UNIVERSITI PUTRA MALAYSIA

GROUP THEORETIC QUANTISATION ON SPHERES AND QUANTUM HALL EFFECT

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By

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GROUP THEORETIC QUANTISATION ON SPHERES AND HALL EFFECT

By

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In this thesis, Isham's group theoretic quantisation technique has been applied to quantise Hall systems with spheres as their underlying configuration spaces. Before doing this, a preliminary mathematical tools needed for this work is given followed by an overview of the above mentioned quantisation scheme.

Beginning with the simple sphere in the first stage, it is found that the part of canonical group which acts on the configuration space when the magnetic field is absent is either the group $SO(3)$ or its covering group $SU(2)$. However when the external field is present there is an obstruction which necessitates the group $SU(2)$ as the canonical group. The representations of the group $SU(2)$ are parameterized by an integer n which could be used to classify the integer Hall states. This however gives only a description for the case of integer quantum Hall effect.

To get the quantisation of a system of a test particle within a "many- particle formalism" punctures are introduced on the sphere. First, the quantisation problem on the punctured sphere is approached using a generalization of the method that

works for the simple sphere. This method seems to show that $SU(2)$ is still the canonical group at first glance, but with the problem of global definition, the right choice of canonical group would be the quotient group $SU(2)/H$ with H as the subgroup of $SU(2)$ which takes points on the sphere to the punctures. Unfortunately, such description is not very illuminating and this group doesn't show clearly the symmetry exchange of the punctures. To overcome a small portion of this problem we use uniformisation theory to get the canonical group directly by Isham's technique of the homogeneous space. Within this approach it is possible to adopt the quotient group$SL(2,\mathbb{R})/SO(2)$ as the canonical group for the case without magnetic field and $SL(2,\mathbb{R})$ for the case with magnetic field. From another perspective we also attempted quantisation on the universal covering, the upper half plane with the hope of projecting it down to the punctured sphere, and we found $SL(2,\mathbb{R})$ to be the canonical group. However the use of representations of $SL(2,\mathbb{R})$ cannot lead to a classification of the fractional Hall state and a twisted representation could be necessary to get such classification.

At the end of this thesis a different technique of approaching the fractional quantum Hall classification has been applied to the special case of the thrice-punctured sphere. First we present a link between the principal congruence subgroup of the modular group of prime level 2, $\Gamma(2)$ as the isomorphic group to the fundamental group of the thrice-puncture sphere and the braid group of three particles on the plane. Then a classification of the Hall states, integer as well as fractional, has been given using the action of the group $\Gamma(2)$ on the cusps of the fundamental region defining the punctured sphere on the upper half plane.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor falsafah.

PENGKUANTUMAN TEORI KUMPULAN ATAS SFERA DAN KESAN HALL

Oleh

BOUKETIR AHMED

Ogos 2000

Pengerusi: Dr Hishamuddin Zainuddin

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Dalam tesis ini, teknik pengkuantuman teori kumpulan oleh Isham diguna untuk mengkuantumkan sistem-sistem Hall yang mempunyai sfera sebagai ruang konfigurasi mereka. Sebelum berbuat demikian, kami berikan peralatan-peralatan yang diperlukan untuk tujuan tersebut dan dituruti pula gambaran kasar skema pengkuantuman.

Bermula dengan sfera ringkas, kami mendapati bahagian kumpulan berkanun yang bertindak ke atas ruang konfigurasi bagi kes tanpa medan magnet ialah sama ada kumpulan $SO(3)$ atau kumpulan litupannya $SU(2)$. Walau bagaimanapun apabila wujud medan luar, terdapat halangan yang hanya dapat diatasi dengan berkehendakkan kumpulan $SU(2)$ sebagai kumpulan berkanun. Perwakilan-perwakilan kumpulan $SU(2)$ mempunyai parameter integer n yang dapat diguna untuk mengkelaskan keadaan-keadaan Hall integer.

Berikutnya, tebukan dimuatkan dalam sfera supaya formalisme satu zarah uji dalam gambaran berbilang zarah diperolehi. Masalah pengkuantuman di atas sfera tertebuk ini mula-mula diselesaikan dengan kaedah yang sama seperti kes sfera

ringkas. Kaedah ini menunjukkan seakan-akan $SU(2)$ masih menjadi kumpulan berkanun pada sekali imbas, tetapi dengan masalah takrifan sejagat, calon yang betul ialah kumpulan hasil bahagi $SU(2)/H$ dengan H sebagai subkumpulan $SU(2)$ yang mengambil titik-titik sfera ke titik tebukan. Malangnya, perihalan sedemikian tidak begitu jelas dan kumpulan ini tidak menunjukkan simetri tukar ganti tebukan-tebukan. Untuk mengatasi sebahagian kecil masalah ini, kita gunakan teori penyeragaman bagi mendapatkan kumpulan berkanun secara terus mengikut teknik pengkuantuman Isham di atas ruang homogen. Dalam kaedah ini, kita boleh mengambil kumpulan hasil bahagi $SL(2,\mathbb{R})/SO(2)$ dan kumpulan $SL(2,\mathbb{R})$ masing-masing sebagai kumpulan berkanun untuk kes tanpa medan magnet dan kes bermedan magnet. Dalam perspektif yang lain pula, kami juga mencuba pengkuantuman ke atas ruang litupan universal iaitu separuh satah atas dengan harapan mengunjurkan semula ke sfera tertebuk dan kami perolehi $SL(2,\mathbb{R})$ sebagai kumpulan berkanun. Namun demikian penggunaan perwakilan $SL(2,\mathbb{R})$ tidak dapat membawa kepada pengkelasan keadaan Hall pecahan dan perwakilan terpilin mungkin diperlukan untuk berbuat demikian.

Di akhir tesis, teknik berbeza digunakan dalam memperolehi pengkelasan Hall kuantum pecahan dengan aplikasi khusus kepada sfera tiga tebuk. Kami tunjukkan hubungan antara subkumpulan kongruen utama bagi kumpulan modulus bertahap utama 2, $\Gamma(2)$ sebagai kumpulan yang isomorf kepada kumpulan asasi sfera tiga tebuk dengan kumpulan sirat tiga zarah di atas satah. Kemudian kami kelaskan keadaan Hall pecahan menggunakan tindakan kumpulan $\Gamma(2)$ ke atas juring rantau asasi yang mentakrifkan sfera tertebuk pada separuh satah atas.

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Finally, may Allah reward the people who helped me directly or indirectly in finishing this work, especially my brother Omrane for his help and encouragement. I wish them every success in this life and hereafter under the guidance of Allah s.w.t.

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LIST OF ABBREVIATION

QHE: Quantum Hall Effect.

IQHE: Integer Quantum Hall Effect.

FQHE: Fractional Quantum Hall Effect.

HVF: Hamiltonian Vector Fields

CHAPTER 1

MATHEMATICAL PRELIMINARIES

Our aim in this chapter is to prepare the way for the study of quantisation problem in the forecoming chapters, by giving a brief account of the mathematical tools that we shall use throughout our work. Before that we shall give a general introduction to our work first.

1.1 General Introduction

The quantum mechanics on two-dimensional spaces capture a lot of attention during the last twenty years after the discovery of quantum Hall effect (QHE) [47,48] with what have been achieved concerning this new phenomenon and the progress that have been obtained in the quantum gravity in two dimensions [110] from another perspective.

Within this trend, studying quantum mechanics of a system of charged particle living on a two-dimensional surface and interacting with a magnetic field normal to the surface has received a renewed interest in the context of the QHE. In fact the QHE appears to be related to many rich physical and mathematical structures, which are worthwhile to be investigated in their various possible configurations. A particular intriguing and interesting case occurs when the two-dimensional surface is a non-simply connected Riemann surface where the topological effects play an important role particularly in the explanation of the fractional Quantum Hall effect.

This thesis is motivated on one part by the wish to quantise a system of charged particle moving on the sphere and on the punctured sphere with and without the influence of an external magnetic field with more focus on the latter case. In another perspective, the quantisation perspective may shed some light into the problem of fractional quantum Hall effect, particularly in the possible (partial) explanation of the filling fractions.

As it is known that the fractional Hall effect is a phenomenon explained in a system of many particles; thus we consider punctures in the configuration space to play the role of other particles seen by the test particle based on the hard-core particle concept. Specifically we will explore the problem from the viewpoint of quantisation and classification, using the group theoretic approach with a geometrical background called group theoretic quantisation developed by Isham in his attempt to quantise gravity [38]. The formalisms of group theoretic quantisation are intended to perform the task easily. It is desirable as far as possible to understand the physics without using information other than the symmetries of the system under study and with the property of keeping the topological effects inherently encoded within the formalism. Those effects will play in important role in the understanding of the quantum Hall effect and other quantum phenomena.

The thesis is organized as follows. In the rest of this chapter we begin by introducing preliminary mathematical tools that shall be used in the rest of the work. The symplectic geometry, fiber bundle, group representation and Riemann surfaces are among the main topics being discussed where the basic facts and theorems are given.

Chapter two introduces the quantisation programme. We first briefly list the programme steps then we go through the details at the second stage followed by an application to the torus case as it is an non-simply connected Riemann surface which was solved previously [40]. Chapter three is devoted to the quantisation problem of a charged particle on the sphere without and with external magnetic field. For the later case we find an obstruction occurs in the action of the canonical group and a covering group has been introduced in order to lift this obstruction. The multiply connected configuration space i.e. the punctured sphere with its rich structure discussed in chapter four with more Riemann surfaces tools together with the quantisation on the upper half plane. Chapter five is devoted to the thrice punctured sphere with focus on the classification of the Hall states for both cases integer as well as fractional using the modular group that is obtained by geometric and algebraic ways. Finally we summarize briefly our findings.

1.2 Mathematical Tools

1.2.1 Symplectic Geometry

As it is a mathematical technique that played a central role in the modern theory of classical mechanics, it is still also an essential material in the group theoretic quantisation scheme [1]. Symplectic geometry [2-4] studies the symplectic manifolds and the symplecto-diffeomorphisms. The relation with mechanics is usually expressed by saying that the phase space of a mechanical system is a symplectic manifold, and time evolution of a conservative dynamical system is a one parameter family of symplectic diffeomorphisms. To be more formal let us give the following definitions.

By a *symplectic structure* ω on a smooth (C^∞) even-dimensional manifold we mean a closed nondegenerate differential 2-form on it.i.e.

i) ω is closed, $d\omega = 0$,

ii) for each $x \in Q$, $\omega_x : T_xQ \times T_xQ \to \mathbb{R}$ is nondegenerate;

where ω_x and T_xQ is the two-form ω and the tangent space at x respectively. One should mention that the tangent space at each point of the symplectic manifold is a symplectic vector space. A manifold Q equipped with a symplectic structure ω is called a *symplectic manifold* (Q, ω). The C^∞-mapping ψ: $Q_1 \to Q_2$ which takes the symplectic structure of one manifold over into the symplectic structure of another manifold is called a *symplectic* (symplectomorphism) or a *canonical transformation*. In other words the map ψ defines a canonical transformation if and only if:

$$\psi^* \omega_1 = \omega_2 \quad ; \qquad (1\text{-}1)$$

ψ^* being the pull-back of ψ. For the transformation on the same manifold one only has: $\omega_1 = \omega_2 = \omega$. This type of diffeomorphisms together with their infinitesimal generators play a key role in the quantisation scheme used in this study. Usually in the mathematical treatment of the finite-dimensional classical systems, the phase space is taken to be the cotangent bundle T^*Q of the configuration space Q in place of the tangent bundle. The motivation for taking the cotangent bundle as a mathematical model for the phase space (state space) lies in the possibility of identifying elements of T^*Q with initial data for the dynamical evolution [1]. Assume that Q is a smooth n-dimensional manifold and pick its local coordinates being

$\{q^1, q^2, q^3, \cdots, q^n\}$, then $\{dq^1, dq^2, dq^3, \cdots, dq^n\}$ forms a basis of T_q^*Q. By writing $\alpha \in T_q^*Q$ as $\alpha = p_i dq^i$ we get the local coordinates $(q^1, q^2, \cdots, q^n; p_1, p_2, \cdots, p_n)$ on T^*Q, which describes α as a point in a fibre of T^*Q. A natural symplectic structure on T^*Q is defined as:

$$\omega = -d\alpha \tag{1-2}$$

which in local coordinates is given by:

$$\omega = dq^i \wedge dp_i \quad ; \text{ (summation over } i \text{)}. \tag{1-3}$$

1.2.2 Observables and Hamiltonian Vector Fields

In physical applications, a symplectic manifold (S, ω) represents the phase space of a classical system. A smooth function $f : S \to \mathbb{R}$ then represents a classical observable. When the system has a configuration space Q and S it's cotangent bundle, f is simply a smooth function of position and momentum. A classical observable plays two roles. First, it is a measurable quantity that takes a definite value for any given state of the system. Secondly, it is an object that plays the role of a generator of one parameter group of canonical transformations. The two roles can be related geometrically as follows:

Given a classical observable $f \in C^\infty(S)$, the vector field X_f determined by:

$$X_f \lrcorner \omega = df \ , \tag{1-4}$$

generates a one parameter family of canonical transformations of S. If the integral curves of X_f are complete, X_f is defined globally and is called the Hamiltonian vector field generated by f. The set of all such vector fields on S is denoted by HVF(S).

1.2.3 Poisson Brackets

The Poisson bracket of two functions $f, g \in C^{\infty}(S)$ is the function $\{f, g\} \in C^{\infty}(S)$ defined by:

$$\{f, g\} = \omega(X_f, X_g) = X_f \rfloor X_g \rfloor \omega = X_f(g) \quad . \tag{1-5}$$

The Poisson bracket has the following properties:

1- it is antisymmetric in f and g since :

$$X_f(g) = X_f \rfloor dg = -X_g \rfloor df \quad ; \tag{1-6}$$

2- it is homomorphic with the Poisson algebra of f and g

$$\{X_f, X_g\} = X_{\{f,g\}} \; ; \qquad f, g \in C^{\infty}(S) \; ; \tag{1-7}$$

3- it satisfies the Jacobi identity (which in fact is equivalent to the condition that ω is closed).

Thus the Poisson bracket makes $C^{\infty}(S)$ into an infinite-dimensional real Lie algebra with the linear map $f \to X_f$ defining a Lie algebra homomorphism of $C^{\infty}(S)$ onto HVF(S), with the kernel defined as the set of constant functions on S, i.e.

$$HVF(S) = C^{\infty}(S)/\mathbf{R} \quad . \tag{1-8}$$

1.2.4 Fiber Bundle

A *bundle* [5-10] consists of a *base space B* (physically it is the ordinary space in which the particle moves), *a total space E* (the phase space), and a *map* π that projects every point in the total space onto a point in the base space. The set of all points in the total space that are mapped onto the same point x in the base space is called the *fiber* over x and is denoted by $\pi^{-1}(x)$. If for all x belongs to B, $\pi^{-1}(x)$ is homeomorphic to a common space F, this F is known as the fiber of the bundle. In a more mathematical language, a fiber bundle is a triplet (E, B, F) of differentiable manifolds and a surjection π: $E \to B$, together with a topological group G of homeomorphisms of F into itself called the structure group. To go into details let's clarify the following points.

1) Locally the bundle is a trivial bundle, i.e. for all $x \in B$ there is an open neighborhood U of x and a diffeomorphism

$$\begin{aligned} \varphi : U \times F &\to \pi^{-1}(U) \\ \pi(\varphi(x,f)) &= x \end{aligned} \quad ; \tag{1-9}$$

hence $\pi^{-1}(U)$ is diffeomorphic to $U \times F$ ($\pi^{-1}(U_i) \cong U_i \times F$) and the map φ is called the local trivialization.

2) An important notion which characterise the topological structure of the bundle E is the transitive function $g_{ij}(x)$ which describes the gluing of the patches $U_i \times F$, and is defined in the following way.

Let $\{U_i\}$ be an open covering of B such that:

$$\varphi_i^{-1} : \pi^{-1}(U_i) \to U_i \times F ; \tag{1-10}$$

If two coverings U_i and U_j have a nonempty intersection, we have two maps φ_i^{-1} and φ_j^{-1} on $U_i \cap U_j$ and two elements of F related to the same point $u \in \pi^{-1}(U_i \cap U_j)$ one assigned by φ_i^{-1} and the other by φ_j^{-1} i.e. as $\varphi_i^{-1}(u) = (x, f_i)$, $\varphi_j^{-1}(u) = (x, f_j)$ then there exists a gluing map,

$$g_{ij} := \varphi_j^{-1} \circ \varphi_i \ : U_i \cap U_j \to G \tag{1-11}$$

which relates f_i and f_j as $f_i = g_{ij}(x) f_j$ which is called the transition function. In order to glue the local pieces of the fiber bundle with consistency the following requirements should be satisfied:

$$\begin{aligned} g_{ii} &= e \ , \\ g_{ij} &= g_{ji}^{-1} \ , \\ g_{ij} g_{jk} &= g_{ik} \end{aligned} \tag{1-12}$$

A ***trivial fiber bundle*** is one for which the total space is the direct product of the base space by the fiber, i.e. $E = B \times F$.

A well known type of fiber bundle is the so called *principle G-bundle* which is a fiber bundle with structure group G as its fiber ($\mathrm{F} \cong \mathrm{G}$). For a principle G-bundle over B, denoted by $P(B,G)$ (or simply by P) with projection map π; the group G acts as a right transformation group on P and acts simply transitively on each fiber. The whole structure satisfies the condition of locality i.e. for each $x \in B$ there exists an open neighborhood U of x and an isomorphism $f : U \times G \to \pi^{-1}(U)$, such that for any point $u \in U$ and elements $g, g' \in G$ we have

$$\pi(f(u,g)) = u; \qquad f(u, gg') = f(u,g)g' \ ; \tag{1-13}$$

(right action of g' on the element $f(u,g) \in P$). The manifold $P_x = \pi^{-1}(x)$ is called the fiber over x and is isomorphic to the Lie group G. The tangent space to the fiber is isomorphic (as a vector space) to the Lie algebra $\mathfrak{g}$ of G.