta} \left[f(x) + \sum_{j=1}^{n} c_j a_j(x) \right], \tag{1.14}$$

where

$$c_j = \int_A b_j(t) y_m(t) dt. \tag{1.15}$$

If multiply (1.13) by $b_i(x)$ and then integrate over A the coefficient $\{c_j\}$ can be found by the following system

$$\eta c_i - \sum_{j=1}^{n} c_j \langle a_j, b_i \rangle = \langle f, b_i \rangle, i = 1, 2, ..., n, \tag{1.16}$$

with

$$\langle a_j, b_i \rangle = \int_A a_j(x) b_i(x) dx. \tag{1.17}$$

By solving the system (1.16), y_m can be obtained from (1.14).

1.1.2 Projection methods

The integral equation (1.7) can be solved approximately by choosing a finite dimensional family of functions which contains a function $\tilde{y}(t)$ close to exact solution $y(t)$. It is supposed that the obtained numerical solution $\tilde{y}(t)$ satisfies (1.7) approximately. $\tilde{y}(t)$ can be said to satisfy (1.7) approximately in various concepts which leads to different types of methods, such as collocation method (Atkinson, 2008).

1.1.3 Collocation method

A collocation method is a approximation method for solving ordinary differential equations, partial differential equations and integral equations numerically. In this method, we need to choose basis functions which are usually polynomials up to certain degree and collocation points as well. Finally, we select an approximation expansion function which satisfies the given problem at the collocation points (Mehrzad et al., 2013).

1.2 Adomian Decomposition Method (ADM)

Adomian decomposition method or Decomposition method (DM) was proposed by a development Adomian method. This method was proposed by Adomian in his book (Adomian, 1994) and other authors have then mentioned this method in their works (Yousif, 2008; Meyers, 2011). The efficiency and validity of this method have been proved for a wide class of equations. In this procedure, the solution $y(x)$ is determined in a series form defined by

$$y(x) = \sum_{n=0}^{\infty} y_n(x). \tag{1.18}$$

The method is a powerful semi analytic technique which can be used for solving problems with strong nonlinearity. This method has some advantages. We can apply ADM to solve ordinary and partial differential equations even if they do not contain small or large parameters. The rate of convergence in the method is high and the convergence of the solution obtained by the method was proved in (Abbaoui and Cherruault, 1994). However, Adomian decomposition method has also some constraints. In this method approximate solutions usually contain polynomials. In general, power series can not be considered as an efficient set of base functions to approximate a nonlinear problem since power series have small convergence intervals thus acceleration methods should usually enlarge convergence regions. In the Adomains decomposition method, we do not have freedom to use different base functions. Like the artificial small parameter method and the δ-expansion method, Adomians decomposition method itself also does not provide us with a convenient way to adjust convergence region and rate of convergence.

1.3 Artifficial Small Parameter Method (ASPM)

The artificial small parameter method was presented by Lyapunov (1992). In this method Lyapunov considered the equation

$$\frac{dx}{dt} = B(t)x \tag{1.19}$$

where $B(t)$ is a time periodic matrix. Lyapunov (1992) defined an artificial parameter ϵ and then replaced (1.19) with the equation

$$\frac{dx}{dt} = \epsilon B(t)x. \tag{1.20}$$

In this method, power series expansions are calculated over ϵ for the solutions. In many cases the convergence of the series was proved for $\epsilon = 1$ by Lyapunov. Thus in the final expression we put $\epsilon = 1$.

1.4 Homotopy Analysis Method (HAM)

Suppose a nonlinear equation

$$g(t) = 0. \tag{1.21}$$

The so-called homotopy can be constructed by

$$\mathcal{H}[t;p] = (1-p)[g(t) - g(t_0)] + pg(t), \tag{1.22}$$

where an initial approximation of t is shown by t_0 and $p \in [0,1]$ denotes an embedding parameter or homotopy parameter. It is clear that,

$$\mathcal{H}[t;p]\Big|_{p=0} = g(t) - g(t_0), \quad \mathcal{H}[t;p]\Big|_{p=1} = g(t). \tag{1.23}$$

So when the embedding parameter $p \in [0,1]$, $\mathcal{H}[t;p]$ changes continuously from $g(t) - g(t_0)$ to $g(t)$. In topology, this kind of continuous variation is known as deformation. Let

$$\mathcal{H}[t;p] = 0, \tag{1.24}$$

i.e.

$$(1-p)[g(t) - g(t_0)] + pg(t) = 0. \tag{1.25}$$

Eq. (1.25) is a family of algebraic equations and the solution depends upon the embedding parameter p. Further (1.25) can be represented as

$$(1-p)[g(\psi(p)) - g(t_0)] + pg(\psi(p)) = 0. \tag{1.26}$$

Eq. (1.26), is the same as the original equation (1.25), when the embedding parameter p alters from 0 to 1, $\psi(p)$ changes from the initial approximation t_0 to the solution t of $g(t) = 0$. The family of equations (1.26) is called *the zeroth-order deformation equation.* Now $\psi(p)$ as a function of the embedding parameter p can be expanded into Maclaurin series.

$$\psi(p) = t_0 + \sum_{n=1}^{+\infty} t_n p^n. \tag{1.27}$$

where $\psi(0) = t_0$ and

$$t_n = \frac{1}{n!}\frac{\partial^n \psi(p)}{\partial p^n}\Big|_{p=0} = D_n(\psi). \tag{1.28}$$

The series (1.27) is named *homotopy series* and $D_n(\psi)$ is called the *nth-order homotopy-derivative* of ψ. Suppose the homotopy series (1.27) is convergent at $p = 1$. Assuming $\psi(1) = t$ in Eq. (1.27), the so-called *homotopy series* solution can be defined as follows:

$$t = t_0 + \sum_{n=1}^{+\infty} t_n. \tag{1.29}$$

However, many functions have Maclaurin series with radius of convergence less than 1. In this method, it is considered that the homotopy-series is convergent at $p = 1$. An auxiliary parameter is introduced to solve this limitation. In view of the basic theorem regarding Taylor series, the homotopy-series (1.27) has unique coefficient t_n. So, the unique governing equation of t_n can be derived from the zeroth-order deformation equation (1.26). If the 1st-order homotopy-derivative is taken on both sides of the zeroth-order deformation equation (1.26), the so-called

1st-order deformation equation is given by

$$t_1 g'(t_0) + g(t_0) = 0, \tag{1.30}$$

with solution

$$t_1 = -\frac{g(t_0)}{g'(t_0)}.$$

Similarly, taking the 2nd-order homotopy-derivative results in the 2nd-order deformation equation as follows:

$$t_2 g'(t_0) + \frac{1}{2} t_1^2 g''(t_0) = 0, \tag{1.31}$$

with the solution

$$t_2 = -\frac{t_1^2 g''(t_0)}{2g'(t_0)} = -\frac{g^2(t_0) g''(t_0)}{2[g'(t_0)]^3}.$$

Following the same procedure, one gets t_n for $n = 1, 2, 3, \ldots$. Obviously, these high-order deformation equations are linear, and so they are easy to solve. The 1st-order homotopy-series approximation can be presented by

$$t \approx t_0 + t_1 = t_0 - \frac{g(t_0)}{g'(t_0)}, \tag{1.32}$$

and also the 2nd-order homotopy-series approximation is shown as follows:

$$t \approx t_0 + t_1 + t_2 = t_0 - \frac{g(t_0)}{g'(t_0)} - \frac{g^2(t_0) g''(t_0)}{2[g'(t_0)]^3}. \tag{1.33}$$

The disadvantage of the homotopy-series (1.27) is that the series is not always convergent at $p = 1$, so the homotopy-series solution (1.29) will be divergent. A nonzero auxiliary parameter h was introduced by Liao (1997) to solve this limitation of the early HAM. Liao (1997) proposed a zeroth-order deformation equation using the auxiliary parameter $h \neq 0$ as follows,

$$(1 - p)[g(\psi(p)) - g(t_0)] = phg(\psi(p)). \tag{1.34}$$

Obviously, when $p = 1$ it holds $hg(\psi(1)) = 0$. Since $h \neq 0$ so $g(\psi(1)) = 0$. In the new homotopy, all other formulas such as Eqs (1.27) and (1.29) are the same, but the high-order deformation equation is different. In the same way, if the 1st-order homotopy-derivative operators on both sides of (1.34), the corresponding 1st-order deformation equation is obtained as

$$t_1 g'(t_0) - hg(t_0) = 0, \tag{1.35}$$

with solution

$$t_1 = h\frac{g(t_0)}{g'(t_0)}. \tag{1.36}$$

By taking the 2nd-order homotopy-derivative on both sides of (1.34) gives the

2nd-order deformation equation:

$$t_2 g'(t_0) - (1+h)t_1 g'(t_0) + \frac{1}{2}t_1^2 g''(t_0) = 0, \tag{1.37}$$

then we have

$$t_2 = (1+h)t_1 - \frac{t_1^2 g''(t_0)}{2g'(t_0)} = h(1+h)g(t_0) - \frac{g_1(t_0)g''(t_0)}{2[g'(t_0)]^3}. \tag{1.38}$$

The corresponding first-order and second-order homotopy-series approximation can be presented as follows,

$$t \approx t_0 + t_1 = t_0 + h\frac{g(t_0)}{g'(t_0)}, \tag{1.39}$$

$$t \approx t_0 + t_1 + t_2 = (1+h+h^2)t_0 + h\frac{g(t_0)}{g'(t_0)} - \frac{g^2(t_0)g''(t_0)}{2[g'(t_0)]^3}, \tag{1.40}$$

respectively. It is clear that when $h = -1$, Eqs. (1.39) and (1.40) are the same as (1.32) and (1.33), respectively. In Eq.(1.34), the auxiliary-parameter h is considered as a iteration factor and if it is chosen properly then the convergence of iteration can be ensured. Also, (Liao (1999), Liao and Sherif (2004), Liao and Magyari (2006), Liao and Tan (2007), Liao (2009)) found that the convergence of the homotopy-series like (1.27) dependent upon the value of h. The auxiliary parameter h provides us with a simple way to ensure the convergence of series solution. Because of this reason, h was renamed to the convergence-control parameter.

1.4.1 Some characteristics of homotopy-derivative

Definition 1.1 *(Liao, 2009) Suppose ψ is a function of the homotopy-parameter p, so the nth-order homotopy-derivative of ψ can be considered as follows:*

$$D_n\psi = \frac{1}{n!}\frac{\partial^n(\psi(p))}{\partial p^n}\Big|_{p=0} \tag{1.41}$$

where n is an positive integer.

Definition 1.2 *(Liao, 2009) Consider the general nonlinear problem as,*

$$N[x(t)] = 0. \tag{1.42}$$

If ψ is a function of p, whose Maclaurin series is

$$\psi(p) = \sum_{n=0}^{+\infty} x_n p^n, \tag{1.43}$$

then the zeroth-order deformation equation of Eq. (1.42) is the family of equations

$$(1-p)L[x(t;p) - x_0(t)] = ph(N[x(t;p)]),\ p \in [0,1]. \tag{1.44}$$

Eq. (1.44) *is equivalent to the original Eq.* (1.42) *at* $p = 1$ *so that*

$$x = \psi(p)\Big|_{p=1} = \sum_{n=0}^{+\infty} x_n, \tag{1.45}$$

and it is also equal to initial guess at $p = 0$. *The Eq.* (1.43) *is called homotopy series solution of* (1.42), *and Eq.* (1.44) *is called the nth-order deformation equation.*

Theorem 1.3 *Suppose* g *and* k *be two independent functions of embedding parameter* p *and* ψ *and* ϕ *be homotopy series. Then the following relation*

$$D_n(g\psi + k\phi) = gD_n(\psi) + kD_n(\phi), \tag{1.46}$$

is true.

Proof. To see the proof of this theorem, interested readers are referred to (Liao, 2009)

Theorem 1.4 *If* ψ *and* ϕ *be two homotopy series,*

$$\psi(p) = \sum_{k=0}^{\infty} x_k p^k, \quad \phi(p) = \sum_{j=0}^{\infty} y_j p^j, \tag{1.47}$$

then it can be concluded that

(i) $D_n(\phi) = y_n$,

(ii) $D_n(p^k\phi) = D_{n-k}(\phi)$,

(iii) $D_n(\psi\phi) = \sum_{j=0}^{n} D_j(\psi)D_{n-j}(\phi) = \sum_{j=0}^{n} D_j(\phi)D_{n-j}(\psi)$,

(iv) $D_n(\psi^r\phi^s) = \sum_{j=0}^{n} D_j(\psi^r)D_{n-j}(\phi^s) = \sum_{j=0}^{n} D_j(\phi^s)D_{n-j}(\psi^r)$,

where $n,\ r,\ s \geq 0$ *and* $0 \leq k \leq n$ *are integer.*

Proof. To see the proof of this theorem, interested readers are referred to (Liao, 2009)

Theorem 1.5 *Suppose* L *is a linear independent operator of embedding parameter* p. *If* ψ *is a homotopy series*

$$\psi = \sum_{i=0}^{\infty} x_i p^i, \tag{1.48}$$

then the following equality can be resulted

$$D_n(L\psi) = L(D_n\psi). \tag{1.49}$$

Proof. To see the proof of this theorem, interested readers are referred to (Liao, 2009)

1.4.2 Some properties of the high-order deformation equations

Lemma 1.1 *Suppose homotopy series* ψ *is defined as*

$$\psi(x) = \sum_{n=0}^{\infty} y_n(x)p^n, \tag{1.50}$$

where $p \in [0,1]$ *is the embedding parameter,* $y_n(x)$ *is a function of variable* x*. Consider* L *as an auxiliary linear operator with respect to* x*, and* $y_0(x)$ *is an initial guess. We have*

$$D_n\{(1-p)L[\psi(x) - y_0(x)]\} = L[y_n(x) - \chi_n y_{n-1}(x)], \tag{1.51}$$

where the operator D_n *is given by* (1.28) *and* χ_n *is defined as*

$$\chi_n = \begin{cases} 0, & n \leq 1 \\ 1, & otherwise. \end{cases} \tag{1.52}$$

Proof. To see the proof of this lemma, interested readers are referred to (Liao, 2009)

Theorem 1.6 *Suppose homotopy series* ψ *is defined as*

$$\psi(x) = \sum_{n=0}^{\infty} y_n(x)p^n, \tag{1.53}$$

where $p \in [0,1]$ *is the embedding parameter. Consider* L *as an auxiliary linear operator* N *is a nonlinear operator,* $y_0(x)$ *is an initial guess* h *is a nonzero auxiliary parameter and* $H(x)$ *a nonzero auxiliary function. If the zeroth-order deformation equation is given by*

$$(1-p)L[\psi(x) - y_0(x)] = phH(x)N[\psi(x)], \tag{1.54}$$

then nth order deformation equation can be defined as follows:

$$L[y_n(x) - \chi_n y_{n-1}(x)] = hH(x)D_{n-1}(N[\psi(x)]), \quad n \geq 1. \tag{1.55}$$

Proof. To see the proof of this theorem, interested readers are referred to (Liao, 2009) In view of Definition 1.1 and Theorem 1.6, the so-called higher-order deformation equation (1.55) can be defined as follows (Liao, 2009).

$$L[y_n(x) - \chi_n y_{n-1}(x)] = hH(x)R_n(\vec{y}_{n-1}), \tag{1.56}$$

where

$$R_n(y_{n-1}) = \frac{1}{(n-1)!}\frac{\partial^{n-1}N[\psi(x)]}{\partial p^{n-1}}\Big|_{p=0}. \tag{1.57}$$

Theorem 1.7 ***(Convergence theorem)***
If the following series

$$y_0(x) + \sum_{n=1}^{+\infty} y_n(x), \tag{1.58}$$

is obtained from Eqs (1.56) *and* (1.57) *is convergent then it converges to the exact solution of the problem.*

Proof. To see the proof of this theorem, interested readers are referred to (Liao and Sherif, 2004).

Theorem 1.8 *If the series*

$$y_0(x) + \sum_{n=1}^{+\infty} y_n(x), \tag{1.59}$$

is obtained from Eqs (1.56) *and* (1.57) *is convergent, it must be a solution of*

$$\sum_{n=1}^{+\infty} R_n(y_{n-1}, x) = 0. \tag{1.60}$$

Proof. To see the proof of this theorem, interested readers are referred to (Liao and Sherif, 2004).

1.4.3 Convergence control parameter h

Suppose that we obtain a family of solution series in terms of the auxiliary parameter h by employing homotopy analysis method. The main question that arises here is the way of choosing the value of h such that the solution series converges as fast as possible in a large region. One way to choose a suitable value of h is to obtain the exact square residual error integrated in the whole region T which is defined as following:

$$R(h) =_T \left[N\left(\sum_{i=0}^{m} y_i(t)\right)\right]^2 dt, \tag{1.61}$$

where N is a nonlinear operator and m is the order of approximation. It is obvious that the more quickly $R(h)$ decreases to zero, the faster the corresponding series solution converges. Therefore, at the given order of approximation m, the corresponding optimal value of the h is given by the minimum of $R(h)$, corresponding to a nonlinear algebraic equation of the form

$$\frac{dR(h)}{dh} = 0.$$

The main disadvantage of the way is being very time-consuming even if for low-order approximation especially if the approximation include unknown converges-control parameters and other physical quantities. On the other hand, many nonlinear problems include some important physical quantities which also depend on the converges-control parameter h. Liao and Sherif (2004) suggested to plot the curves of these quantities verses h for example, we can plot $y'(0)$ or $y''(0)$ as a function of h to determine the region of h in which homotopy series-solution is convergent. Due to Theorems 1.7 and 1.8 all convergent series-solution given by different values of h converge to its exact solution. Therefore, providing to existence of unique solution, all approximations converge to the same value and therefore the curve of that quantity verse h contains a horizontal line segment $R(h)$ which is called the valid region of h and the curve is called $h-$curve. Having chosen the so called valid region of h, we make sure that the corresponding solution series converge. It should be mentioned that the more h-curves are graphed, the easier it is to determine the value of h. Abbasbandy et al. (2011) proved it mathematically that the horizontal line occurs in the plot of homotopy series solution at some points corresponding to the convergence control parameter h. However, it should be noted that h-curves can not provide us with the optimal value of h. Yabushita et al. (2007) employed the so-called optimization method to minimize the $R(h)$ in Eq. (1.61). Marinca et al. (2008) established the optimal homotopy-asymptotic method. They obtained a set of nonlinear algebraic equations about $h_1, h_2, ..., h_m$, by minimizing the $R(h)$ in Eq. (1.61). In theory, if we have the more convergence-control parameters, we get the better approximation by this optimal homotopy analysis method. However, for a complicated nonlinear problem, it is time-consuming to obtain the corresponding square residual errors as there are so many unknown parameters especially at high-order of approximations. It was stated that the optimal approach developed by Marinca et al. (2008) often does not work in practice (Niu and Wang, 2010). In Liao (2010), a modification to the work of Marinca et al. (2008) is proposed. His method contains at most three unknown convergence-control parameters h_1, h_2, h_3 at any order of approximations. He showed that even if $h_2 = h_3 = 0$ (means that $R(h)$ in Eq. (1.61) depends to h_1 only), it needs too much CPU time to calculate $R(h)$ in Eq. (1.61). It takes $68.13s$, $272.7s$ and $1089.5s$ to obtain the corresponding exact residual error (1.61) for $M = 6, 8$ and 10, respectively. Therefore, to decrease the CPU time, he considered the following so-called averaged residual error:

$$R(h) = \frac{1}{L}\sum_{i=0}^{L}\left[N\left(\sum_{j=0}^{m} y_j(i\Delta t)\right)\right]^2, \tag{1.62}$$

where $\Delta t = \frac{10}{L}, L = 20$ for Blasius flow problem. It should be emphasized that the determination of optimal convergence-control parameter h is suggested for a given complicated physics problem.

1.4.4 Some basic rules in homotopy analysis method

Rule of solution expression

The main part of approximating a function is to choose an appropriate set of base functions. The better approximation is achived by choosing the more suitable basis functions. In order to solve a given nonlinear problem by using HAM, we approximate the solution $y(x)$ of the governing equation in terms of a set of base functions as follows,

$$y(x) \approx \sum_{m=0}^{M} c_m \phi_m(x). \tag{1.63}$$

The rule of solution expression states that having been selected a set of base functions, the auxiliary linear operator L, the initial approximation $y_0(x)$, and the auxiliary function $H(x)$ must be determined with the condition that all solutions of the higher order deformation equations (1.55) exists and can be presented by this set of base functions.

Rule of coefficient ergodicity

In order to have further restriction on the choosing of the auxiliary function $H(x)$, it seems that we should consider another rule. The rule of coefficient ergodicity expresses that all coefficients in Eq. (1.63) should be modified such that the set of base functions $\{\phi_m(x) \mid m = 0, 1, 2, ...\}$ is complete.

Rule of solution existence

The rule of solution existence states that the higher order deformation equation should be closed and have solutions.

1.5 Motivation and problem statement

An integro-differential equation is an equation where both differential and integral operator are included in the equation. In this kind of equations one or more unknown functions $y(x)$ and its nth order derivatives appear both outside and under the integral operator. As a general form, the nth order non linear Fredholm integro-differential equations can be considered as follows:

$$\begin{cases} F(x, y(x), y'(x), y''(x), \dots, y^{(n)}(x)) = f(x) + \int_a^b k(x,t) G(y(t)) dt, \\ H_r(y(\tau_0), \dots, y^{(n-1)}(\tau_0), \dots, y(\tau_n), \dots, y^{(n-1)}(\tau_n)) = 0, \quad r = 0, \dots, n-1, \end{cases} \tag{1.64}$$

and also nth order non linear Volterra integro differential equations can be presented as follows,

$$\begin{cases} F(x, y(x), y'(x), y''(x), \dots, y^{(n)}(x)) = f(x) + \int_a^x k(x,t) G(y(t)) dt, \\ H_r(y(\tau_0), \dots, y^{(n-1)}(\tau_0), \dots, y(\tau_n), \dots, y^{(n-1)}(\tau_n)) = 0, \quad r = 0, \dots, n-1, \end{cases} \tag{1.65}$$

where $k(x,t), f(x)$ and $y(x)$ are analytic functions, x is a variable, $H_r, r = 0, \dots, n-1$, are linear functions, and the points $\tau_0, \tau_1, \dots, \tau_n$ lie in $[a, b]$. Non linear equations are much more difficult to solve than linear ones, especially by means of analytic methods. In general, there are two standards for a satisfactory analytic method of nonlinear equations:

(a) It should be able to provide efficient analytical approximations.

(b) It should guarantee the accuracy of analytic approximations for all physical parameters. These two standards can be used as criteria in order to compare different analytical techniques for solving nonlinear problems.

In Perturbation techniques small or large parameters called perturbation quantities which appear in governing equations or initial or boundary conditions play a main role. Perturbation approximations can be generally represented in a series of perturbation quantities, and the main nonlinear equations are altered by some linear or nonlinear sub-problems. The sub-problems are specified by the main governing equation and also by the place where perturbation quantities appear. Perturbation methods are simple and easy to understand. Especially, based on small physical parameters, perturbation approximations often have clear physical meanings. There is not such kind of perturbation quantity in every nonlinear problem. In addition, although a small parameter can exist in the problem, the sub-problem might have no solutions, or might be rather complicated so that only a few of the sub-problems can be solved. Thus, there is no guarantee that any nonlinear problem can always get perturbation approximations efficiently. Also most perturbation approximations are accurate only for small physical parameters. In general, a perturbation result is not always credible in the whole region of all physical parameters (Liao and Sherif, 2004).

Some traditional non perturbation methods have been developed to improve the restrictions of perturbation techniques. For example, Lyapunovs artificial small parameter method, the δ-expansion method, Adomian decomposition method, and so on. In these methods, the solution of a given problem is approximated as a series of a so-called artificial parameter. One of the drawbacks of non-perturbation techniques is that there is no instruction to determine a suitable place for the artificial small parameter to get a better approximation. Another disadvantage of these methods is that there is not guarantee that the approximation series will converge to the exact solution. For example in Adomian decomposition method, we employ the linear operator $\frac{d^k}{dx^k}$ in most cases, where k is considered as the highest order of derivative of governing equations, and so it is quiet simple to get solutions of the related sub-problems using integration k times with regard to x. However, using such simple linear operator we get approximate solutions as a power-series with a finite radius of convergence. Therefore it is not guaranteed the approximation solution converges to the exact solution for all physical parameter.

Homotopy analysis method (HAM) is proposed in Liao (1992). In recent years, many authors have considered this method for solving different type of integral and integro- differential equations. Unlike the traditional non perturbation methods such as Adomian decomposition method, and the δ-expansion method, which are the special cases of HAM, this scheme does not need a small perturbation parameter. In the HAM, the original nonlinear problem is converted to infinite number of linear problems without using the perturbation techniques. Homotopy analysis method is powerful than traditional perturbation methods since it is applicable for solving problems with strong nonlinearity even if they do not have any small or large parameters. This method can help to adjust the convergence region and the rate of approximation series solution and allow us to choose different base functions to approximate a nonlinear problem.

In using the HAM, and in order to effectively control the region and the rate of convergence of the HAM series solution, one has to carefully select an initial approximation, an auxiliary linear operator, an auxiliary function and a convergence controlling auxiliary parameter. (Shidfar et al., 2010)
One of the main limitation of the homotopy analysis method is the necessity that the solution we are looking for needs to comply with the so-called rule of solution expression and the rule of coefficient ergodicity that lead us in selecting the appropriate initial approximations, the auxiliary linear operators and the auxiliary functions. These parameters are chosen in such a way that the obtained higher order deformation equations can be simply solved using high-performance computers and symbolic computation software. Convenient initial approximation may indeed not be a good guess of the solution. On the other hand, complicated initial approximations and linear operators might result in the higher order deformation equations that are not easy or even possible to integrate.

Spectral homotopy analysis method is a combination of the HAM with the Chebyshev spectral method so in this method we have larger freedom to choose auxiliary linear operators. In theory, any continuous function in a bounded interval can be best approximated using Chebyshev polynomial. So, the SHAM provides larger freedom to choose the auxiliary linear operator and initial guess.

The useful properties of Chebyshev polynomials and block-pulse functions make it a computationally efficient method to approximate the solution of Fredholm integro–differential equations. In composite Chebyshev finite difference method, the given problem is converted to a system of algebraic equations by using collocation points. The main advantage of the present method is the ability to represent smooth and especially piecewise smooth functions properly. In this approach the accuracy can be improved either by increasing the number of subintervals or by increasing the number of collocation points in subintervals.

1.6 Objectives of the Research

In this thesis the following integral equations, are considered

1. Second order linear Fredholm integro– differential equation:

$$\begin{cases} \sum_{j=0}^{2} a_j(x)y^{(j)}(x) = g(x) + \int_a^b k(x,t)y(t)dt, \\ y(a) = \alpha, \ y(b) = \beta. \end{cases} \tag{1.66}$$

2. Second order nonlinear Fredholm integro– differential equation:

$$\begin{cases} \sum_{j=0}^{2} a_j(x)y^{(j)}(x) = g(x) + \int_a^b k(x,t)f(y(t))dt, \\ y(a) = \alpha, \ y(b) = \beta. \end{cases} \tag{1.67}$$

3. Second order linear Volterra integro– differential equation:

$$\begin{cases} \sum_{j=0}^{2} a_j(x)y^{(j)}(x) = g(x) + \int_a^x k(x,t)y(t)dt, \\ y(a) = \alpha, \ y(b) = \beta. \end{cases} \tag{1.68}$$

4. Second order nonlinear Volterra integro– differential equation:

$$\begin{cases} \sum_{j=0}^{2} a_j(x)y^{(j)}(x) = g(x) + \int_a^x k(x,t)f(y(t))dt, \\ y(a) = \alpha, \ y(b) = \beta. \end{cases} \tag{1.69}$$

5. Second order linear Volterra-Fredholm integro– differential equation:

$$\begin{cases} \sum_{j=0}^{2} a_j(x)y^{(j)}(x) = g(x) + \int_a^x k_1(x,t)y(t)dt + \int_a^b k_2(x,t)y(t)dt, \\ y(a) = \alpha, \ y(b) = \beta. \end{cases} \tag{1.70}$$

6. Second order linear and nonlinear system of Fredholm integro– differential equation:

$$\begin{cases} H_1(x,v,v',v'',w,w',w'') = g_1(x) + \int_a^b k_1(x,t,v(t),w(t))dt, \\ H_2(x,v,v',v'',w,w',w'') = g_2(x) + \int_a^b k_2(x,t,v(t),w(t))dt, \\ v(a) = \alpha_1, \ v(b) = \alpha_2, \ w(a) = \beta_1, \ w(b) = \beta_2. \end{cases} \tag{1.71}$$

The specific objectives of the research are addressed as follows:

1. To combine the homotopy analysis method and Chebyshev pseudo spectral transformation to solve linear Fredholm (1.66) and Volterra integro– differential equations (1.68) and comparing the obtained results with homotopy analysis method.

2. To apply spectral homotopy analysis method to solve nonlinear Fredholm (1.67) and Volterra integro– differential equations (1.69).

3. To use spectral homotopy analysis method to solve Volterra-Fredholm integro - differential equations (1.70).

4. To prove the convergence of spectral homotopy analysis method and to investigate the error analysis and the rate of convergence of the method.

5. To apply spectral homotopy analysis method to solve linear and nonlinear system of Fredholm integro- differential equations (1.71).

6. Solving linear and nonlinear Fredholm integro– differential equations (1.66)-(1.67) using composite Chebyshev finite difference method, compare the results with some other methods such as Chebyshev finite difference method.

7. Applying composite Chebyshev finite difference method to solve linear and nonlinear system of Fredholm integro– differential equations (1.71).

1.7 Outline of the Thesis

This thesis is structured as follows. In Chapter 1, a brief introduction to the research topic is given. The problems under consideration for solving in the succeeding chapters are stated and the main objectives of the thesis are summarized. Chapter 2 includes some notations, definitions and preliminary facts that will be used further in this research work. Some main concepts like approximation theory, orthogonal polynomials and numerical integration are explained in this chapter.

In Chapter 3, spectral homotopy analysis method (SHAM) is employed to get an accurate and efficient solution to linear Fredholm, Volterra and Volterra- Fredholm integro–differential equations. The convergence of Spectral homotopy analysis method is proved for solving linear integro differential equations, the rate of convergence is obtained and the error estimation is done.

In Chapter 4, the spectral homotopy analysis method is used to solve nonlinear Volterra and Fredholm integro–differential equations. Convergence of spectral homotopy analysis method is proved for solving nonlinear integro-differential equations and the error estimation is done.

In Chapter 5, the spectral homotopy analysis method is used to solve linear and nonlinear system of Fredholm integro–differential equations .

In Chapter 6, we employ Composite Chebyshev Finite Difference Method for solving linear and nonlinear Fredholm and linear and nonlinear system of Fredholm integro-differential equations and the error bound of solution is obtained for composite Chebyshev finite difference method.

Finally, in Chapter 7, a short analysis of the work done in this research is made. There are some suggestions for possible extensions to this work to be carried in future studies.

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BIODATA OF STUDENT

Zohreh Pashazadeh Atabakan was born in 1973 in Tehran, the capital city of Iran. She finished her pre-university education in Tehran. In September 1992, she went to Kharazmi University (previously called Tehran University of Teacher Training) to study Mathematics where she received her Bachelor of Science in 1996. She taught Mathematics in high school for several years and then pursued her higher education. She obtained her Master's degree in applied mathematics in 2003 from Iran University of Science and Technology .

After teaching Mathematics at Islamic Azad University and Elmi Karbordi University for seven years, she enrolled as a PhD student at University Putra Malaysia in December 2010. The main results of the research on Applied Mathematics (spectral homotopy analysis method, Integral equations) during the past three years and a half has formed the content of the thesis which is now at your disposal.

LIST OF PUBLICATIONS

1. **Pashazadeh Atabakan, Z.**, Kılıçman, A., and Kazemi Nasab, A. 2012. On Spectral Homotopy Analysis Method for Solving Linear Volterra and Fredholm Integrodifferential Equations. *Abstract and Applied Analysis*, Volume 2012 (2012), Article ID 960289, 16 pages.

2. **Pashazadeh Atabakan, Z.**, and Kılıçman, A. solving integro-differential boundary value problems by using spectral homotopy analysis method. In *Proceedings of the Fundamental Science Congress*, 17 – 18 July (2012), Universiti Putra Malaysia, 34–35.

3. **Pashazadeh Atabakan, Z.**, Kazemi Nasab, A., Kılıçman, A., and Eshkuvatov, Zainidin K. 2013. Numerical solution of nonlinear Fredholm integro-differential equations using Spectral Homotopy Analysis method. *Mathematical Problems in Engineering*, Volume 2013 (2013), Article ID 674364, 9 pages.

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5. **Pashazadeh Atabakan, Z.**, Kazemi Nasab, A., and Kılıçman, A. Spectral Homotpy Analysis Method for solving Volterra– Fredholm Integro–Differential Equations. In *Proceedings of the The 9th East Asia SIAM Conference and The 2nd Conference on Industrial and Applied Mathematics (Journal of Advances in Applied Mathematics and Mechanics)*, 18 – 20 June (2013), Bandung, Indonesia.

6. **Pashazadeh Atabakan, Z.**, Kazemi Nasab, A., and Kılıçman, A. Spectral Homotopy Analysis Method for Solving Nonlinear Volterra Integro Differential Equations. In Proceedings of the international conference on mathematical sciences and statistics, Malaysian Journal of Mathematical Sciences 8(S): 153–161 (2014).

7. **Pashazadeh Atabakan, Z.**,Kılıçman, A. and F. Ismail, A numerical method for solving system of Fredholm integro-differential equations, submitted.

8. **Pashazadeh Atabakan, Z.**, Kılıçman, A.,Eshkuvatov, Zainidin K., A. Kazemi Nasab and F. Ismail. Spectral homotopy analysis method for solving linear and nonlinear system of Fredholm integro–differential equations, submitted.