UNIVERSITI PUTRA MALAYSIA

ON THE DIOPHANTINE EQUATION px+qmny = z2

HALIMATUN SAADIAH BINTI BAKAR

IPM 2018 6

ON THE DIOPHANTINE EQUATION $p^x + q^m n^y = z^2$

By

HALIMATUN SAADIAH BINTI BAKAR

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Master of Science

June 2018

COPYRIGHT

DEDICATIONS

To
my lovely parents,
Bakar bin Zakaria and Rasidah binti Husin,
my brothers,
Muhammad Amiruddin, Hassan Basri and Abdul Qhahar,
my grandmother,
Siah binti Jelani,
Gadis-Gadis INSPEM,
families and friends
who always support me . . .

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Master of Science

ON THE DIOPHANTINE EQUATION $p^x + q^m n^y = z^2$

By

HALIMATUN SAADIAH BINTI BAKAR

June 2018

Chairman : **Siti Hasana Sapar, PhD**

Faculty : **Institute for Mathematical Research**

Diophantine equation is a polynomial equation with two or more unknowns for which only integral solutions are sought. Exponential Diophantine equation is a Diophantine equation that has additional variable or variables occurring as exponents.

This research concentrates on finding an integral solution to the Diophantine equation $p^x + q^m n^y = z^2$ for x, q, m, n, y, z are positive integers and p is odd prime. Firstly, the equation is solved for $p = q$ and $y = 1, 2$. By considering the values of x, m, n and by using substitution method, the integral solution will be obtained. Some of the results are $(x, m, n, y, z) = (2r, 2r, s(s+2), 1, p^r(s+1))$, $(2r-1, 2r-1, ps^2-1, 1, p^r s)$, $(2(r+t), 2t, s(2p^r+s), 1, p^t(p^r+s))$, $(2(r+t)-1, 2t, s^2-p^{2r-1}, 1, p^t s)$, $(2r, 2r+t, p^t s^2 \pm 2s, 1, p^r(p^t s \pm 1))$, $\left(2(t+u-1)+r, 2t, p^{u-1}\left(\frac{p^r-1}{2}\right), 2, p^{t+u-1}\left(\frac{p^r+1}{2}\right)\right)$ for $r, s, t, u \in \mathbb{N}$.

The integral solutions to the Diophantine equation $p^{2x} + q^m n^y = z^2$ for $q > p$ and $y = 1, 2$ are $(x, m, n, y, z) = \left(r, t, q^t s^2 \pm 2p^r s, 1, q^t s \pm p^r\right)$ and $\left(r, 2t, \frac{p^{2r-\alpha} - p^{\alpha}}{2q^t}, 2, \frac{p^{2r-\alpha} + p^{\alpha}}{2}\right)$ where $0 \leq \alpha < r$ for $r, s, t \in \mathbb{N}$ respectively. Then, for y an even number, the fundamental solution (n, z) with

$(x,m,y) = (r, 2t-1, 2u)$ for $r,t,u \in \mathbb{N}$ is in the following bound:

$$0 < n \leq \left(\frac{p^r b_1}{\sqrt{2(a_1+1)}} \right)^{\frac{1}{u}} \text{ and}$$

$$0 < |z| \leq \sqrt{\frac{p^{2r}(a_1+1)}{2}},$$

where (a_1, b_1) is a fundamental solution of $z^2 - Dn^2 = 1$ and $D = q^{2t-1}$.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Master Sains

PERSAMAAN DIOFANTUS $p^x + q^m n^y = z^2$

Oleh

HALIMATUN SAADIAH BINTI BAKAR

Jun 2018

Pengerusi : **Siti Hasana Sapar, PhD**

Fakulti : **Institut Penyelidikan Matematik**

Persamaan Diofantus merupakan satu persamaan polinomial dengan dua atau lebih pembolehubah yang mana hanya penyelesaian integer yang akan dicari. Persamaan Diofantus eksponen ialah persamaan Diofantus yang mempunyai pembolehubah tambahan yang bertindak sebagai eksponan.

Kajian ini tertumpu kepada mencari penyelesaian integer kepada persamaan Diofantus $p^x + q^m n^y = z^2$ dengan x, q, m, n, y ialah integer positif dan p ialah nombor perdana ganjil. Pertama sekali, persamaan tersebut diselesaikan bagi $p = q$ dan $y = 1, 2$. Dengan mempertimbangkan nilai x, m, n dan menggunakan kaedah penggantian, penyelesaian integer tersebut akan diperoleh. Sebahagian daripada hasilnya ialah $(x,m,n,y,z) = (2r, 2r, s(s+2), 1, p^r(s+1)), (2r-1, 2r-1, ps^2-1, 1, p^r s), (2(r+t), 2t, s(2p^r+s), 1, p^t(p^r+s)), (2(r+t)-1, 2t, s^2-p^{2r-1}, 1, p^t s), (2r, 2r+t, p^t s^2 \pm 2s, 1, p^r(p^t s \pm 1)), \left(2(t+u-1)+r, 2t, p^{u-1}\left(\frac{p^r-1}{2}\right), 2, p^{t+u-1}\left(\frac{p^r+1}{2}\right)\right)$ bagi $r, s, t, u \in \mathbb{N}$.

Penyelesaian integer kepada persamaan Diofantus $p^{2x} + q^m n^y = z^2$ bagi $q > p$ dan $y = 1, 2$ adalah $(x,m,n,y,z) = \left(r, t, q^t s^2 \pm 2p^r s, 1, q^t s \pm p^r\right)$ dan $\left(r, 2t, \frac{p^{2r-\alpha} - p^\alpha}{2q^t}, 2, \frac{p^{2r-\alpha} + p^\alpha}{2}\right)$ dengan $0 \le \alpha < r$ bagi $r, s, t \in \mathbb{N}$. Kemudian, bagi y nombor genap, penyelesaian asasi (n, z) dengan $(x, m, y) = (r, 2t-1, 2u)$ bagi

$r, t, u \in \mathbb{N}$ berada dalam sempadan berikut:

$$0 < n \leq \left(\frac{p^r b_1}{\sqrt{2(a_1+1)}} \right)^{\frac{1}{u}} \text{ dan}$$

$$0 < | z | \leq \sqrt{\frac{p^{2r}(a_1+1)}{2}},$$

dengan (a_1, b_1) merupakan penyelesaian asasi bagi $z^2 - Dn^2 = 1$ dan $D = q^{2t-1}$.

ACKNOWLEDGEMENTS

In the Name of Allah, the Most Beneficient and the Most Merciful. Praised be to Allah the Almighty for His blessings showered upon me for making the writing of the thesis a successful one. I would like to express my deepest gratitude to my supervisor, Assoc. Prof. Dr. Siti Hasana Sapar and my co-supervisor, Dr. Mohamat Aidil Mohamat Johari for their excellent guidance, patience, supports and ideas that helped me a lot in completing this study.

Besides, I would like to thank the Ministry of Higher Education Malaysia and Universiti Putra Malaysia who support me financially during this study. My sincere thanks also goes to Institute for Mathematical Research and all the staffs for providing me with excellent and comfortable atmosphere for completing this research.

Last but not least, a milion thanks to all my family and friends for their continuous love, support and encouragement. I am very thankful to have them in my life.

This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Master of Science. The members of the Supervisory Committee were as follows:

Siti Hasana Sapar, PhD
Assosiate Professor
Faculty of Science
Universiti Putra Malaysia
(Chairperson)

Mohamat Aidil Mohamat Johari, PhD
Senior Lecturer
Faculty of Science
Universiti Putra Malaysia
(Member)

____
ROBIAH BINTI YUNUS, PhD
Professor and Dean
School of Graduate Studies
Universiti Putra Malaysia

Date:

Declaration by graduate student

I hereby confirm that:

- this thesis is my original work;
- quotations, illustrations and citations have been duly referenced;
- this thesis has not been submitted previously or concurrently for any other degree at any other institutions;
- intellectual property from the thesis and copyright of thesis are fully-owned by Universiti Putra Malaysia, as according to the Universiti Putra Malaysia (Research) Rules 2012;
- written permission must be obtained from supervisor and the office of Deputy Vice-Chancellor (Research and Innovation) before thesis is published (in the form of written, printed or in electronic form) including books, journals, modules, proceedings, popular writings, seminar papers, manuscripts, posters, reports, lecture notes, learning modules or any other materials as stated in the Universiti Putra Malaysia (Research) Rules 2012;
- there is no plagiarism or data falsification/fabrication in the thesis, and scholarly integrity is upheld as according to the Universiti Putra Malaysia (Graduate Studies) Rules 2003 (Revision 2012-2013) and the Universiti Putra Malaysia (Research) Rules 2012. The thesis has undergone plagiarism detection software.

Signature:____________________ Date:____________________

Name and Matric No: Halimatun Saadiah Binti Bakar, GS44314

Declaration by Members of Supervisory Committee

This is to confirm that:
- the research conducted and the writing of this thesis was under our supervision;
- supervision responsibilities as stated in the Universiti Putra Malaysia (Graduate Studies) Rules 2003 (Revision 2012-2013) are adhered to.

Signature: ______________________________
Name of Chairman of Supervisory Committee:
Siti Hasana Sapar

Signature: ______________________________
Name of Member of Supervisory Committee:
Mohamat Aidil Mohamat Johari

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

1.1 Preliminary

This research will be focused on finding an integral solution to the Diophantine equation $p^x + q^m n^y = z^2$. In this chapter, we will brief some background of Diophantine equation. Then, we state the objectives and methodology of this research followed by the literature reviews from the previous researchers. Lastly, the organization of the thesis according to each chapter will be given.

1.2 Mathematical Background

In this section, we will give some background of Number theory and Diophantine equation. James et al. (2010) stated that Number theory is the study of natural numbers and called "the queen of mathematics" by Carl Friedrich Gauss. The beautiful patterns and theorems that emerge have fascinated many of the greatest mathematical minds throughout the centuries. Yet, give challenges to the mathematicians to solve the problems.

An equation with the restriction that only integer solutions are sought is called Diophantine equation. The main focus of this research is to solve exponential Diophantine equation. Exponential Diophantine equation is an equation that has additional variable or variables occuring as exponents. The simple expression of exponential Diophantine equation is $x^a + y^b = z^c$ where all the unknowns must be natural numbers.

There is no general method for solving Diophantine equation. Some of the equations can be solved by applying Pell's equation. Pell's equation is a special type of Diophantine equation in the form of $x^2 - Dy^2 = N$ where D is nonsquare integer and N is nonzero integer. McCune and Clark (2012) stated the following definition:

Definition 1.1 : *(Perfect square) A number that is an exact square of another number is a perfect square.*

Nagell (1964) stated the following theorems:

Theorem 1.1 : *If D is a natural number which is not a perfect square, the Diophantine equation*

$$x^2 - Dy^2 = 1 \tag{1.1}$$

has infinitely many solutions $x+y\sqrt{D}$. All solutions with positive x and y are obtained by the formula

$$x_n + y_n\sqrt{D} = \left(x_1 + y_1\sqrt{D}\right)^n$$

where $x_1 + y_1\sqrt{D}$ is the fundamental solution of (1.1)*, where n runs through all natural numbers, and where*

$$x_n = x_1^n + \sum_{k=1} \binom{n}{2k} x_1^{n-2k} y_1^{2k} D^k \ \ \textit{and}$$
$$y_n = \sum_{k=1} \binom{n}{2k-1} x_1^{n-2k+1} y_1^{2k-1} D^{k-1}.$$

Now, Diophantine equation of the form

$$\xi^2 - D\eta^2 = -1 \tag{1.2}$$

is solvable for all odd prime factors of D be of the form $4n+1$.

Theorem 1.2 : *Let D be a natural number which is not a perfect square. Suppose that* (1.2) *is solvable and $\xi_1 + \eta_1\sqrt{D}$ is its fundamental solution. Then the number*

$$x_1 + y_1\sqrt{D} = \left(\xi_1 + \eta_1\sqrt{D}\right)^2 = \xi_1^2 + D\eta_1^2 + 2\xi_1\eta_1\sqrt{D} \tag{1.3}$$

is the fundamental solution of equation (1.1)*.*
Further, if we put

$$\xi_n + \eta_n\sqrt{D} = \left(\xi_1 + \eta_1\sqrt{D}\right)^n \tag{1.4}$$

with

$$\xi_n = \xi_1^n + \sum_{k=1} \binom{n}{2k} \xi_1^{n-2k} \eta_1^{2k} D^k,$$
$$\eta_n = \sum_{k=1} \binom{n}{2k-1} \xi_1^{n-2k+1} \eta_1^{2k-1} D^{k-1},$$

formula (1.4) *gives:*

1. *all the solutions with positive ξ and η of equation* (1.2) *when n runs through all positive odd integers.*

2. *all the solutions with positive $x = \xi_n$ and $y = \eta_n$ of equation* (1.1) *when n runs through all positive even integers.*

Theorem 1.3 : *If p is a prime $\equiv 1 \ (mod\ 4)$, the Diophantine equation*

$$\xi^2 - p\eta^2 = -1$$

is solvable in integers ξ and η.

Now, consider the Diophantine equation of the form

$$u^2 - Dv^2 = C, \tag{1.5}$$

where D be a natural number which is not a perfect square and C is nonzero integer. Suppose that the equation is solvable and $u + v\sqrt{D}$ is a solution of it. If $x + y\sqrt{D}$ is any solution of the equation

$$x^2 - Dy^2 = 1, \tag{1.6}$$

the number

$$\left(u + v\sqrt{D}\right)\left(x + y\sqrt{D}\right) = ux + vyD + (uy + vx)\sqrt{D}$$

is also a solution of (1.5). This solution is said to be associated with the solution $u + v\sqrt{D}$. The set of all solutions associated with each other forms a class of solutions of (1.5).

Theorem 1.4 : *If $u + v\sqrt{D}$ is the fundamental solution of the equation*

$$u^2 - Dv^2 = N, \tag{1.7}$$

where N is positive integer and if $x_1 + y_1\sqrt{D}$ is the fundamental solution of equation (1.6)*, we have the inequalities*

$$0 \leq v \leq \frac{y_1}{\sqrt{2(x_1+1)}}\sqrt{N} \ \text{ and}$$

$$0 < \mid u \mid \leq \sqrt{\frac{1}{2}(x_1+1)N}.$$

Besides the above theorems, we also consider the following definitions, theorem and propositions to find the integral solution to the Diophantine equation. The following results were obtained by Kumanduri and Romero (1998):

Definition 1.2 : *(Divisibility). If a and b are integers, we say that a divides b (denoted as $a \mid b$) if there exists an integer c such that $b = ac$. If no such c exists, then a does not divide b (denoted by $a \nmid b$). If a divides b, we say that a is a divisor of b and b is divisible by a.*

Theorem 1.5 : *There are infinitely many prime numbers.*

Proposition 1.1 : *(Primality Test). A number p is prime if and only if it is not divisible by any prime q.*

Proposition 1.2 : *Let a and b be integers. If p is a prime number such that $p \mid ab$, then $p \mid a$ or $p \mid b$.*

Definition 1.3 : *(Greatest Common Divisor). The greatest common divisor (gcd) of two numbers a and b, not both zero, is the largest integer dividing both a and b. It will be denoted by gcd(a,b) or (a,b).*

Definition 1.4 : *Let a,b,m be integers, we say that a is congruent to b modulo m denoted by $a \equiv b \pmod{m}$ if $m \mid a-b$. If $m \nmid a-b$, we write $a \not\equiv b \pmod{m}$ and say that a is not congruent or incongruent to b modulo m.*

Definition 1.5 : *(Quadratic Residue Modulo). Let a and m be integers such that $(a,m)=1$. If the congruence $x^2 \equiv a \pmod{m}$ has an integer solution, then a is a quadratic residue modulo m. Otherwise, it is a quadratic nonresidue modulo m.*

Example 1.1 : *Let $m=7$, then $1,2,4$ are quadratic residues and $3,5,6$ are non-residue. This follows from the equations*

$$\begin{aligned} 1^2 \equiv 6^2 \equiv 1 \pmod{7} \\ 2^2 \equiv 5^2 \equiv 4 \pmod{7} \\ 3^2 \equiv 4^2 \equiv 2 \pmod{7}. \end{aligned}$$

Here $3,5,6$ are nonresidues, as we have squared all the invertible elements and obtained $1,2,4$. Note that $a \equiv b \pmod{m}$ implies that $a^2 \equiv b^2 \pmod{m}$. Hence, it is enough to consider elements in a complete residue system to determine the quadratic residues modulo m.

Definition 1.6 : *(Legendre Symbol). Let p be an odd prime and a an integer, the Legendre symbol $\left(\frac{a}{p}\right)$ is defined to be 1 if a is a quadratic residue modulo p, -1 if a is a quadratic nonresidue modulo p and 0 if $p \mid a$.*

Example 1.2 :

1. *From Example 1.1, we have*

$$\begin{aligned} \left(\frac{1}{7}\right) = \left(\frac{2}{7}\right) = \left(\frac{4}{7}\right) = 1 \ \text{ and} \\ \left(\frac{3}{7}\right) = \left(\frac{5}{7}\right) = \left(\frac{6}{7}\right) = -1. \end{aligned}$$

2. *It is clear that* $\left(\frac{1}{p}\right) = 1$ *for all* p.

Definition 1.7 : *(Euler's Criterion). Let p be an odd prime and a an integer such that $(a,p) = 1$, then*

$$a^{\frac{p-1}{2}} \equiv \left(\frac{a}{p}\right) \ (mod\ p).$$

Proposition 1.3 : *Let p be an odd prime, then*

$$\left(\frac{-1}{p}\right) = \begin{cases} 1 \text{ if } p \equiv 1 \ (mod\ 4), \\ -1 \text{ if } p \equiv 3 \ (mod\ 4). \end{cases}$$

1.3 Objective and Methodology

In this section, we will state the objective and methodology of this research. The main objectives of this research are:

(i) to find the integral solutions (x,m,n,z) to the Diophantine equation

$$p^x + q^m n^y = z^2$$

for $p = q$ an odd prime and $y = 1,2$.

(ii) to find the integral solutions (x,m,n,z) to the Diophantine equation

$$p^{2x} + q^m n^y = z^2 \tag{1.8}$$

for p is odd prime, q any integer with $q > p$ and $y = 1,2$.

(iii) to provide a bound for the fundamental solutions (n,z) to the Diophantine equation (1.8) where p is odd prime, q any integer with $q > p$ and y any even number.

Now, we will present the methodology to determine the integral solution for the Diophantine equation. In order to find the integral solution for $p^x + q^m n^y = z^2$ when $y = 1,2$, we consider the parity of x,m and n. By using substitution method and parameterization technique, the integral solutions of (x,m,n,z) will be obtained.

1.4 Literature Review

The difference form of exponential Diophantine equations have been solved and studied by many researchers. Acu (2007) discussed on the Diophantine equation $2^x + 5^y = z^2$ and proved that the equation has exactly two solutions in non-negative integer $(x,y,z) = (3,0,3),(2,1,3)$.

Suvarnamani et al. (2011) showed that Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$ have no solution in non-negative integer. Wang and Wang (2011) and Gokhan and Ismail (2014) studied on the Diophantine equation $nx^2 + 2^{2m} = y^n$. They found that the equation has no positive integer solution (x,y,m) for any odd integer $n > 1$ with $\gcd(x,y) = 1$.

Sroysang investigated the Diophantine equation $a^x + b^y = z^2$ for some $a,b \in \mathbb{N}$ and (x,y,z) are non-negative integers. His results are given as follows:

(i) Sroysang (2012a) found that the non-negative integer solution to the Diophantine equation $8^x + 19^y = z^2$ is $(x,y,z) = (1,0,3)$.

(ii) Sroysang (2012c) found that the non-negative integer solution to the Diophantine equation $3^x + 5^y = z^2$ is $(x,y,z) = (1,0,2)$.

(iii) Sroysang (2012b) found that there is no non-negative integer solution to the Diophantine equation $31^x + 32^y = z^2$.

(iv) Sroysang (2013f) found that the non-negative integer solution to the Diophantine equation $7^x + 8^y = z^2$ is $(x,y,z) = (0,1,3)$.

(v) Sroysang (2013a) found that there is no non-negative integer solution to the Diophantine equation $23^x + 32^y = z^2$.

(vi) Sroysang (2013b) found that the non-negative integer solution to the Diophantine equation $3^x + 17^y = z^2$ is $(x,y,z) = (1,0,2)$.

(vii) Sroysang (2013e) found that there is no non-negative integer solution to the Diophantine equation $5^x + 7^y = z^2$.

(viii) Sroysang (2013d) found that there is no non-negative integer solution to the Diophantine equation $5^x + 23^y = z^2$.

(ix) Sroysang (2013c) found that there is no non-negative integer solution to the Diophantine equation $47^x + 49^y = z^2$.

(x) Sroysang (2013g) found that there is no non-negative integer solution to the Diophantine equation $89^x + 91^y = z^2$.

(xi) Sroysang (2014a) found that there is no non-negative integer solution to the Diophantine equation $131^x + 133^y = z^2$.

(xii) Sroysang (2014e) found that the non-negative integer solution to the Diophantine equation $8^x + 13^y = z^2$ is $(x,y,z) = (1,0,3)$.

(xiii) Sroysang (2014d) found that the non-negative integer solution to the Diophantine equation $3^x + 85^y = z^2$ is $(x,y,z) = (1,0,2)$. This result implies that $(x,u,v,z) = (1,0,0,2)$ is the result for Diophantine equation $3^x + 5^u 17^v = z^2$.

(xiv) Sroysang (2014b) found that the non-negative integer solution to the Diophantine equation $143^x + 145^y = z^2$ is $(x, y, z) = (1, 0, 12)$.

(xv) Sroysang (2014c) found that the non-negative integer solution to the Diophantine equation $3^x + 45^y = z^2$ is $(x, y, z) = (1, 0, 2)$.

Liu (2013) consider the Diophantine equation $x^4 - q^4 = py^n$ for (n, p, q) an odd primes and (x, y) are positive integers. The author proved that if $n > 3$ and $p \equiv 3 \pmod 4$, then the equation has no positive integer solution with $\gcd(x, y) = 1$ and $2 \nmid y$. Chotchaisthit (2013) studied on the Diophantine equation $p^x + (p+1)^y = z^2$ for (x, y, z) are non-negative integers and p is Mersenne prime where $p = 2^n - 1$. The only solutions to the equation are $(p, x, y, z) = (7, 0, 1, 3)$ and $(3, 2, 2, 5)$.

Su and Li (2014) considered the exponential Diophantine equation $(4m^2 + 1)^x + (5m^2 - 1)^y = (3m)^z$ for (m, x, y, z) are positive integers. They showed that if $m > 90$ and $3 \mid m$, then the only solution to the equation is $(x, y, z) = (1, 1, 2)$. The Diophantine equation $m^x + m^{2s}n^y = z^{2t}$ where (n, s, t) are non-negative integers and $n \equiv 5 \pmod{20}$ was studied by other researchers and the results obtained by them for certain m are as follow:

(i) Sarasit and Chotchaisthit (2014) proved that all non-negative integer solutions (x, y, z) to the Diophantine equation $3^x + 3^{2s}n^y = z^{2t}$ are

$$(x, m, z) = \begin{cases} (1+2s, 0, 2(3)^s) & ; t = 1 \\ \text{No solution} & ; \text{otherwise.} \end{cases}$$

(ii) Chotchaisthit and Worawiset (2015a) proved that all non-negative integer solutions (x, y, z) to the Diophantine equation $143^x + 143^{2s}n^y = z^{2t}$ are

$$(x, m, z) = \begin{cases} (1+2s, 0, 12(143)^s) & ; t = 1 \\ \text{No solution} & ; \text{otherwise.} \end{cases}$$

(iii) Chotchaisthit and Worawiset (2015b) proved that all non-negative integer solutions (x, y, z) to the Diophantine equation $323^x + 323^{2s}n^y = z^{2t}$ are

$$(x, m, z) = \begin{cases} (1+2s, 0, 18(323)^s) & ; t = 1 \\ \text{No solution} & ; \text{otherwise.} \end{cases}$$

(iv) Chotchaisthit and Worawiset (2015c) proved that all non-negative integer solutions (x, y, z) to the Diophantine equation $483^x + 483^{2s}n^y = z^{2t}$ are

$$(x, m, z) = \begin{cases} (1+2s, 0, 22(483)^s) & ; t = 1 \\ \text{No solution} & ; \text{otherwise.} \end{cases}$$

Goedhart and Grundman (2015) proved that the Diophantine equation $(a^2cx^k - 1)(b^2cy^k - 1) = (abcz^k - 1)^2$ has no positive integer solution with $x,y,z > 1, k \geq 7$ and $a^2x^k \neq b^2y^k$. Tatong and Suvarnamani (2015) investigated on the Diophantine equation $(p+1)^{2x} + q^y = z^2$ where (x,y,z) are non-negative integers. They found that the equation has no solution for (x,y,z) where p is a Mersenne prime and $q - p = 2$.

Bacani and Rabago (2015) studied on the complete set of solutions of Diophantine equation $p^x + q^y = z^2$ for twin prime p and q. They showed that the equation has infinitely many solution (p,q,x,y,z) in positive integers. Furthermore, they showed that if the sum of p and q is a square, then the equation has unique solution $(x,y,z) = (1,1,\sqrt{p+q})$ in non-negative integers. Trojovský (2015) proved that there is no solution to the Diophantine equation $p^a + (p+1)^b = z^2$ for an odd prime $p > 3$, $b \geq 2$ and z an even number.

1.5 Organization of Thesis

In this research, we concentrate on finding the integral solution to the Diophantine equation $p^x + q^m n^y = z^2$. In Chapter 2, we focus on finding the integral solution to the Diophantine equation $p^x + q^m n^y = z^2$ for $y = 1$ and $p = q$ an odd prime. We find the integral solution for $p = 3$ and then for any odd prime p.

In Chapter 3, we find the integral solution to the Diophantine equation $p^x + q^m n^y = z^2$ for $y = 2$ and $p = q$ an odd prime. In Chapter 4, we discuss on finding the integral solution to the Diophantine equation $p^{2x} + q^m n^y = z^2$ for p,q are odd prime where $q > p$ and $y = 1,2$. Then, followed by finding the bound for the fundamental solutions (n,z) of the Diophantine equation where y an even number.

In Chapter 5, we will give the summary and the conclusion of this research. Finally, we give some suggestions for the future research.

REFERENCES

Acu, D. (2007). On a Diophantine Equation $2^x + 5^y = z^2$. *General Mathematics*, 15(4):145–148.

Bacani, J. B. and Rabago, J. F. T. (2015). The Complete Set of Solutions of the Diophantine Equation $p^x + q^y = z^2$ for Twin Primes p and q. *International Journal of Pure and Applied Mathematics*, 104(4):517–521.

Chotchaisthit, S. (2013). On the Diophantine Equation $p^x + (p+1)^y = z^2$ where p is a Mersenne Prime. *International Journal of Pure and Applied Mathematics*, 88(2):169–172.

Chotchaisthit, S. and Worawiset, S. (2015a). On the Diophantine Equation $143^x + 143^{2s}n^y = z^{2t}$ where s,t,n are Non-negative Integers and $n \equiv 5 \ (mod\ 20)$. *International Journal of Pure and Applied Mathematics*, 100(3):405–412.

Chotchaisthit, S. and Worawiset, S. (2015b). On the Diophantine Equation $323^x + 323^{2s}n^y = z^{2t}$ where s,t,n are Non-negative Integers and $n \equiv 5 \ (mod\ 20)$. *International Journal of Pure and Applied Mathematics*, 100(3):435–442.

Chotchaisthit, S. and Worawiset, S. (2015c). On the Diophantine Equation $483^x + 483^{2s}n^y = z^{2t}$ where s,t,n are Non-negative Integers and $n \equiv 5 \ (mod\ 20)$. *International Journal of Pure and Applied Mathematics*, 100(4):461–468.

Goedhart, E. G. and Grundman, H. G. (2015). Diophantine Approximation and the Equation $(a^2cx^k - 1)(b^2cy^k - 1) = (abcz^k - 1)^2$. *Journal of Number Theory*, 154:74–81.

Gokhan, S. and Ismail, N. C. (2014). Note on "On the Diophantine equation $nx^2 + 2^{2m} = y^n$". *Journal of Number Theory*, 140:425–426.

James, E. P., Tim, K. M., and Erica, L. F. (2010). *Number Theory a Lively Introduction with Proofs, Applications and Stories*. Laurie Rosatone, United States of America.

Kumanduri, R. and Romero, C. (1998). *Number Theory with Computer Applications*. Pearson College Division.

Liu, Y. (2013). On the Diophantine Equation $x^4 - q^4 = py^n$. *Expositiones Mathematicae*, 31:196–203.

McCune, S. L. and Clark, W. D. (2012). *Easy Algebra Step-by-Step*. Mc Graw Hill, United States of America.

Nagell, T. (1964). Diophantine Equation of the Second Degree. In *Introduction to Number Theory*, pages 188–226, Chelsea, New York. Chesea Publishing Company. Chapter 6.

Sarasit, N. and Chotchaisthit, S. (2014). On the Diophantine Equation $3^x + 3^{2s}n^y = z^{2t}$ where n,s,t are Non-negative Integers and $n \equiv 5 \ (mod\ 20)$. *International Journal of Pure and Applied Mathematics*, 97(2):211–218.

Sroysang, B. (2012a). More on the Diophantine Equation $8^x + 19^y = z^2$. *International Journal of Pure and Applied Mathematics*, 81(4):601–604.

Sroysang, B. (2012b). On the Diophantine Equation $31^x + 32^y = z^2$. *International Journal of Pure and Applied Mathematics*, 81(4):609–612.

Sroysang, B. (2012c). On the Diophantine Equation $3^x + 5^y = z^2$. *International Journal of Pure and Applied Mathematics*, 81(4):605–608.

Sroysang, B. (2013a). On the Diophantine Equation $23^x + 32^y = z^2$. *International Journal of Pure and Applied Mathematics*, 84(3):231–234.

Sroysang, B. (2013b). On the Diophantine Equation $3^x + 17^y = z^2$. *International Journal of Pure and Applied Mathematics*, 89(1):111–114.

Sroysang, B. (2013c). On the Diophantine Equation $47^x + 49^y = z^2$. *International Journal of Pure and Applied Mathematics*, 89(2):279–282.

Sroysang, B. (2013d). On the Diophantine Equation $5^x + 23^y = z^2$. *International Journal of Pure and Applied Mathematics*, 89(1):119–122.

Sroysang, B. (2013e). On the Diophantine Equation $5^x + 7^y = z^2$. *International Journal of Pure and Applied Mathematics*, 89(1):115–118.

Sroysang, B. (2013f). On the Diophantine Equation $7^x + 8^y = z^2$. *International Journal of Pure and Applied Mathematics*, 84(1):111–114.

Sroysang, B. (2013g). On the Diophantine Equation $89^x + 91^y = z^2$. *International Journal of Pure and Applied Mathematics*, 89(2):283–286.

Sroysang, B. (2014a). On the Diophantine Equation $131^x + 133^y = z^2$. *International Journal of Pure and Applied Mathematics*, 90(1):65–68.

Sroysang, B. (2014b). On the Diophantine Equation $143^x + 145^y = z^2$. *International Journal of Pure and Applied Mathematics*, 91(2):265–268.

Sroysang, B. (2014c). On the Diophantine Equation $3^x + 45^y = z^2$. *International Journal of Pure and Applied Mathematics*, 91(2):269–272.

Sroysang, B. (2014d). On the Diophantine Equation $3^x + 85^y = z^2$. *International Journal of Pure and Applied Mathematics*, 91(1):131–134.

Sroysang, B. (2014e). On the Diophantine Equation $8^x + 13^y = z^2$. *International Journal of Pure and Applied Mathematics*, 90(1):69–72.

Su, J. and Li, X. (2014). The Exponential Diophantine Equation $(4m^2+1)^x + (5m^2-1)^y = (3m)^z$. *Hindawi*, 2014:1–5.

Suvarnamani, A., Singta, A., and Chotchaisthit, S. (2011). On Two Diophantine Equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$. *Science and Technology RMUTT Journal*, 1(1):25–28.

Tatong, M. and Suvarnamani, A. (2015). On the Diophantine Equation $(p+1)^{2x} + q^y = z^2$. *International Journal of Pure and Applied Mathematics*, 103(2):155–158.

Trojovský, P. (2015). On the Diophantine Equation $p^a + (p+1)^b = z^2$. *International Journal of Pure and Applied Mathematics*, 105(4):745–749.

Wang, Y. and Wang, T. (2011). On the Diophantine Equation $nx^2 + 2^{2m} = y^n$. *Journal of Number Theory*, 131:1486–1491.