UNIVERSITI PUTRA MALAYSIA

KILLING TENSOR OF FIVE DIMENSIONAL MELVIN'S SPACETIME

GANESH A/L SUBRAMANIAM

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KILLING TENSOR OF FIVE DIMENSIONAL MELVIN'S SPACETIME

By

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Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia,
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DEDICATIONS

To the spirits of the nature that hold everything in order.

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Master of Science

KILLING TENSOR OF FIVE DIMENSIONAL MELVIN'S SPACETIME

By

GANESH A/L SUBRAMANIAM

February 2018

Chairman: Hishamuddin Zainuddin, PhD
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In this research, (3+1)D Melvin's spacetime was extended into (4+1)D Melvin's spacetime using hypersphere coordinates. Later, the method to compute Killing tensor as introduced by Garfinkle for (3+1)D spacetime was studied to compute Killing tensor in (4+1)D Melvin's spacetime. The method introduced by Garfinkle needs the existence of commuting Killing vectors or Killing vectors in a particular coordinate direction. It is not well understood that the method introduced by Garfinkle worked well for extended (4+1)D Melvin's metric because the fifth component of the metric canceled one of the commuting Killing vector in 3-dimension. Therefore, Killing tensors are obtained using the method introduced by Garfinkle and they are compared with the ones obtained by symmetrically multiplying the Killing vectors. For the later case, we found Killing vectors and Killing tensors for (4+1)D Melvin's spacetime. At the end it is concluded that the Killing tensor found for (4+1)D consist of products of some noncommuting Killing vector. Killing tensor found in this research might be applicable in separability of Hamilton-Jacobi and Klein-Gordon equation.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Master Sains

TENSOR ”KILLING” UNTUK RUANGMASA MELVIN BERMATRA LIMA

Oleh

GANESH A/L SUBRAMANIAM

Februari 2018

Pengerusi: Hishamuddin Zainuddin, PhD
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Dalam kajian ini, ruangmasa Melvin bermatra (3+1) diubahsuai kepada matra (4+1) dengan menggunakan koordinat-koordinat hipersfera. Selepas itu, kaedah yang diusulkan oleh Garfinkle untuk mencari tensor Killing dalam ruangmasa bermatra (3+1) dikaji untuk mencari tensor Killing dalam ruangmasa Melvin bermatra (4+1). Kaedah yang diusulkan oleh Garfinkle memerlukan kewujudan vektor Killing kalis tukar tertib. Kaedah yang diusulkan oleh Garfinkle tidak beberapa jelas bagi ruangmasa Melvin bermatra (4+1) kerana componen metrik kelima yang kita memperkenalkan dalam ruangmasa Melvin bermatra (3+1) menghapuskan salah satu vektor Killing dalam ruang Melvin bermatra tiga. Oleh sebab ini, tensor Killing dicari mengunakan kaedah yang diusulkan oleh Garfinkle dan lepas itu dibandingkan dengan tensor Killing yang ditemui melalui pendaraban vektor Killing secara simetri. Akhirnya, vektor Killing dan tensor Killing telah ditemui untuk ruangmasa Melvin bermatra (4+1). Kesimpulanya, tensor Killing mempunyai hasil darab vector Killing yang tidak kalis tukar tertib. Tensor Killing yang ditemui dalam kajian ini mungkin boleh digunakan untuk menyelesaikan persamaan Hamilton-Jacobi dan persamaan Klein-Gordon.

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This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Master of Science. The members of the Supervisory Committee were as follows:

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LIST OF ABBREVIATIONS

EFE	Einstein Field Equation
KV	Killing Vector
KT	Killing Tensor
w.r.t	with respect to

LIST OF SYMBOLS

$\mathbb{R}$	Euclidean space
M^n	n-dimensional Minkowski space or n-dimensional manifold
$\mathbf{g}, g_{ab}, h_{ab}$	Metric tensor
U_a, U_b	Open neighbourhood
$\cap$	Intersection
$\mathscr{F}$	Collection of smooth function from M^n to $\mathbb{R}$
ρ	Embedding map
x^a	Coordinate on the manifold where $a = 1, 2, 3, \cdots, n$
y^α	Coordinate on the hyperspace where $\alpha = 1, 2, 3, \cdots, (n-1)$
∂_x	Partial derivative with respect to x
$\mathrm{X}_{a;b}$ or $\nabla_b X_a$	Covariant derivative with respect to coordinate b
$\mathscr{L}_\xi\ g_{ab}$	Lie derivative of metric tensor g_{ab} with respect to vector field ξ

CHAPTER 1

INTRODUCTION

Nowadays, higher dimensional spacetime is one of the active field of research in theoretical physics. This higher dimensional spacetime is always related to the unification of four fundamental interactions of nature; weak and strong nuclear forces, electromagnetism and gravity. Even though grand unified theory (GUT) has unified electromagnetism, strong and weak nuclear forces but unifying gravity (weakest force in short range) is difficult. Some fields like string theory, M-theory and field theory emerge to achieve the goal of unification. Today, such a unification has achieved through string theory and M theory but it happened only in 10, 11 and 26 dimension. Long before an attempt was taken by Kaluza to unify electromagnetism with Einstein's general theory of relativity but it is only achievable by extending gravity to five-dimension. Since the cylindrical condition assumed by Kaluza is unnatural, it leads to the consideration of fifth-dimension as length like which was proposed by Klein in 1926. If one considers fifth-dimension as length like then extra dimension will be compactified (Overduin and Wesson, 1998; Piercey, 2008). This extra dimension can not be observed because it is either very small (Klein, 1926) or our observable universe is confined. Recently, in 1999 Randall and Sundrum proposed a model (Randall Sundrum model) in which extra dimensions can be larger (Randall and Sundrum, 1999a,b; Hamed et al., 2002). Apart from all this higher dimensional problems in fundamental physics, gravity act strange when explained through quantum mechanics. So far we do not understand quantum nature of gravity yet as general relativity is nonlinear and difficult to reconcile with quantum theory. Because of this, it is difficult to conduct experiment to understand quantum nature of gravity.

To further investigate quantum aspects of gravity, it is customary to consider gravitational objects or fields in curved spacetime. Since black hole is an astronomical object formed under strong gravity, quantum feature may appear important such as Hawking radiation (Hawking, 1975a,b). Other possible avenue of investigation is inclusion of electromagnetic sources in combination with gravity. The field equations in such situations are called Einstein-Maxwell equations. Melvin's (magnetic) spacetime (Melvin, 1964) or sometimes called magnetic universe is one of the interesting spacetime which is the cylindrically symmetric solution of Einstein-Maxwell vacuum spacetime. Magnetic fields are found in all galaxies, galaxy clusters and also present in smooth low density intergalactic medium. Furthermore, it is expected that primordial magnetic field is generated in the early universe (Naoz and Narayan, 2013). Furthemore, motion of particles are studied broadly in fourth and higher dimensional spacetime such as Kerr spacetime, Kerr-Newman spacetime, Euclidean Taub-NUT spacetime, Taub-NUT spacetime, Kerr-NUT-Ads spacetime, static spherically symmetric spacetime, stationary axisymmetric vacuum (SAV) spacetime and pp wave spacetime.

Killing vector field are said to be infinitesimal generator of isometry in a spacetime (Bianchi, 2001). For a n-dimensional maximally symmetric spacetime, there are $n(n+1)/2$ Killing vector fields in the spacetime. Killing vector fields are used to study symmetries of a spacetime. Studying about symmetries of a spacetime is important to solve Einstein field equation (EFE). Since Killing vector field have structures of Lie algebra, spacetime can also be studied algebraically. Killing tensors (sometimes together with Killing-Yano tensors) as a generalization of Killing vector fields is often called hidden symmetries. Killing tensors provide conserved quantities of geodesic motion for a particular spacetime or mechanical system (Sommers, 1973). Existence of Killing tensor enable one to solve equation of motion for a particular spacetime, a notable example is the Kerr spacetime which admits rank two Killing tensor (Carter, 1968; Walker and Penrose, 1970). Even though separation of variable in Hamilton-Jacobi equation and Klein-Gordon equation is possible due to the existence of Killing tensors, for this research we did not go into that direction of study. In this thesis, we will study Killing tensors for Melvin's magnetic universe in five dimensional spacetime.

1.1 Problem Statement

It is not easy to find Killing tensors and Killing vectors in a particular spacetime. For example for a four dimensional spacetime, one need to solve ten Killing equation with sixty four Christoffel symbols. One have to compute more Killing equations and Christoffel symbols for five and other higher dimensions. One can not skip this calculations to compute Killing vectors and after computing Killing vectors one can use Killing vectors to find Killing tensors. A method has been introduce by (Garfinkle and Glass, 2010) in which one can find Killing tensors without computing Christoffel symbols. The method they introduce need the existence of commuting Killing vectors and the Killing vectors should be hypersurface orthogonal. Our interest lie on computing Killing tensors for five dimensional Melvin's spacetime. The fifth component of the metric that we introduced canceled the commuting Killing vector in the other direction.

Recently in (Lim, 2015), motion of charged particle around magnetized or electrified black hole was studied without dealing with Killing tensors. Apart from motion of particles in spacetime, in (Paliathanasis, 2016) they use Killing tensor of minisuperspace to specify functional form of f(R) in f(R)-gravity of Friedmann-Lemaitre-Robertson-Walker spacetime. Furthermore, Killing tensors of minisuperspace are studied in generalized Brans-Dicke cosmology to determine the unspecified potential of scalar tensor gravity theory (Papagiannopoulos et al., 2017). Since Killing tensors are not much used to study in the Magnetic universe or Melvin spacetime, we are interested to initiate such a study of Killing tensors in Magnetic or Melvin universe.

1.2 Objectives

1. Construct Killing tensors for Melvin Universe from lower dimensions up to five dimensions using the method introduced by Garfinkle.

2. Compute Killing vectors for five dimensional Melvin's spacetime.

3. Construct Killing tensor for five dimensional Melvin's spacetime by symmetrically multiplying Killing vectors found in step two and add five dimensional Melvin's metric into the Killing tensor.

4. Finally, compare the Killing tensor found by symmetrically multiplying the Killing vector and the one obtained by using the method introduced by Garfinkle.

1.3 Organization of present work

This thesis contains five chapters. Chapter 1 introduces our work, problems in the field and our objective of studies. In Chapter 2, we review relevant topic in our studies and in Chapter 3, we discuss about the mathematics needed to understand the thesis. Chapter 4 contains results and discussion while in chapter 5 we conclude our findings and suggest some future work.

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