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Numerical Solution of Volterra Integro-Differential Equations by Hybrid Block with Quadrature Rules Method

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ABSTRACT

In this paper, the implementation of one-step hybrid block method with quadrature rules will be proposed for solving linear and non-linear first order Volterra Integro-Differential Equations (VIDEs) of the second kind. VIDEs have important applications in many branches of sciences and engineering, such as analysing rhythmic biological data can be conducted by utilizing a curve fitting technique based on solutions of the VIDEs. The formulation of the hybrid block method is based on the Lagrange interpolation polynomial. The approximation of the integral part in the VIDEs will be estimated using the quadrature rules. The proposed hybrid block method of order five will compute the numerical solutions at two points simultaneously at each integration steps. The stability analysis such as order of the method, consistency, zero stable and stability region of the method are deliberated. The fixed step size is used to generate the results and the code is written in C language. Numerical simulations are presented to show the efficiency and accuracy of the hybrid method when compared to the Runge-Kutta of order four and five in terms of accuracy, total steps, and total function calls.

Keywords: Volterra integro-differential equations, hybrid block method, quadrature rules, rhythmic biological data.

1. Introduction

Nowadays, VIDEs become one of the important parts in solving real world problems especially in science and engineering fields. [Phipps and Jenner (1977)] showed that a method of analysing rhythmic biological data can be conducted by utilizing a curve fitting technique based on solutions of the Volterra Integro-Differential Equation (VIDE). The technique involves the use of difference equations together with the method of moments as the fitting criterion. It is conjectured that combinations of Volterra integro-differential equations can provide a more meaningful fit for such data rather than through harmonic analysis.

The standard form of first order VIDEs are given below:

$$\begin{aligned} y'(x) &= F(x) + \int_0^x K(x,s)y(s)ds, \\ y(x_0) &= y_0, 0 \le s \le x, \end{aligned} \tag{1}$$

where

$F(x)$= is a function,
$K(x,s)$=Kernel,
$y(s)$=solution of the function.

It is assume that F and K are uniformly continuous in all variables and that the following Lipschitz condition are satisfied:

$$\begin{aligned} |F(x,y_1,z) - F(x,y_2,z)| &\le L_1 |y_1 - y_2|, \\ |F(x,y,z_1) - F(x,y,z_2)| &\le L_2 |z_1 - z_2|, \\ |K(x,s,y_1) - K(x,s,y_2)| &\le L_3 |y_1 - y_2|. \end{aligned}$$

Under these conditions equation (1) has a unique solution in $0 \le x \le a$ [Linz (1969)].

Several researchers have discussed numerical method for solving first order of VIDEs such as Chebyshev wavelets method [Biazar and Ebrahimi (2011)], linear multistep method [Linz (1969)] and also analytical method for solving first and higher order VIDEs as in [Arikoglu and Ozkol (2005)]. [Alahmadi et al. (2018)] have investigated the qualitative analysis of solutions for the nonlinear VIDEs. In the paper, the combination of Lyapunov functional with Laplace transform will be used in order to obtain the boundedness and stability results of VIDEs. Later, [Tunç and Tunç (2018)] have proposed the second method of Lyapunov for solving nonlinear scalar and system of first order VIDEs. The results indicate that the technique used is suitable to solve VIDEs.

In general, VIDEs are very hard to solve using analytical methods compared to numerical methods. Regarding (1), there are two cases of the kernel which are $K(x, s)$ equals to 1 and $K(x, s)$ does not equals to 1. In [Filiz (2014a)] has implemented the combination of Runge-Kutta method with different quadrature rules for solving an integral part of VIDEs. The quadrature rules such as trapezoidal and Simpson's rule are applied in the papers. The concept was then extended by [Mohamed and Majid (2015)], [Mohamed and Majid (2016)] and [Majid and Mohamed (2019)] for solving VIDEs using block method.

2. Formulation

2.1 Derivation of the method

Derivation of the two-point block method with two off-step point is discussed in this section that based on numerical integration. Figure 1 shows the interval of $[a, b]$ is divided within a block where every block carrying two values.

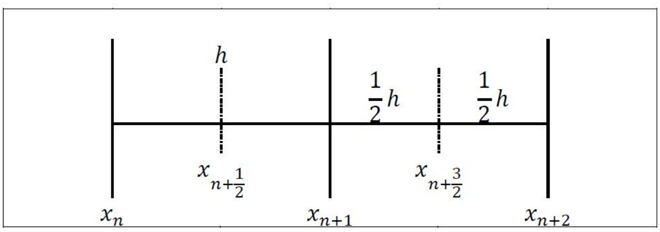

Figure 1: Two-point one-step block.

The derivation of the formulae for solving first order VIDEs subject to the initial value problem (IVP) will be discussed. The two values of y_{n+1} and y_{n+2} is approximated simultaneously at points x_{n+1} and x_{n+2}. The first and second point of the formulae will use the set of points $\left\{x_n, x_{n+\frac{1}{2}}, x_{n+1}, x_{n+\frac{3}{2}}, x_{n+2}\right\}$. The points of y_{n+1} and y_{n+2} is obtained by integrating $y' = F(x, y)$ at the interval of $[x_n, x_{n+1}]$ and $[x_{n+1}, x_{n+2}]$ as follows,

$$\begin{aligned}\int_{x_n}^{x_{n+1}} y^{'}(x)dx &= \int_{x_n}^{x_{n+1}} F(x,y)dx,\\ \int_{x_{n+1}}^{x_{n+2}} y^{'}(x)dx &= \int_{x_{n+1}}^{x_{n+2}} F(x,y)dx.\end{aligned} \tag{2}$$

The function, $F(x, y)$ in (2) will be replaced by Lagrange polynomial. The application of two-point hybrid block method with two off-step point (2PHBM) is based on predictor and corrector PE(CE) mode. To predict the initial values of y_{n+1} and y_{n+2}, the Euler method of order one is applied to calculate the points at each block. The corrector formulae is as the following:

$$\begin{aligned}y_{n+1} &= y_n + \frac{h}{180}\left(29F_n + 124F_{n+\frac{1}{2}} + 24F_{n+1} + 4F_{n+\frac{3}{2}} - F_{n+2}\right),\\ y_{n+2} &= y_{n+1} + \frac{h}{180}\left(-F_n + 4F_{n+\frac{1}{2}} + 24F_{n+1} + 124F_{n+\frac{3}{2}} + 29F_{n+2}\right).\end{aligned} \tag{3}$$

The off-step point formulae is as follows,

$$\begin{aligned}y_{n+\frac{1}{2}} &= y_n + \frac{h}{24}(F_n + 4F_{n+1} + F_{n+2}),\\ y_{n+\frac{3}{2}} &= y_n + \frac{h}{24}(3F_n + 9F_{n+1} + 2F_{n+2}).\end{aligned} \tag{4}$$

3. Stability Analysis

In this section, the order of the method will be discussed, and the stability region was plotted.

3.1 Order of the method

A linear difference operator L is defined by

$$L\left[y(x);h\right] = \sum_{i=0}^{k}[\alpha_j y(x+jh) - h\beta_j y'(x+jh) - h\beta v_j y'(x+jh)] \tag{5}$$

for any function $y(x)$. Then, the right hand side in (5) will be expanded as Taylor series and x gives

$$L\left[y(x);h\right] = C_0 y(x) + C_1 h y^1(x) + ... + C_q h^q y^q(x) + ..., \tag{6}$$

where the C_q are constants.

Definition 1: The difference operator (5) and the associated linear multistep method are said to be of order q if $C_0 = C_1 = ... = C_q = 0$ and $C_{q+1} \neq 0$ [Lambert (1973)].

Definition 2: The method is consistent if it possesses an order $p \geq 1$[Lambert (1973)].

The general formula for the order of the method is defined as follows:

$$\begin{aligned}
C_0 &= \sum_{j=0}^{k} \alpha_j, \\
C_1 &= \sum_{j=0}^{k} j\alpha_j - \sum_{j=0}^{k} \beta_j - \sum_{j=1}^{k} \beta_{vj}, \\
&. \\
&. \\
&. \\
C_q &= \frac{1}{q!}\left[\sum_{j=0}^{k} j^q \alpha_j - q\left(\sum_{j=0}^{k} j^{q-1}\beta_j + \sum_{j=1}^{k} v j^{q-1}\beta_{vj}\right)\right],
\end{aligned} \tag{7}$$

where $q = 2, 3, 4, ...$ The order of the method in (3) is determined by applying the formula in (7).

For $q = 0$,

$$C_0 = \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

For $q = 1$,

$$\begin{aligned}
C_1 = &\begin{bmatrix} -1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&- \left(\begin{bmatrix} \frac{29}{180} \\ -\frac{1}{180} \end{bmatrix} + \begin{bmatrix} \frac{24}{180} \\ \frac{24}{180} \end{bmatrix} + \begin{bmatrix} -\frac{1}{180} \\ \frac{29}{180} \end{bmatrix} + \begin{bmatrix} \frac{124}{180} \\ \frac{4}{180} \end{bmatrix} + \begin{bmatrix} \frac{4}{180} \\ \frac{124}{180} \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\end{aligned}$$

For $q = 2$,

$$C_2 = \frac{1}{2}\Big[\Big(\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 1 \end{bmatrix}\Big) - 2\Big(\begin{bmatrix} \frac{24}{180} \\ \frac{24}{180} \end{bmatrix} + 2\begin{bmatrix} -\frac{1}{180} \\ \frac{29}{180} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \frac{124}{180} \\ \frac{4}{180} \end{bmatrix} + \begin{bmatrix} \frac{4}{180} \\ \frac{124}{180} \end{bmatrix}\Big)\Big] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

For $q = 3$,

$$C_3 = \frac{1}{6}\Big[\Big(\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 8\begin{bmatrix} 0 \\ 1 \end{bmatrix}\Big) - 3\Big(\begin{bmatrix} \frac{24}{180} \\ \frac{24}{180} \end{bmatrix} + 4\begin{bmatrix} -\frac{1}{180} \\ \frac{29}{180} \end{bmatrix} + \frac{1}{4}\begin{bmatrix} \frac{124}{180} \\ \frac{4}{180} \end{bmatrix} + \begin{bmatrix} \frac{4}{180} \\ \frac{124}{180} \end{bmatrix}\Big)\Big] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

For $q = 4$,

$$C_4 = \frac{1}{24}\Big[\Big(\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 16\begin{bmatrix} 0 \\ 1 \end{bmatrix}\Big) - 4\Big(\begin{bmatrix} \frac{24}{180} \\ \frac{24}{180} \end{bmatrix} + 8\begin{bmatrix} -\frac{1}{180} \\ \frac{29}{180} \end{bmatrix} + \frac{1}{8}\begin{bmatrix} \frac{124}{180} \\ \frac{4}{180} \end{bmatrix} + \begin{bmatrix} \frac{4}{180} \\ \frac{124}{180} \end{bmatrix}\Big)\Big] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

For $q = 5$,

$$C_5 = \frac{1}{120}\Big[\Big(\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 32\begin{bmatrix} 0 \\ 1 \end{bmatrix}\Big) - 5\Big(\begin{bmatrix} \frac{24}{180} \\ \frac{24}{180} \end{bmatrix} + 16\begin{bmatrix} -\frac{1}{180} \\ \frac{29}{180} \end{bmatrix} + \frac{1}{16}\begin{bmatrix} \frac{124}{180} \\ \frac{4}{180} \end{bmatrix} + \begin{bmatrix} \frac{4}{180} \\ \frac{124}{180} \end{bmatrix}\Big)\Big] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

For $q = 6$,

$$C_6 = \frac{1}{240}\Big[\Big(\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 64\begin{bmatrix} 0 \\ 1 \end{bmatrix}\Big) - 6\Big(\begin{bmatrix} \frac{24}{180} \\ \frac{24}{180} \end{bmatrix} + 32\begin{bmatrix} -\frac{1}{180} \\ \frac{29}{180} \end{bmatrix} + \frac{1}{32}\begin{bmatrix} \frac{124}{180} \\ \frac{4}{180} \end{bmatrix} + \begin{bmatrix} \frac{4}{180} \\ \frac{124}{180} \end{bmatrix}\Big)\Big] = \begin{bmatrix} \frac{17}{3600} \\ \frac{467}{3600} \end{bmatrix},$$

where $C_6 \neq 0$. Here, we can conclude that the 2PHBM method is order five by referring to Definition 1 and the error constant for the method is $(\frac{17}{3600}, \frac{467}{3600})^T$. Since the order of the method is five, therefore it was confirmed that the method was consistent by referring to Definition 2. The corrector formulae are one order higher than the predictor formulae and the implementation is in PECE mode where P is predictor, E is the evaluation and C is the corrector.

Definition 3: A block method is to be zero stable if and only if providing the roots of $R_j, j = 1(1)k$ of the first characteristic polynomial, $\rho(R)$ specified as

$$\rho(R) = det\left[\sum_{j=0}^{k} A^i R^{(k-1)}\right] = 0, \tag{8}$$

satisfies with $|R_j| \leq 1$ and those roots with $|R_j| = 1$ [Fatunla (1995)].

Another way of representing formulae in (3) is of the form:

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} y_{n+1} \\ y_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} y_{n-1} \\ y_n \end{bmatrix} + h\begin{bmatrix} 0 & \frac{29}{180} \\ 0 & -\frac{1}{180} \end{bmatrix}\begin{bmatrix} F_{n-1} \\ F_n \end{bmatrix}$$

$$+h\begin{bmatrix} \frac{124}{180} & \frac{24}{180} \\ \frac{4}{180} & \frac{24}{180} \end{bmatrix}\begin{bmatrix} F_{n+\frac{1}{2}} \\ F_{n+1} \end{bmatrix} + h\begin{bmatrix} \frac{4}{180} & -\frac{1}{180} \\ \frac{124}{180} & \frac{29}{180} \end{bmatrix}\begin{bmatrix} F_{n+\frac{3}{2}} \\ F_{n+2} \end{bmatrix},$$

which is equaivalent to the difference equations:

$$A_0 Y_m - A_1 Y_{m-1} - h(B_0 F_m + B_1 F_{m-1} + B_2 F_{m-2}) = 0, \tag{9}$$

where

$$A_0 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$B_0 = \begin{bmatrix} \frac{4}{180} & -\frac{1}{180} \\ \frac{124}{180} & \frac{29}{180} \end{bmatrix}, B_1 = \begin{bmatrix} \frac{124}{180} & -\frac{24}{180} \\ \frac{4}{180} & \frac{24}{180} \end{bmatrix}, B_2 = \begin{bmatrix} 0 & \frac{29}{180} \\ 0 & -\frac{1}{180} \end{bmatrix},$$

and the matrix of $Y_m, Y_{m-1}, F_m, F_{m-1}$ and F_{m-2} are given below:

$$\begin{aligned} Y_m &= \begin{bmatrix} y_{n+1} \\ y_{n+2} \end{bmatrix}, Y_{m-1} = \begin{bmatrix} y_{n-1} \\ y_n \end{bmatrix}, \\ F_m &= \begin{bmatrix} F_{n+\frac{3}{2}} \\ F_{n+2} \end{bmatrix}, F_{m-1} = \begin{bmatrix} F_{n+\frac{1}{2}} \\ F_{n+1} \end{bmatrix}, F_{m-2} = \begin{bmatrix} F_{n-1} \\ F_n \end{bmatrix}. \end{aligned} \tag{10}$$

Regarding explanation in (5) and (6), it follows that:

$$\rho(R) = det\begin{bmatrix} RA^0 & -A^1 \end{bmatrix}, det = \left[\begin{pmatrix} R & 0 \\ -R & R \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right],$$

$$det = \begin{bmatrix} R & 1 \\ -R & R \end{bmatrix} = \begin{bmatrix} R(R-1) \end{bmatrix},$$

where $R(R-1) = 0$ and $R_1 = 0, R_2 = 1$. Therefore, regarding Definition 3 the 2PHBM is zero stable.

3.2 Stability region

In this part, the stability property of the 2PHBM with modified composite Boole's rule is considered. The test equation is given as:

$$y'(x) = \xi y(x) + \eta \int_0^x y(t)dt. \tag{11}$$

By setting $\xi = \lambda + \mu$ and $\eta = -\lambda\mu$ hence, employed the following alternative form of (11),

$$y'(x) = (\lambda + \mu)y(x) + (-\lambda\mu) \int_0^x y(t)dt, \tag{12}$$

and all solutions of (12) tend to zero as $x \to 0$ if and only if,

$$\lambda < 0,\ \mu < 0 \text{ when } \lambda, \text{ and } \mu \text{ are real,}$$
$$Re(\lambda) < 0, \text{ when } \lambda(\overline{\mu}) \text{ is complex.}$$

Then the stability polynomial of linear VIDEs [Brunner and Lambert (1974)] can be considered as:

$$\pi(r, h\xi, h^2\eta) = \tilde{\rho}(r)[\rho(r) - h\xi\sigma(r)] - h^2\eta\tilde{\sigma}(r)\sigma(r), \tag{13}$$

by letting $H_1 = h\xi$ and $H_2 = h^2\eta$, thus the stability polynomial of linear VIDE of the second kind will be:

$$\pi(r, H_1, H_2) = \tilde{\rho}(r)[\rho(r) - H_1\sigma(r)] - H_2\eta\tilde{\sigma}(r)\sigma(r). \tag{14}$$

Definition 4: The method is A-stable if and only if the region contains at the quarter plane $h\xi < 0, h^2\eta < 0$ [Brunner and Lambert (1974)].

In this part, we will need to determine the characteristics polynomial $\rho(r), \sigma(r), \tilde{\rho}(r)$ and $\tilde{\sigma}(r)$ as follows.

The point at y_{n+1} of corrector formula:

$$\begin{aligned} \rho(r) &= r^2 - 1, \\ \sigma(r) &= -\frac{1}{180}r^4 + \frac{4}{180}r^3 + \frac{24}{180}r^2 + \frac{124}{180}r + \frac{29}{180}. \end{aligned} \tag{15}$$

The point at y_{n+2} of corrector formula:

$$\begin{aligned} \rho(r) &= r^4 - r^2, \\ \sigma(r) &= \frac{29}{180}r^4 + \frac{124}{180}r^3 + \frac{24}{180}r^2 + \frac{4}{180}r - \frac{1}{180}. \end{aligned} \tag{16}$$

Boole's rule:

$$\begin{aligned} \tilde{\rho}(r) &= r^2 - 1, \\ \tilde{\sigma}(r) &= \frac{14}{45}r^4 + \frac{64}{45}r^3 + \frac{24}{45}r^2 + \frac{64}{45}r + \frac{14}{45}. \end{aligned} \tag{17}$$

The stability polynomial is obtained via combination of 2PHBM and modified composite Boole's rule. Figure 2 illustrates the region and based on Definition 4, the 2PHBM method is A-stable inside the shaded region. Thus, the region of absolute stability obtained is plotted below:

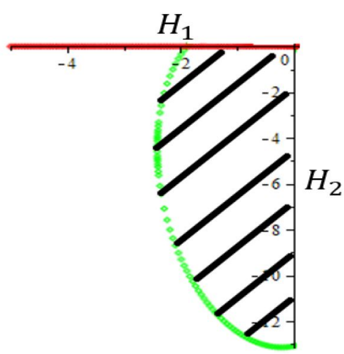

Figure 2: Stability region for 2PHBM method

4. Implementation

In this section, the implementation of 2PHBM and Boole's rule [Filiz (2014b)] to solve VIDE problems. Integrating equation (1) from x_n to x_{n+h} for $s =$

1,2,..., we have:

$$\begin{aligned} y(x_{n+h}) &= y(x_n) + \int_{x_n}^{x_{n+s}} F(x, y(x), z(x))dx, \\ z(x) &= \int_0^x K(x, s, y(s))ds. \end{aligned} \tag{18}$$

The implementation of corrector formulae for differential part is as the following:

$$\begin{aligned} y_{n+1} - y_n = \frac{h}{180}\big(&29F(x_n, y_n, z_n) + 124F(x_{n+\frac{1}{2}}, y_{n+\frac{1}{2}}, z_{n+\frac{1}{2}}) \\ &+ 24F(x_{n+1}, y_{n+1}, z_{n+1}) + 4F(x_{n+\frac{3}{2}}, y_{n+\frac{3}{2}}, z_{n+\frac{3}{2}}) \\ &- F(x_{n+2}, y_{n+2}, z_{n+2})\big), \\ y_{n+2} - y_{n+1} = \frac{h}{180}\big(&- F(x_n, y_n, z_n) + 4F(x_{n+\frac{1}{2}}, y_{n+\frac{1}{2}}, z_{n+\frac{1}{2}}) \\ &+ 24F(x_{n+1}, y_{n+1}, z_{n+1}) + 124F(x_{n+\frac{3}{2}}, y_{n+\frac{3}{2}}, z_{n+\frac{3}{2}}) \\ &+ 29F(x_{n+2}, y_{n+2}, z_{n+2})\big). \end{aligned} \tag{19}$$

Then, the integration part will be calculated using modified Boole's rule as follows:

$$z_{n+1} = z_n + \frac{h}{90}(7y_n + 32y_{n+\frac{1}{4}} + 12y_{n+\frac{1}{2}} + 32y_{n+\frac{3}{4}} + 7y_{n+1}),$$

where

$$\begin{aligned} z_{n+1} - z_n = \frac{h}{90}\big(&7K(x_{n+1}, x_n, y_n) + 32K(x_{n+1}, x_{n+\frac{1}{4}}, y_{n+\frac{1}{4}}) \\ &+ 12K(x_{n+1}, x_{n+\frac{1}{2}}, y_{n+\frac{1}{2}}) + 32K(x_{n+1}, x_{n+\frac{3}{4}}, y_{n+\frac{3}{4}}) \\ &+ 7K(x_{n+1}, x_{n+1}, y_{n+1})\big), \end{aligned} \tag{20}$$

and given $n = 0, 1$.

For the next block, values of z_{n+1} and z_{n+2} are computed by applying modified composite Boole's rule with interpolation schemes. Given $n = 2, 4, 6, \ldots$,it follows that:

$$
\begin{aligned}
z_{n+1} &= \frac{2h}{45}\sum_{i=0}^{n}\omega_i^s K(x_{n+1},x_i,y_i) + \frac{h}{90}\big(7K(x_{n+1},x_n,y_n) \\
&\quad + 32K(x_{n+1},x_{n+\frac{1}{4}},y_{n+\frac{1}{4}}) + 12K(x_{n+1},x_{n+\frac{1}{2}},y_{n+\frac{1}{2}}) \\
&\quad + 32K(x_{n+1},x_{n+\frac{3}{4}},y_{n+\frac{3}{4}}) + 7K(x_{n+1},x_{n+1},y_{n+1})\big), \\
z_{n+2} &= \frac{h}{90}\sum_{i=0}^{n+2}\omega_i^s K(x_{n+2},x_i,y_i),
\end{aligned} \tag{21}
$$

where ω_i^s are Boole's rule of weights $7, 32, 12, 32, ..., 32, 12, 32, 7$.

Here, $y_{n+\frac{1}{4}}, y_{n+\frac{1}{2}}$ and $y_{n+\frac{3}{4}}$ are unknown values and they can be formulated using Lagrange interpolating polynomial at points $\{x_n, x_{n+1}, x_{n+2}\}$. Hence we obtain:

$$
\begin{aligned}
y_{n+\frac{1}{4}} &= 3F_n + F_{n+1} + 6F_{n+2}, \\
y_{n+\frac{1}{2}} &= F_n + 4F_{n+1} + F_{n+2}, \\
y_{n+\frac{3}{4}} &= 6F_n + F_{n+1} + 3F_{n+2}.
\end{aligned} \tag{22}
$$

5. Algorithm

The following is the algorithm on the detail approach of our method.

Algorithm:

1. `INPUT: endpoint` a, b; `value of` N; `initial condition` $y(a) = \alpha, z(0) = 0$.
2. `OUTPUT: approximation` y `at` N, `values of` x.
3. `Set` $h = \frac{b-a}{N}; x_0 = a; y_0 = \alpha; z_0 = 0;$ `OUTPUT` (x_0, y_0, x_0).
4. `Set` $x + ih$.
5. `For` $i = 1;$ `for` $i = 2, .., (N/2)$, `do step 6-8.`
6. `Calculate for predictor formulae` y_{n+1}, z_{n+1} `and` y_{n+2}, z_{n+2}; `calculate for off-step point` $y_{n+\frac{1}{2}}, z_{n+\frac{1}{2}}$ `and` $y_{n+\frac{3}{2}}, z_{n+\frac{3}{2}}$.
7. `Compute the solution (corrector formulae) for` y_{n+1}, z_{n+1} `and` y_{n+2}, z_{n+2}.
8. `OUTPUT:` (x, y, z) `and calculate` $|y(x_0) - y(x)|$.
9. `STOP.`

6. Numerical Results

Four problems of VIDEs have been solved and the results will be compared with the existing method. The following notations are used in Tables 1-4:

h	Step size used
MAXE	Maximum error
TFC	Total function calls
TS	Total of step
Time (s)	Execution time in second
2PHBM	Two-point hybrid block method with two off-step point method
RK4	Runge-Kutta method of order four [Filiz (2013)]
RK5	Runge-Kutta method of order five [Hossain et al. (2017)]

Problem 1: The linear of VIDE

$$y'(x) = 1 + 2x - y(x) + \int_0^x x(1+2x)e^{t(x-t)}y(t)dt,$$
$$y(0) = 1, 0 \leq x \leq 1.$$

Exact solution: $y(x) = e^{x^2}$.
Source: Agbolade and Anake (2017).

Table 1: Comparison between RK4, RK5 and 2PHBM for Problem 1

h	METHOD	MAXE	TFC	TS	Time (s)
0.1	RK4	3.5549E-05	40	10	0.0888
	RK5	1.3355E-06	50	10	0.0915
	2PHBM	7.3365E-06	35	5	0.0695
0.01	RK4	4.3656E-06	400	100	0.1669
	RK5	7.9659E-07	500	100	0.1733
	2PHBM	1.2033E-07	350	50	0.1101
0.001	RK4	8.6655E-07	4000	1000	1.8667
	RK5	2.0366E-07	5000	1000	2.0114
	2PHBM	4.3365E-08	3500	500	1.6336

Problem 2: The nonlinear of VIDE

$$y'(x) = 2x - \frac{1}{2}sin(x^4) + \int_0^x x^2tcos(x^2y(t))dt,$$
$$y(0) = 0, 0 \leq x \leq 1.$$

Exact solution: $y(x) = x^2$.
Source: Majid and Mohamed (2019).

Table 2: Comparison between RK4, RK5 and 2PHBM for Problem 2

h	METHOD	MAXE	TFC	TS	Time (s)
0.1	RK4	4.3696E-04	40	10	0.0888
	RK5	7.8133E-05	50	10	0.1051
	2PHBM	4.2217E-05	35	5	0.0833
0.01	RK4	1.2558E-05	400	100	0.1297
	RK5	9.3345E-06	500	100	0.1995
	2PHBM	4.1200E-05	350	50	0.1351
0.001	RK4	4.9249E-06	4000	1000	1.9191
	RK5	4.0030E-06	5000	1000	2.1131
	2PHBM	1.3369E-07	3500	500	1.7995

Problem 3: The nonlinear of VIDE

$$y'(x) = 1 + y(x) - xe^{-x^2} + 2\int_0^x xte - y^2(t)dt,$$
$$y(0) = 1, 0 \leq x \leq 1.$$

Exact solution: $y(x) = e^x$.
Source: Makroglou (1982).

Table 3: Comparison between RK4, RK5 and 2PHBM for Problem 3

h	METHOD	MAXE	TFC	TS	Time (s)
0.1	RK4	8.5779E-04	40	10	0.0918
	RK5	3.4494E-04	50	10	0.0991
	2PHBM	4.6697E-05	35	5	0.0829
0.01	RK4	7.6695E-05	400	100	0.1447
	RK5	7.6655E-06	500	100	0.1911
	2PHBM	9.1200E-06	350	50	0.1031
0.001	RK4	2.6468E-05	4000	1000	1.9787
	RK5	1.0032E-06	5000	1000	2.0933
	2PHBM	3.2219E-07	3500	500	1.7787

Problem 4: The nonlinear of VIDE

$$y'(x) = 1 - xe^{-x^2} + y(x) + \int_0^x -2xte^{-y^2(t)}dt,$$
$$y(0) = 1, 0 \leq x \leq 1.$$

Exact solution: $y(x) = e^x$.
Source: Hashemi et al. (2018).

Table 4: Comparison between RK4, RK5 and 2PHBM for Problem 4

h	METHOD	MAXE	TFC	TS	Time (s)
0.1	RK4	4.6697E-04	40	10	0.0977
	RK5	4.8886E-05	50	10	0.1003
	2PHBM	4.3364E-05	35	5	0.0844
0.01	RK4	7.6655E-05	400	100	0.1299
	RK5	1.3795E-05	500	100	0.2011
	2PHBM	2.2213E-05	350	50	0.1231
0.001	RK4	1.6655E-05	4000	1000	2.0048
	RK5	3.3321E-06	5000	1000	2.1011
	2PHBM	1.1133E-07	3500	500	1.8000

The 2PHBM is compared with RK4 and RK5 in terms of maximum error, total function calls and execution times. Based on the numerical results obtained in Tables 1-4, the maximum error became smaller when the step size decrease and we conclude that 2PHBM is better in terms of accuracy compared to RK4 and RK5. The total steps taken and the number of function calls of 2PHBM are much lesser than RK4 and RK5. The proposed method needs a smaller number of function evaluations because for each step taken it produced two approximate solutions. While for the existing methods, each step taken will produce one approximate solution. In terms of execution time, it is clearly shown that the time taken for 2PHBM is less than RK4 and RK5 as in Tables 1-4. It is obvious that the 2PHBM is applicable in solving VIDEs as it gives advantage in decreasing the cost per step. In general, we can conclude that when h is reduced the approximate solutions are more accurate.

7. Conclusion

The main objective of this research is to solve the first order linear and non-linear VIDEs using 2PHBM with modified composite Boole's rule. Numerical results have shown that the 2PHBM of order five gave comparable results in term of accuracy but smaller number of total steps and function evaluations. The 2PHBM is faster in terms of timing compared to RK4 and RK5. The numerical computation reveal that the 2PHBM is more efficient and less costly than RK4 and RK5.

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