UNIVERSITI PUTRA MALAYSIA

COMMUTING GRAPH OF SOME PRIME ORDER ELEMENTS IN SYMPLECTIC AND MATHIEU GROUPS

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By

SUZILA BINTI MOHD KASIM

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Doctor of Philosophy

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DEDICATIONS

I am blessed to have the most best-loved persons in my heart:

To my respected *Mak* who always too busy to make prayers for her children
To my wise late *Abah* who worked hard to make our dreams come true
To my inspirational *Ibu* who shows beautiful spirit in many ways
To my amazing *Ayah* who supports me to reach my goals
To my best buddies who cheer me up more times
To my awesome siblings who I know I have got your back
To my wonderful unbiological siblings who get our bond stronger
To my loving husband who stay with me through thick & thin each day
To my little son who fills my life with joy & may your future shine bright!

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfillment of the requirement for the degree of Doctor of Philosophy

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SUZILA BINTI MOHD KASIM

January 2020

Chairman : Athirah binti Nawawi, PhD
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This thesis concentrates on an investigation to describe the properties of groups in graph-theoretic context such as connectivity, distance and many more. One of the ways to achieve this is by studying a commuting graph. Let G be a group and X is a subset of G. The commuting graph $\mathcal{C}(G, X)$ is defined by taking a vertex set X, and letting two distinct vertices $x, y \in X$ be adjacent whenever $xy = yx$. We construct the commuting graph by selecting G as a symplectic group or a Mathieu group, and X denotes a conjugacy class of G for prime order elements. The main contribution of this thesis deals with various aspects of the structure of the commuting graph $\mathcal{C}(G, X)$. We choose $t \in X$ to be a fixed vertex of $\mathcal{C}(G, X)$, then we continue to determine the connectivity of the graph whether it is connected or disconnected. If $\mathcal{C}(G, X)$ is found to be connected, then we further the analysis in the commuting graph $\mathcal{C}(G, X)$ to obtain the distance of $x \in X$ from t, the disc size and its diameter. However, if the commuting graph $\mathcal{C}(G, X)$ is disconnected, then there are more than one subgraph can be obtained. These subgraphs which are named as subgraphs $\mathcal{D}(G, X)$ of $\mathcal{C}(G, X)$ are then found to be isomorphic with each other and the diameter of $\mathcal{D}(G, X)$ is either 1 or 2 for the cases we consider. Consequently the number of the subgraphs can also be determined and for each subgraph $\mathcal{D}(G, X)$, we describe its structure as we treat the connected commuting graph $\mathcal{C}(G, X)$. Next, the study of the commuting graph takes naturally into consideration of the suborbits of G on X, that is the orbits under the action of a centralizer $C_G(t)$. Apart from finding the sizes of $C_G(t)$-orbits of X, we obtain representatives $x \in X$ for each of these orbits. Moreover, we also include some properties which in many cases aid speedy identification of the given orbit. We obtain the subgroup $\langle t, x \rangle$ generated by t and x. Then, we conduct a deeper analysis on the $C_G(t)$-orbits by specifying the connectedness between every two orbits. We visualize the interaction of the $C_G(t)$-orbits in a collapsed ad-

jacency diagram, showing a line to join two circles of so-called $C_G(t)$-orbits. We also compute how many vertices a $C_G(t)$-orbit representative is connected to and record those data in a collapsed adjacency matrix. The entries of that matrix may indicate whether two distinct orbits are adjacent or not in the graph. Finally, we present the spectrum of the collapsed adjacency matrix of the commuting graph, that is the multiset of eigenvalues of its collapsed adjacency matrix and of course is one of many important invariants from which much information about the graph can be ascertained. The study of the spectrum of the graph, $\mathrm{Spec}(G, X)$ relates to the graph energy, $\mathcal{E}(G, X)$. We obtain $\mathcal{E}(G, X) > 0$ if the graph is connected, otherwise $\mathcal{E}(G, X) = 0$.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

GRAF KALIS TUKAR TERTIB DARIPADA BEBERAPA ELEMEN PERINGKAT PERDANA DI DALAM KUMPULAN SIMPLEKTIK DAN MATHIEU

Oleh

SUZILA BINTI MOHD KASIM

Januari 2020

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Tesis ini tertumpu kepada sebuah penyiasatan untuk menerangkan ciri-ciri kumpulan di dalam konteks teori-graf seperti keterkaitan, jarak dan banyak lagi. Ini boleh dicapai dengan mengkaji graf kalis tukar tertib. Andaikan G adalah kumpulan terhingga dan X adalah subset bagi G, graf kalis tukar tertib, ditandakan sebagai $\mathcal{C}(G, X)$ yang mempunyai set bucu X dengan dua bucu berbeza $x, y \in X$ disambung oleh satu garis bucu apabila $xy = yx$. Kami membina graf kalis tukar tertib dengan memilih G sebagai satu kumpulan simplektik atau kumpulan Mathieu, dan X merujuk kepada kelas konjugasi G terhadap elemen peringkat perdana. Sumbangan utama tesis ini adalah bagi mengetengahkan kepelbagaian aspek dalam struktur sebuah graf kalis tukar tertib $\mathcal{C}(G, X)$. Kami memilih $t \in X$ sebagai titik pemula bagi $\mathcal{C}(G, X)$ dan kami terus menentukan keterkaitan di dalam graf sama ada ianya berkait atau tidak. Jika $\mathcal{C}(G, X)$ didapati berkait, maka kami melanjutkan analisis dalam graf $\mathcal{C}(G, X)$ untuk menentukan jarak $x \in X$ dari t, saiz cakera dan diameter graf. Walau bagaimanapun, jika graf kalis tukar tertib $\mathcal{C}(G, X)$ adalah tidak berkait, maka terdapat lebih daripada satu subgraf yang boleh dijumpai. Subgraf ini dinamakan sebagai subgraf $\mathcal{D}(G, X)$ daripada $\mathcal{C}(G, X)$ yang kemudian didapati isomorfik dengan satu sama lain dan diameter$\mathcal{D}(G, X)$ adalah sama ada 1 atau 2 bagi kes-kes kami pertimbangkan. Oleh itu bilangan subgraf juga boleh ditentukan dan bagi setiap subgraf $\mathcal{D}(G, X)$, kami akan menghuraikan strukturnya sebagaimana yang kami kendalikan terhadap graf kalis tukar tertib berkait $\mathcal{C}(G, X)$. Seterusnya, kajian terhadap graf kalis tukar tertib secara khususnya mempertimbangkan suborbit dari G ke atas X, iaitu orbit yang melalui tindakan satu titik penstabil $C_G(t)$. Selain mencari saiz orbit-$C_G(t)$ di dalam X, kami mencari wakil-wakil daripada kesemua orbit. Tambahan pula, kami juga menyertakan beberapa ciri yang membantu mem-

percepatkan pencarian orbit tertentu. Kami mencari subkumpulan $\langle t, x\rangle$ yang dihasilkan daripada t dan x. Kemudian, kami mengendalikan analisis mendalam ke atas orbit-$C_G(t)$ dengan menentukan hubungan antara setiap dua orbit. Kami menggambarkan interaksi orbit-$C_G(t)$ ke dalam diagram bersebelahan runtuh, yang menunjukkan terdapat satu garis menghubung dua bulatan yang dipanggil sebagai orbit-$C_G(t)$. Kami mengira bilangan bucu yang disambungkan dengan satu wakil orbit-$C_G(t)$ dan mencatat data yang diperoleh ke dalam sebuah matriks bersebelahan runtuh. Setiap data dalam matriks itu akan menunjukkan sama ada dua orbit adalah bersebelahan runtuh atau tidak di dalam graf. Akhir sekali, kami menilai spektrum daripada graf kalis tukar tertib, iaitu nilai eigen pelbagai set yang terdapat daripada matriks bersebelahan runtuh dan sudah semestinya ia antara salah satu fungsi penting yang dapat dipastikan daripada maklumat sebuah graf. Kajian mengenai graf spektrum, $\mathrm{Spec}(G, X)$ berkait rapat dengan graf tenaga, $\mathcal{E}(G, X)$. Kami mendapati $\mathcal{E}(G, X) > 0$ jika graf adalah berkait, sebaliknya $\mathcal{E}(G, X) = 0$.

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This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Doctor of Philosophy. The members of the Supervisory Committee were as follows:

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LIST OF ABBREVIATIONS

G	a finite group
X	a subset (or a conjugacy class) of G
H	a subgroup of G
Γ	a simple graph
V	a non-empty finite set of vertices
E	a finite set of edges
$\mathbb{V}$	a vector space
$\mathbb{F}$	a field is a set together with two operations called addition and multiplication
$\mathbb{N}$	a natural number
M_i	a Mathieu group on point $i \in \mathbb{N}$, where $i \in \{11, 12, 22, 23, 24\}$
$S_{2m}(q)$	a symplectic group of dimension $2m$ over a a field $q(=\mathbb{F})$ where $m, q \in \mathbb{N}$
p	a prime number
p^n	an elementary abelian group of order p to the power of $n \in \mathbb{N}$ or a direct product of n copies of $\mathbb{Z}_p$, a cyclic group of order p
p_+^{1+2n}	an extra special group of order p to the power of $1 + 2n$
$A \times B$	a direct product, with normal subgroups A and B
$A : B$	a semi-direct product (or a split extension), with a normal subgroup A and a subgroup B
$A \wr B$	a wreath product A and B
$A \cdot B$	a non-split extension, with a normal subgroup A and quotient B, but no subgroup B
$A. B$	an upward extension or unspecified extension of A and B
A_n	an alternating group of degree n
D_n	a dihedral group of order $2n$
$PGL_n(\mathbb{F})$	a projective general linear group of dimension n over a field F
$PSL_n(\mathbb{F})$	a projective special linear group of dimension n over a field F
$SL_n(\mathbb{F})$	a special linear group of dimension n over a field F
$\mathbb{Z}_n$, C_n or n	a cyclic group of order n

CHAPTER 1

INTRODUCTION

1.1 Commuting Graph $\mathcal{C}(G, X)$

This thesis is an exploration of the relationship between two kinds of mathematical objects, groups and graphs. Some problems concerning groups are best attacked by using graphs. Given any two elements of the group, the rule yields another group element, which depends on the two elements chosen. The information in a group can be represented by a graph, which is a collection of points, called vertices, and lines between them, called edges. In the case of the graph encoding a group, the vertices are elements of the group and the edges are determined by the combination rule.

One can associate a graph to a group in many different ways such as the order of the group and the set of element orders. As a reason for this, a special type of graph is studied to examine the behaviours of that graph associated with a group. In this work, a commuting graph $\mathcal{C}(G, X)$ is considered, with G, a finite group and X, a subset of elements in G. The translation of a commuting graph should be in both ways from group to graph and vice versa. A commuting graph is not just a representation of groups but it is also a graph.

Definition 1.1.1 *Let G be a group and X a subset of G. The commuting graph of G on X, denoted by $\mathcal{C}(G, X)$, has X as its vertex set where $x, y \in X$ are connected by an edge whenever x and y commute and $x \neq y$.*

Figure 1.1 expresses an overview of a commuting graph. In general, $t \in G$ is picked randomly as a fixed vertex of $\mathcal{C}(G, X)$ and let X be the set of vertices in that graph. A group G of course acts by conjugation, inducing graph automorphisms of $\mathcal{C}(G, X)$ and is transitive on the vertices of the graph. Throughout this chapter, X will denote a conjugacy class of prime order element of G where G is either a symplectic or Mathieu group. We shall let t denote a fixed element in X. For $x \in X$ and $i \in \mathbb{N}$, the distance on the commuting graph or the i^{th} disc of t is written by $\Delta_i(t) = \{x \in X : d(t, x) = i\}$. The diameter of $\mathcal{C}(G, X)$ is denoted by Diam $\mathcal{C}(G, X)$.

The group can be described as a graph by exploiting the commutative law in that group so the graph is called a commuting graph. If two elements of a group G commute, then this is represented by a line. The next section provides the groundwork for understanding the concept of commuting graph $\mathcal{C}(G, X)$.

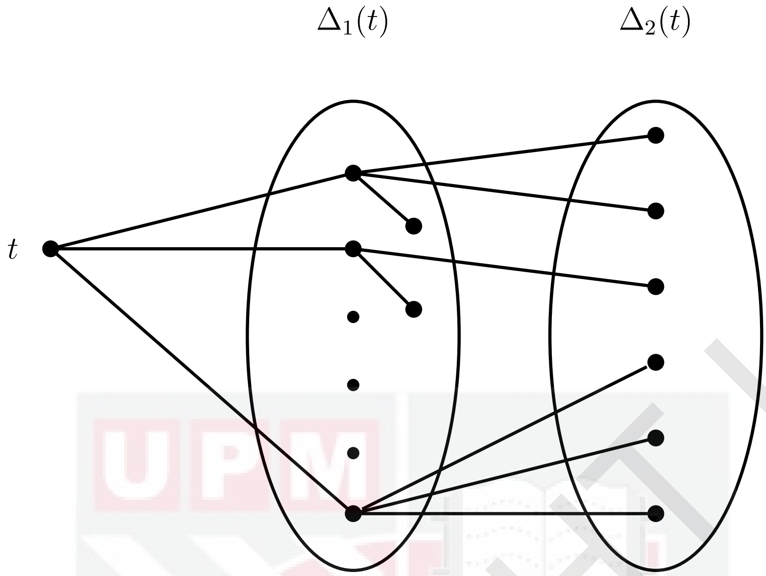

Figure 1.1: Commuting Graph $\mathcal{C}(G, X)$

1.2 Preliminaries

This section includes the preliminary definitions and notions necessary for the later discussion of group and graph theories that will be applied in $\mathcal{C}(G, X)$.

1.2.1 Group Theory

An algebraic structure or commonly known as a group is primarily concerned in this study. The theory of group is pervasive as it is widely used in numerous areas within and outside mathematics, making it a central organizing principle of contemporary mathematics. The object of this subsection is to show a general overview of group theory. To facilitate the study of group, it is convenient to introduce some terminologies and notations. Now we proceed to define the terms in group theory mentioned by Rose (2009) and Malik et al. (1997).

Definition 1.2.1 *(Rose, 2009) A group is an ordered pair $(G, *)$, where G is a non-empty set and $*$ is a binary operation on G such that the following axioms hold:*

1. *Closure: G is closed under the operation of $x, y \in G \Rightarrow xy \in G$.*
2. *Associativity: $(xy)z = x(yz)$ for all $x, y, z \in G$.*

3. *Identity: There exists $e \in G$ such that $xe = ex = x$ for all $x \in G$.*

4. *Inverses: For every element $x \in G$ there exists an element $x^{-1} \in G$ such that $xx^{-1} = x^{-1}x = e$.*

These properties led us to define a few important terms relevant to an abstract group. In this case of a finite group, the set is finite. A group G is said to be abelian if the binary operation is commutative or $xy = yx$ for all $x, y \in G$. The number of elements of a group is called its order and $|G|$ is used to denote the order of G.

In the evolution of group theory, one of the specialized theories of groups is the permutation group which arose from the source, classical algebra. The study of this type of group is initiated by defining what a permutation is. Let X be a nonempty set. A permutation π of X is one-to-one function from X onto X. A group $(G, *)$ is called a permutation group on a nonempty set X if the elements of G are permutations of X and the operation $*$ is the composition of two functions.

A permutation group usually refers to a group that is acting on a set. This includes the symmetric group and every subgroup of a symmetric group. The symmmetric group on set $\Omega = \{1, 2, \ldots, n\}$, written $\mathrm{Sym}(\Omega)$ is the set of all permutations on Ω. It forms a group with the operation of composition. We write permutations on the right and compose from left to right. Technically, the image of $\alpha \in \Omega$ under the permutation f is αf, and the composition of f and g is fg, so that $\alpha\big(fg\big) = \big(\alpha f\big)g$. If $\big|\Omega\big| = n$, it is customary to choose $\Omega = \{1, 2, \ldots, n\}$ as the set of first n natural numbers and it is called the symmetric group of degree n. $\mathrm{Sym}(n)$ is said to be a noncommutative group if $n \geq 3$. The number of elements in $\mathrm{Sym}(n)$ is symbolized and counted by $|\mathrm{Sym}\,(n)\,| = n!$.

In addition, we can display permutation f using cycle notation. The cycle $\big(\alpha_1, \alpha_2, \ldots, \alpha_k\big)$ in f where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are distinct elements of Ω. This means that f send α_1 to α_2, α_2 to α_3, ... and α_k back to α_1. A k-cycle, or a cycle of length k, is a cycle containing k elements. The cycle type of f is an expression of the form $\big(1^{m_1}, 2^{m_2}, \ldots, n^{m_k}\big)$, where m_k is the number of cycles of length k in f.

From the point of view of group theory, groups belonging to the same class are considered to be identical. Now all the groups can be described up to isomorphism. An isomorphism f from a group G_1 to a group G_2 is a one-to-one and onto function from G_1 to G_2 that preserves the group operation. That is, $f\,(xy) = f\,(x)\,f\,(y)$ for all $x, y \in G_1$. If there is an isomorphism from G_1 to G_2, then G_1 and G_2 are isomorphic and written as $G_1 \cong G_2$.

A certain kind of isomorphism is referred to so often that it has been given a special name as an automorphism of G. An automorphism of a group G is an isomorphism from a group G onto itself. The set of all automorphisms of G is denoted by $\mathrm{Aut}(G)$. The reason the set of automorphisms of a group is noteworthy because the group is under the operation of composition function.

Informally, G acting on X is that elements of the group G may be applied to elements of X to give a new element of X. This thesis focuses on the connections between a group and a graph which in some sense represents it. One way the group relates to the graph is through group action, so we recall some facts of group actions Malik et al. (1997). A group G acts on X if there is a map $\cdot : G \times X \to X$, so that if $g \in G$ and $x \in X$, then $g \cdot x \in X$, such that: For every $g, h \in G$, $x \in X$, $(gh)\cdot x = g\cdot(h \cdot x)$ and $e\cdot x = x$, where $e \in G$ is the identity.

The underlying set of one group is a subset of the underlying set of another group. This leads to the concept of a subgroup. Let $(G, *)$ be a group and H be a nonempty subset of G. Then $(H, *)$ is called a subgroup of $(G, *)$ if $(H, *)$ is a group. This is denoted by $H \leq G$. Moreover, H is called a normal subgroup of G and denoted by $H \lhd G$ if $xH = Hx$ for all $x \in G$. Then, a subgroup H of G is normal in G if and only if $xHx^{-1} \subseteq H$ for all x in G.

A question may arise on why normal subgroups are of special significance. When the subgroup H is said to be normal in G, then the set of left (or right) cosets of H in G is itself a group. Thus, this is called the factor group (or the quotient group) of G by H. Let G be a group and H is a normal subgroup of G. The set $G/H = \{xH | x \in G\}$ is a group under the operation $(xH)(yH) = xyH$. Since G is finite and $H \neq \{e\}$, the set G/H is smaller than G. This gives one feasible way to deduce properties of G by examining the less complicated structure of group G/H instead.

However, one of the most important subgroups is $Z(G)$, the center of G as it is always be a normal subgroup of G and closed under conjugation, whenever all elements commute. The center, $Z(G)$, or simply denoted by Z of a group G is the subset of elements in G that commute with every element of G: $Z(G) = \{a \in G | ax = xa \text{ for all } x \in G\}$.

Although an element from a non-abelian group does not necessarily commute with every element of the group, there are always some elements with which it will commute. For example, every element x commutes with all powers of x. This observation prompts the next definition. Let x be a fixed element of a group G. The centralizer of x in G, $C_G(x)$, is the set of all elements in G

that commute with x: $C_G(x) = \{g \in G | gx = xg\}$. Notice that $Z(G)$ is also the intersection of all the centralizers of each element of G. As centralizers are subgroups, this again shows that $Z(G)$ is a subgroup.

However, we also put the definition of the stabilizer of G and the orbit of x under G. Let G be a group of a set X. For each x in X, let $G_x = \{g \in G | gx = x\}$. G_x is called the stabilizer subgroup of x, that is, the set of all elements in G that fix x. For each $x \in X$, let $\mathcal{O}(x) = \{gx | g \in G\}$. The set $\mathcal{O}(x)$ is a subset of X called the orbit of x under G. $|\mathcal{O}(x)|$ denotes the number of elements in $\mathcal{O}(x)$. If G is transitive on Ω, the rank of G on Ω is the number of orbits G_x on Ω. So, these orbits are called the suborbits.

For any group G, two permutations a and b are in the same conjugacy class if and only if they have the same cycle type and the same order. So, two elements $a, b \in X$ lie in the same orbit of an action of G if and only if $y = gx$ for some $g \in G$. We say that a and b are conjugate in G if $xax^{-1} = b$ for some x in G. The conjugacy class of a is the set $\{xax^{-1} | x \in G\}$.

The notion of simple group was introduced by Galois (1811 - 1832). A group G is said to be a simple group if $G \neq \{1\}$ and G contains non-trivial normal subgroup. The abelian simple groups are the group order 1 and the cyclic groups of prime order, while the non-abelian simple groups generally have very complicated structures. Just as prime numbers can be thought of as the building blocks in number theory or the elements in chemistry, in a similar fashion, simple groups may be considered as the building blocks for all finite groups in the following way by defining first a maximal normal subgroup and composition series.

A normal subgroup H of a group G is said to be a maximal normal subgroup of G if $H < G$ and there is no normal subgroup K of G such that $H < K < G$. In addition, H is a maximal normal subgroup if and only if the factor group G/H is simple. Every finite group G of order greater than one possesses a finite series of subgroups, called a composition series, such that

$$\{e\} = H_n \lhd H_{n-1} \lhd \ldots \lhd H_2 \lhd H_1 \lhd H_0 = G,$$

where H_{i+1} is a maximal subgroup of H_i and $H \lhd G$ means that H is a normal subgroup of G.

Given a finite group G, choose a maximal normal subgroup H_1 of $G = H_0$ of largest order. Then the factor group H_0/H_1 is simple and next a maximal normal subgroup H_2 of H_1 of largest order is chosen. Then H_1/H_2 is also simple and continues along in this fashion until it arrives at $H_n = \{e\}$. The simple groups G/H_0, H_0/H_1, H_1/H_2, ..., H_{n-1}/H_n are called the composition factors of G.

A group that is not simple can be broken into two smaller groups in terms of a non-trivial normal subgroup and the corresponding factor group, called a group extension. If H and K are two subgroups, then G is an extension of K by H (or an extension of H by K) if there is a short exact sequence $\{e\} \rightarrow H \rightarrow G \rightarrow K \rightarrow \{e\}$. This process can be repeated, and for finite groups, one eventually arrives at uniquely determined simple groups by the Jordan-Hölder Theorem.

Theorem 1.2.2 (Jordan-Hölder Theorem) *(Malik et al., 1997) Let G be a finite group. Suppose that there are two chains of subgroups H and K in G. If $\{e\} \subset H_s \subset \ldots \subset H_2 \subset H_1 \subset G$ is one composition series and $\{e\} \subset K_t \subset \ldots \subset H_K \subset K_1 \subset G$ is another, then $t = s$ and corresponding to any composition group K_j/K_{j+1}, there is a composition factor H_i/H_{i+1} such that $\frac{K_j}{K_{j+1}} \cong \frac{H_i}{H_{i+1}}$.*

A group may have more than one composition series. On the other hand, the Jordan-Hölder Theorem asserts that any two composition series of a given group are equivalent. That is, they have the same composition length and the same composition factors, up to permutation and isomorphism. In addition, the Jordan-Hölder Theorem states that these composition factors are independent of the choices of the normal subgroups made in the process described. In some cases, a group can be reconstructed from its composition factors and many of the properties of a group are determined by the nature of its composition factors.

According to the extension problem, where the subgroups H and K are known and the properties of G are to be determined, all finite groups may be constructed as a series of extensions with finite simple groups. This led to the complete classification of finite simple groups (CFSG), completed in 2004.

Theorem 1.2.3 (CFSG) *(Wilson, 2009) A non-abelian finite simple groups G is one of:*

1. *a cyclic group C_p of prime order p,*
2. *an alternating group A_n, for $n \geq 5$,*
3. *a classical group:*
   - (a) *linear: $PSL_n(q)$, $n \geq 2$, except $PSL_2(2)$ and $PSL_2(3)$,*
   - (b) *unitary: $PSU_n(q)$, $n \geq 3$, except $PSU_3(2)$,*

(c) *symplectic:* $PSp_{2n}(q)$, $n \geq 2$, *except* $PSp_4(2)$,

(d) *orthogonal:* $P\Omega_{2n+1}(q)$, $n \geq 3$, q *odd*, $P\Omega^{+}_{2n}(q)$, $n \geq 4$, $P\Omega^{-}_{2n}(q)$, $n \geq 4$ *where* q *is a power* p^a *of a prime* p,

4. *an exceptional group of Lie type:*

$$G_2(q),\ q \geq 3,\ F_4(q),\ E_6(q),\ {}^2E_6(q),\ {}^3D_4(q),\ E_7(q),\ E_8(q)$$

where q *is a prime power, or*

$${}^2B_2\left(2^{2n+1}\right),\ n \geq 1,\ {}^2G_2\left(3^{2n+1}\right),\ n \geq 1,\ {}^2F_4\left(2^{2n+1}\right),\ n \geq 1$$

or the Tits group ${}^2F_4(2)'$,

5. *one of* 26 *sporadic simple groups:*

(a) M_{11}, M_{12}, M_{22}, M_{23}, M_{24} *(the Mathieu groups),*

(b) Co_1, Co_2, Co_3, McL, HS, Suz, J_2 *(the Leech lattice groups),*

(c) Fi_{22}, Fi_{23}, $Fi^{'}_{24}$ *(the Fischer groups),*

(d) $\mathbb{M}$, $\mathbb{B}$, Th, HN, He *(the Monstrous groups),*

(e) J_1, J_3, J_4, $O'N$, Ly, Ru *(the pariahs).*

The 26 sporadic simple groups are in many ways the most interesting of the finite simple groups, yet the most difficult to construct. Emile Mathieu discovered the Mathieu group in the 19^{th} century which was the first family of sporadic simple groups, listed as M_{11}, M_{12}, M_{22}, M_{23} and M_{24}.

M_{24} is the largest Mathieu group and may be defined as an automorphism group of the Steiner system $S(5, 8, 24)$. It follows that the order of M_{24} is $|M_{24}| = 244\,823\,040$. The group M_{24} is 5-transitive and acts on 24 points of the permutation group. Meanwhile, the groups M_{23} and M_{22} are defined to be the point and pointwise stabilizer subgroups in M_{24}, which therefore has order $|M_{24}|/24 = 10\,200\,960$, while the stabilizer of two points has order $|M_{23}|/23 = 443\,520$, respectively. The Steiner system $S(5, 8, 24)$ gives rise to a Steiner system $S(4, 7, 23)$, and so M_{23} is 4-transitive on 23 points. The stabilizer of a point in M_{23}, M_{22} is 3-transitive on 22 points, and preserves a Steiner system $S(3, 6, 22)$. These three groups M_{22}, M_{23} and M_{24} are collectively known as the large Mathieu groups.

The group M_{24} also acts transitively on 12 points, so that the stabilizer of one of them is a group of order $|M_{24}|/2\,576 = 95\,040$, and is the group M_{12}. In other words, M_{12} preserves a Steiner system $S(5, 6, 12)$. The point stabilizer in M_{12} shows that there is a group of order $|M_{12}|/12 = 7\,920$ and acts 4-transitively on a set of 11 points which preserves a Steiner system $S(4, 5, 11)$.

This type of group is one of the small Mathieu groups called M_{11}.

One of the six families of the so-called classical simple groups is the symplectic group which is defined in terms of a certain form on the vector space $\mathbb{V}$. The other five groups are the linear, unitary and the three families of orthogonal groups. All the classical simple groups are defined in terms of groups of matrices over fields. A field $\mathbb{F}$ is a set with two binary operations called addition and multiplication which satisfy the required axioms. Without much loss of generality, let $\mathbb{V}$ be a vector space of dimension $2m$ over the finite field $\mathbb{F}_{q^{2m}}$ that has q^{2m} elements.

The symplectic group, denoted by $S_{2m}(q)$ is the isometry group of a non-singular alternating bilinear form f on $\mathbb{V} \cong \mathbb{F}_{q^{2m}}$, and is said to be the subgroup of $GL_{2m}(q)$ consisting of those elements g such that $f(u^g, v^g) = f(u, v)$ for all $u, v \in \mathbb{V}$. Any such matrix in $S_{2m}(q)$ has a determinant of 1. Vector space $\mathbb{V}$ has a symplectic basis $\{e_1, \ldots, e_m, f_1, \ldots, f_m\}$ such that all basis vectors are perpendicular to one another except $f(e_i, f_i) = 1$. The order of the symplectic group can be calculated by counting the number of ways of choosing an (ordered) symplectic basis $e_1, \ldots, f_m$. Assume that e_1 be any non-zero vector, thus, it can be chosen in $q^{2m} - 1$ ways. Then, $e^{\perp}$ consists of dimension $2m - 1$ and so it has q^{2m-1} vectors. Thus, there are $q^{2m} - q^{2m-1} = (q-1)\,q^{2m-1}$ vectors v with $f(u, v) \neq 0$. With sets of $q - 1$ scalar multiples, one with each possible value of $f(u, v)$, there are just q^{2m-1} choices for f_1. By induction, the order of symplectic group is

$$|S_{2m}(q)| = \prod_{i-1}^{m} \left(q^{2i} - 1\right) q^{2i-1} = q^{m^2} \prod_{i=1}^{m} \left(q^{2i} - 1\right).$$

Observe that $f(\lambda u, \lambda v) = \lambda^2 f(u, v) = f(u, v)$ if and only if $\lambda = \pm 1$. Hence the only scalars in $S_{2m}(q)$ are ± 1. The group $PS_{2n}(q)$ is the quotient of $S_{2m}(q)$ by the subgroup (of order 1 or 2) of scalar matrices. These are in many ways the easiest of the classical groups to understand.

1.2.2 Graph Theory

This subsection provides a summary about the graph theory in order to consolidate our subsequent work and all basic notations of graph theory have been adapted to Malik et al. (2014).

A graph is an ordered pair $\Gamma(V, E)$ comprising a non-empty finite set of vertices V and a finite set of edges E connecting vertices in V. Note that Γ is a simple graph (that is, Γ does not contain parallel edges, directed graph and loops). Let $u, v \in \Gamma$. Two vertices u and v of a graph Γ are adjacent if there is an edge joining them. A graph Γ is connected if for any two distinct vertices $u, v \in \Gamma$, there is a $u - v$ path in Γ, otherwise the graph is called a disconnected graph. Then $N(u) = \{v \in V | \, v \, \text{is adjacent to} \, u\}$. That is, $N(u)$ is the set of all vertices in V that are adjacent to u. Furthermore, $N(u)$ is called the set of neighbour of u.

Furthermore, the distance between two vertices u, v of Γ, written as $d(u, v)$, is the length of a shortest path, if any exists, from u to v. The diameter of Γ, denoted by Diam Γ, is defined by $\text{Diam}\,\Gamma = \max\{d(u, v) \, | u, v \in V\}$. The degree of u, written $\deg(u)$ is the number of edges incident with u. A regular Γ is a graph where each vertex has the same degree. Besides, we denote Γ_1 be a subgraph of a graph Γ. If any two vertices of Γ_1 are connected in Γ_1, then Γ_1 is not properly contained in any connected subgraph of Γ. A graph Γ is considered as a complete graph in which there is an edge between every pair of distinct vertices on n vertices and it is denoted by K_n.

1.2.3 Spectral Theory

By looking at a graph, one can identify the vertices and the edges connect to, and a matrix associated with the graph is a way of recording this. Matrices prove so useful in applications because of the insight, one gains from their eigenvalues. For more detail, the basic concepts from elementary linear algebra particularly on eigenvalues of matrices can be found in Ju et al. (2010). The question then arises, given that the eigenvalues of some matrices, what can we say about the graph. Spectral graph theory looks at answering this type of question. In addition, one of the background materials on spectral graph theory can be found in Biggs (1993) which shows how eigenvalues and structures of graphs are interrelated.

Theoretically, the calculations to produce the multiset of eigenvalues of a matrix are as below. Assume that the following is a matrix $A = [a_{ij}]$ of a graph Γ and has size k.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1k} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2k} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & a_{k3} & \cdots & a_{kk} \end{bmatrix}$$

In doing this, we find the values of λ which satisfy the characteristic equation of the matrix A, namely those values of λ for which $\det(\lambda I - A) = 0$, where I is the $k \times k$ identity matrix. Then, we form the matrix $\lambda I - A$. Notice that this matrix is just equal to $-A$ with λ subtracts each entry on the main diagonal. We have to calculate $\det(\lambda I - A)$. Therefore, the spectrum of Γ is the set of roots λ and written in the form $\left\{\lambda_1^{m_1}, \lambda_2^{m_2}, \ldots, \lambda_i^{m_i}\right\}$, where $i, m \in \mathbb{N}$, λ_i are the eigenvalues of the graph with multiplicities $m_1, m_2, \ldots, m_i$, respectively.

1.3 Computational Tools

This section will introduce the use of the Atlas of Finite Group and MAGMA in our study. For nomenclature of the representations, the Atlas of Finite Group will impart data sources. The use of a mathematical software package is vital to display the output promptly. For this reason, the MAGMA software was deemed to be able to bring computational problems to light and be highlighted in the last part of this section.

1.3.1 ATLAS of Finite Groups

When constructing the commuting graphs, the Atlas of Finite Group and the Atlas of Group Representations are two mediums that have been greatly used. Both of these are simply called and referred in the Atlas (Conway et al., 1985) and ATLAS (Wilson et al., 2017), respectively. They supply some information on conjugacy classes X of finite simple groups G with different classes and orders. For example, a class of elements of X is given a name in the form $2A$, where 2 is the order of an element of that class, and the letter A distinguishes the classes of elements of the same order. However, it can be seen that every group G may have the same notation of conjugacy classes X. Again, for instance, $X = 2A$ exists in all G. Bear in mind that they are different from one another, either the cycle type or the number of fixed points of their permutations.

All five Mathieu groups and several symplectic groups $S_{2m}(q)$ of dimension $2m$ over finite field $\mathbb{F}_q$ have been chosen in this study. All groups are obtained from ATLAS (Wilson et al., 2017) and may be viewed in the smallest degree of permutation representations. The groups used throughout this thesis can be searched in Section A.1 of Appendix A.

1.3.2 MAGMA

Hand calculations are somewhat cumbersome and time-consuming. In order to have mathematically and practically clear and concise methods, MAGMA software (Cannon, 1997) is one of the best tools can be applied in this study. The MAGMA handbook provides codes that help with the learning of the methods that suited us best.

The computer algebra system MAGMA was designed to provide a software environment for computing with the structures which arise in areas such as algebra, number theory, algebraic geometry and combinatorics. Such software enables users to define and compute with structures such as groups, rings, fields, modules, algebras, schemes, curves, graphs, designs, codes and many others. Most of the major algorithms currently installed in the MAGMA kernel are the latest and give performance similar to, or better than, specialized programs.

1.4 Scope of Study

The commuting graph $\mathcal{C}(G, X)$ of a finite group G has a vertex set X which is the subset of the group. We say that two distinct vertices $x, y \in X$ being joined if and only if $xy = yx$. If G is abelian, the graph is called a complete graph. This is the reason why we are interested in non-abelian groups. Our research concentrates on G as subgroups of $\mathrm{Sym}(n)$ where we focus on symplectic group - $S_4(2)'$, $S_4(3)$ and $S_6(2)$, or a Mathieu group - M_{11}, M_{12}, M_{22}, M_{23} and M_{24}. These groups are finite and nonabelian. Their orders are not too big and the computation on them are possible. Furthermore, we only focus on prime order elements (three and above) since many researchers have described commuting graphs of elements of order two (also known as involutions). Thus, this leads into the study of the structure of $\mathcal{C}(G, X)$.

1.5 Research Objectives and Methodology

We create this section by conducting five research objectives. Then, the brief explanations of the method used towards achieving each objective are described. We aimed to:

Objective 1: Identify the connectivity of $\mathcal{C}(G, X)$.

Methodology: We study the structure of commuting graph that is constructed by $t \in G$ and $X = t^G$. We choose G as one of symplectic group - $S_4(2)'$, $S_4(3)$ and $S_6(2)$, or a Mathieu group - M_{11}, M_{12}, M_{22}, M_{23} and M_{24}. Let $t \in G$ is an element of prime order and $X = t^G$, a G-conjugacy class of t.

As we are studying the commuting graph using the elements of permutation group, we require to determine the properties of those elements $x \in X$. For this reason, we can identify the connectivity of graph structures whether there is a path between every pair of vertices. Equivalently, the graph is connected, otherwise we call it the disconnected graph.

Objective 2: Determine the distance and the diameter of the commuting graph, $\mathcal{C}(G, X)$.

Methodology: Let an arbitrary $t \in X$ be an initial vertex to construct the commuting graph, $\mathcal{C}(G, X)$ and x be any other vertices in X. Note that disc of t, $\Delta_i(t)$ define the distance of x from t, where $i \in \mathbb{N}$. The elements in the first disc of t are the set of $x \in X$ such that $x \in C_G(t)$, excluding t. We write $\Delta_1(t) = \{x | x \in (X \cap C_G(t))\} \setminus \{t\}$. Let $y \in \Delta_1(t)$. Then, the elements in the second disc are obtained such that $\Delta_2(t) = \{x \in X | xy = yx\} \setminus (\Delta_1(t) \cup \{t\})$. The same process in $\Delta_2(t)$ is repeated to discover the distance of remaining elements X from t. We terminate the process when $|t| + \sum_i |\Delta_i(t)| = |X|$. Otherwise, there is the disconnected graph and every component or subgraph of this kind of graph is denoted by $\mathcal{D}(G, X)$. If the last disc of t is $\Delta_j(t)$ where $j \in \mathbb{N}$, then the diameter of the connected graph (or the subgraph of disconnected graph) equals j. The diameter of the connected graph may denote as Diam $\mathcal{C}(G, X)$, however Diam $\mathcal{D}(G, X)$ represents the subgraph of the disconnected $\mathcal{C}(G, X)$. Note that G is transitive permutation representation on the set of vertices X. Let $x, y \in X$, we consider t as an initial point of the graph with $x \neq t$ and $y \neq t$, then $d(x, y) \leq$ Diam $\mathcal{C}(G, X)$ (or Diam $\mathcal{D}(G, X)$).

Objective 3: Analyze the $C_G(t)$-orbits of X.

Methodology: Apart from analyzing each element of X, it is also possible to break X into smaller parts under the action of the centralizer of t in G, $C_G(t)$, which can be called as $C_G(t)$-orbits. In fact, $C_G(t)$ is actually the stabilizer of t in G. Then, a random $x \in X$ is picked in each $C_G(t)$-orbit to be the representative and henceforth, together with t, generate the subgroup $\langle t, x \rangle$ for that orbit.

Objective 4: Describe the interactions between $C_G(t)$-orbits in the collapsed adjacency diagram and the collapsed adjacency matrix.

Methodology: The way a $C_G(t)$-orbit interacts with any other $C_G(t)$-orbits in their respective $\Delta_i(t)$ can also be described by the construction of collapsed adjacency diagram and collapsed adjacency matrix. Rows and columns of the collapsed adjacency matrix are labelled by $C_G(t)$-orbits according to the ordering of $\Delta_i(t)$. The entries of the matrix are based on how many a vertex in one orbit is connected to, in another orbit or the same orbit.

Objective 5: Compute the spectrum of $\mathcal{C}(G, X)$, which is extracted by the collapsed adjacency matrix.

Methodology: Since G acts vertex-transitively, the collapsed adjacency diagram and the collapsed adjacency matrix of the graph $\mathcal{C}(G, X)$ can be used to gain more information and applications of the graph. Whether the graph is connected or disconnected, our target is to compute the spectrum of $\mathcal{C}(G, X)$ using its collapsed adjacency matrix. Then, the spectrum of $\mathcal{C}(G, X)$ is determined which gives outputs for the graph energy. However, the validity of graph energy is only evaluated for the connected $\mathcal{C}(G, X)$.

1.6 Overview

This section presents a brief outline of the material used in each chapter. Seven chapters are covered in this thesis with the following contents.

Chapter 1 provides some mathematical backgrounds such as the concept of groups , fundamental of graphs and spectral theory. This study is a mixture of theoretical and computational aspects. We then highlight the computational tools of this reseach which are MAGMA (Cannon, 1997) and ATLAS (Wilson et al., 2017). Also, we clearly define the research objectives and methodology that can be a direction of the study.

In Chapter 2, we review some materials related to the construction of commuting graph and towards its applications. Since this study focus on G-conjugacy class for elements of prime order of finite group, then in the middle of this chapter we give the transition ideas arisen from the previous studies that related to commuting graphs for involution (or elements of order two) and commuting graphs for elements of order three either using $G = \mathrm{Sym}(n)$ or $G \leq \mathrm{Sym}(n)$.

Our main contributions in this study begin with Chapter 3 until Chapter 6. The work presented in Chapter 3 contributes to the study of connected commuting graph for involution in G. Elements $x \in X$ that are connected to t are identified in terms of set of points in Ω, and these will indicate the connectivity in the graph. So, we briefly describe the disc of $t \in G$ in the graph, hence the size of disc $\Delta_i(t)$ and the Diam $\mathcal{C}(G, X)$ are obtained. This chapter is ended by analyzing the $C_G(t)$-orbits with the $C_G(t)$-orbit representatives $x \in X$.

Then, in Chapter 4, this research is continued in studying the connected commuting graph for elements of order three in G. We investigate the set of points in Ω in order to define the connectivity between t and $x \in X$. In this way, the disc of $t \in G$ could be determined, so we can notice the size of disc $\Delta_i(t)$ and the Diam $\mathcal{C}(G, X)$. Besides that, we classify $\Delta_i(t)$ into $C_G(t)$-orbits and each $C_G(t)$-orbit is represented by a $C_G(t)$-orbit representative $x \in X$.

An extensive study on the $C_G(t)$-orbits is discussed in Chapter 5 where this chapter focuses on the connected $\mathcal{C}(G,X)$ for involution and elements of order three. We target to see the connections of vertices between two $C_G(t)$-orbits. This can be achieved by constructing the collapsed adjacency diagram and the collapsed adjacency matrix. We will find out some properties on how they are being constructed. Furthermore, we gain some ideas based on the spectral theory to define the spectrum of commuting graph, $\mathrm{Spec}(G,X)$. Thus, the total of $\mathrm{Spec}(G,X)$ will eventually equal to the graph energy, $\mathcal{E}(G,X)$.

Chapter 6 provides a study on the disconnected commuting graph $\mathcal{C}(G,X)$ which is consisting the vertices $X = t^G$ and t is a prime order element in G. For every subgraph in this kind of graph, we will denote as $\mathcal{D}(G,X)$. Our focus is to identify some properties that will make the graph disconnected. We continue this work by concerning on the connected subgraphs $\mathcal{D}(G,X)$. The size of $\Delta_i(t)$ is determined alongside with the Diam $\mathcal{D}(G,X)$. With the aid of $C_G(t)$-orbit representatives $x \in X$, we will figure out in which $\Delta_i(t)$ they are. After that, we build the collapsed adjacency diagram and the collapsed adjacency matrix of $\mathcal{D}(G,X)$ to represent interactions among those $C_G(t)$-orbits. We then compute the spectrum of $\mathcal{D}(G,X)$ and glean some information from the collapsed adjacency matrix. Note that, in this chapter we will use a slightly different notation for $\mathrm{Spec}\big(\mathcal{D}(G,X)\big)$.

Finally, we will summarize our study in Chapter 7. Throught this chapter, we make an effort to provide some extended work from this research that can be carried out in the future.

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