NUMERICAL METHODS FOR SOLVING FIRST AND SECOND ORDER FUZZY DIFFERENTIAL EQUATIONS

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IPM 2015 22

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By

REZA AFSHARINAFAR

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Doctor of Philosophy

October 2014

DEDICATIONS

To

My Dears Dad, Mum and Wife

for their encouragement

and

My respected teachers

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Doctor of Philosophy

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October 2014

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Fuzzy differential equations (FDEs) with fuzzy initial conditions are studied as a suitable setting for the modeling of problems in science and engineering in which uncertainties or vagueness prevails. In this thesis, numerical methods are extended for solving first-order and second-order Fuzzy Initial Value Problems (FIVPs), which are interpreted by using the strongly generalized differentiability concepts.

There are several interpretations of FDEs depending on the types of differentiability involved. Hukuhara difference is the starting point of fuzzy derivative and has been studied by several researchers. However, it has its drawbacks which resulted in the development of new ideas using different approaches for the solutions of FDEs with initial conditions. Consequently, it has inspired some researchers to present the analytical and numerical methods for the solutions of first-order and also, analytical approach for the solutions of Nth-order FIVPs. All the works are based on the Zadeh's extension principle and Hukuhara differentiability.

In the first part, utilizing the characterization theorems, we transformed the FIVPs into an equivalent system of ordinary differential equations (ODEs). Then we generalized some explicit and implicit one-step methods such as Midpoint, Trapezoidal and Runge-Kutta (RK) methods for solving first-order FIVPs under strongly generalized differentiability. The results under strongly generalized differentiability are more accurate compared to the Hukuhara differentiability.

Next, we extend the multistep methods particularly the third-order Adams Moulton and fourth-order Adams Bashforth method as well as the Predictor-Corrector method of order three and four. They are used for solving FIVPs under strongly generalized differentiability, which clearly shown that the results under strongly generalized differentiability is more accurate compared to other existing approaches.

In the third part of the thesis, we extend the Diagonally Implicit RK method (DIRK) for FDEs under Hukuhara differentiability and the strongly generalized differentiability. Using some mathematical Lemmas, we showed that the approximate fuzzy solutions are convergent to the exact fuzzy solutions. The numerical results are compared with other existing methods and a complete error analysis, which guarantees pointwise convergence is also given.

Finally, we extend some definitions of strongly generalized differentiability for the solutions of special second-order fuzzy differential equations. The second-order FDEs are transformed to an equivalent system of ODEs using characterization theorem. Then, the multiple solutions of the second-order FDEs are obtained using Runge-Kutta-Nyström (RKN) method based on the stacking theorem.

All the numerical methods are validated using several examples to depict their applicability and effectiveness.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

KAEDAH BERANGKA UNTUK MENYELESAIKAN PERSAMAAN PEMBEZAAN KEBUR PERINGKAT PERTAMA DAN KEDUA

Oleh

REZA AFSHARINAFAR

Oktober 2014

Chair : **Professor Fudziah Ismail, Ph.D.**

Fakulti : **Institut Penyelidikan Matematik**

Persamaan pembezaan kabur (PPK) dengan syarat awal yang kabur dikaji sebagai suatu masalah pemodelan yang sesuai dalam sains dan kejuruteraan di mana ketidak-tentuan atau kekaburan wujud. Dalam tesis in, kaedah berangka diperluaskan untuk menyelesaikan persamaan nilai awal kabur peringkat pertama dan kedua. Persamaan ini ditafsirkan menggunakan konsep kebolehbezaan umum yang kukuh.

Ada beberapa tafsiran bagi PPK bergantung kepada jenis kebolehbezaan yang terlibat, pembezaan Hukuhara adalah satu titik permulaan bagi terbitan kabur dan telah dikaji oleh beberapa penyelidik. Walaubagaimanapun ia mempunyai kelemahannya tersendiri yang telah menghasilkan ide baru menggunakan pendekatan yang berbeza telah dibangunkan bagi penyelesaian PPK dengan nilai awal. Oleh yang demikian, ia memberi ilham kepada penyelidik untuk mempersembahkan kaedah analitik dan berangka untuk penyelesaian PPK peringkat pertama dan pendekatan analitik bagi PPK peringkat ke N. Semua kajian ini berdasarkan perluasan prinsipal Zadeh dan kebolehbezaan Hukuhara.

Dalam bahagian pertama, dengan menggunakan teorem pencirian, kami ubah PPK ini kepada sistem PPB yang setara. Kemudian kami perluaskan beberapa kaedah satu langkah tak tersirat dan tersirat seperti kaedah titik tengah, kaedah trapezoidal dan kaedah Runge-Kutta untuk menyelesaikan PPK peringkat pertama menggunakan kebolehbezaan umum yang kukuh. Keputusan berangka menggunakan kebolehbezaan umum yang kukuh adalah lebih jitu berbanding menggunakan kebolehbezaan Hukuhara.

Kemudian kami perluaskan kaedah multilangkah terutamanya kaedah Adams Moulton peringkat ketiga, kaedah Adams Bashforth peringkat keempat dan juga ka edah peramal pembetul peringkat ketiga dan keempat. Kaedah tersebut digunakan dengan kebolehbezaan umum yang kukuh, yang menujukkan keputusan menggunakan kebolehbezaan umum yang kukuh adalah lebih jitu berbanding pendekatan sedia ada.

Di bahagian kedua tesis in, kami perluaskan kaedah RK pepenjuru tersirat (RKPT) untuk menyelesaikan PPK dibawah kebolehbezaan Hukuhara dan kebolehbezaan umum yang kukuh. Dengan menggunakan beberapa Lemma matematik, kami menunjukkan penyelesaian kabur hampiran adalah menumpu kepada penyelesaian kabur yang tepat. Keputusan berangka dibandingkan dengan kaedah sedia ada dan analisis ralat yang menjamin penumpuan titik demi titik turut diberikan.

Akhir sekali, kami perluaskan beberapa takrif bagi kebolehbezaan umum yang kukuh kepada penyelesaian persamaan pembezaan kabur peringkat kedua. PPK peringkat kedua tersebut diubah kepada sistem PPB menggunakan teorem pencirian. Kemudian, penyelesaian berganda bagi PPK peringkat kedua tersebut diperolehi menggunakan kaedah Runge-Kutta Nystrom (RKN) berdasarkan teorem susunan.

Kesemua kaedah berangka tersebut ditentusahkan menggunakan beberapa contoh untuk menunjukkan kebolehgunaan dan keberkesanannya.

ACKNOWLEDGEMENTS

In the name of God,

the most merciful, the most beneficent

First and foremost, I would like to thank God for giving me the ability, patience and health to complete this research study and all of blessings in my life.

I would like to express my sincere gratitude to my supervisor Professor Dr. Fudziah for continuous supporting of my Ph.D study and research, for her patience, motivation, enthusiasm, and immense knowledge during these four years. Her guidance helped me in all the time of research and writing of this thesis. I could not have imagined having a better advisor and mentor for my Ph.D study. I appreciate all her contributions of time, ideas, and funding to make my Ph.D. experience productive and stimulating.

I wish to thank my advisor, Professor Mohammad Suleiman for his encouragement, insightful comments, and advice. He always lead me to the right research direction. His guidance has served me well and I owe him also my heartfelt appreciation.

My sincere thanks also goes to Dr. Norazak Senu for all his support and useful comments. As other member of my supervising Committee, his friendship and assistance has meant more to me than I could ever express.

Last but not least, I would like to thank my family: my patient parents if not for them I will not be here today and for supporting me spiritually throughout my life, I would like to give a special gratitude to my wife for her love and emotional support through of my studies. A special thanks to my friends for the stimulating discussions, for the sleepless nights we were working together before deadlines, and for all the fun we have had in the last four years.

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CHAPTER 1

INTRODUCTION

1.1 Introduction

The concept of fuzzy set theory presented by Lotfi A. Zadeh has been a powerful tool for modeling of uncertainty and for processing vague or subjective information in mathematical models. It has developed in many main directions and has been applied in many different real problems. In fuzzy set theory with possibilistic uncertainty, the fuzzy differential equations (FDEs) used to be a normal method for modeling dynamical systems. The solutions of FDEs are utilized within most of the complications in science and engineering and they need to satisfy the fuzzy initial conditions; so finding the solutions of fuzzy initial value problem (FIVP) has become very important. However, since finding the exact solutions of FIVPs is too complicated or sometimes impossible, numerical methods are applied under several approaches to find the behavior of the solutions.

Hukuhara differentiability was the starting point and the most used technique in fuzzy number value mapping which, however, has a disadvantage that the solution turns into fuzzier while time moves. This is due to the fuzzy solution reacts in a different way from the crisp solution. To avoid the problem, different interpretation such as differential inclusion are used, but that interpretations suffer from different disadvantages such as the existence of a derivative of fuzzy-number-valued functions. Therefore, a more general derivative of Hukuhara derivative named as the strongly generalized Hukuhara differentiability is defined to discuss the FDEs with its parametrical form.

In all approaches, the FDEs are substituted by an equivalent ODE system or its parametrical equations. Consequently, the numerical methods for solving FIVPs are also classified into

- One-step methods: the approximated solution is evaluated using the information of only one previous point
- Multi-step methods: the approximated solution is evaluated using the information of k previous point

1.2 The objective of thesis

The main objective of the research is to find the fuzzy solutions of first-order and second-order fuzzy initial value problems under the strongly generalized differentiability (named as generalized differentiability) using one-step and multi-step methods. This goal can be attained by:

1. Extension of one-step methods consist of Midpoint, Trapezoidal and Runge-Kutta methods under Generalized Hukuhara differentiability. The convergence of the methods are proven using Taylor series.
2. Extension of Diagonally Runge-Kutta method based on the Hukuhara differentiability and Generalized Hukuhara differentiability. The convergence of the method are proven using Taylor series.
3. Construction of generalized Predictor-Corrector method by extension of Adams-Bashford and Adams-Moulton methods as multi-step methods under the Generalized Hukuhara differentiability. The convergence of the methods are also proven using Taylor series.
4. Generalization of some definitions of the solutions to second-order fuzzy differential equations (FDE) which are substituted by an equivalent ODE systems.
5. Discussion of the numerical approximation of the second-order fuzzy differential equations by means of Runge-Kutta-Nyström method of order four under the Hukuhara differentiability and Generalized Hukuhara differentiability. Convergence of the proposed method is also proven.

1.3 Scope of thesis

The scope of thesis will focus on solving first and second order fuzzy differential equations by generalizing one-step and multi-step methods under Hukuhara and Generalized Hukuhara differentiability. In addition, by extending some theorems and definitions, several types of possible derivatives of second order fuzzy number valued functions are defined for the fuzzy solutions of second order FDEs.

1.4 Outline of thesis

In Chapter 1, a brief introduction on fuzzy differential equations and different types of fuzzy derivatives with application of numerical methods for solving first and second order fuzzy initial value problems are given.

Chapter 2 consists of earlier researchers and related study on several proposed approaches for solving first order FDEs. Some basic definitions and theorems on numerical methods for solving FIVPs will also be given.

Chapter 3 presents the extension of Midpoint, Trapezoidal and Runge-Kutta methods under the Generalized Hukuhara differentiability. Utilizing the characterization theorems, approximate solutions of FDEs are obtained by an equivalent system of ODEs followed by the convergence of the approximate solutions. Also, the numerical results are compared with other existing methods and provided with complete error analysis, which guarantees pointwise convergence.

Chapter 4 shows the construction of generalized predictor-corrector (GPC) method combining the extended Adams-Bashford method as a predictor and extended Adams-Moulton method as a corrector method. Convergence of the constructed method is also discussed in detail. Applicability of the method is illustrated by solving some numerical examples in comparison with other existing methods.

Chapter 5 introduces a generalized fourth order Diagonally Implicit Runge-Kutta method (DIRK) as an A-stable Runge-Kutta method for finding the fuzzy solutions of first order fuzzy differential equations (FDEs) under the Hukuhara and Generalized Hukuhara differentiability. Also, the convergence of the method is proven and numerical results are compared with existing Runge-Kutta method.

In Chapter 6, we consider the multiple solutions of second-order fuzzy differential equations with initial conditions. Some definitions for the fuzzy solutions of first-order fuzzy differential equations are extended to second-order FDEs. Utilizing the extended characterization theorem, the numerical approximation of the second-order fuzzy differential equations are obtained by means of Runge-Kutta-Nyström method of order four under Hukuhara and Generalized Hukuhara differentiability. Convergence of the proposed method is also proven. An example is provided to show the applicability and accuracy of the proposed method.

Finally, the conclusion of the thesis is summarized in the last chapter. Future work on the research is also suggested in the last chapter.

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