UNIVERSITI PUTRA MALAYSIA

RUNGE-KUTTA TYPE DIRECT METHODS FOR SOLVING SECOND AND THIRD ORDER BOUNDARY VALUE PROBLEMS

ATHRAA ABDULSALAM JASIM AL-ITHAWI

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By

ATHRAA ABDULSALAM JASIM AL-ITHAWI

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfillment of the Requirements for the Degree of Master of Science

May 2019

DEDICATIONS

To
My beloved and supportive parents,
Mr. Abdulsalam Jasim and Mrs. Jameelah Mohammed,
my loyal brother and sisters,
my little prince Adam.

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfillment of the requirement for the degree of Master of Science

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May 2019

Chairman: Associate Professor Norazak Senu, PhD
Faculty: Science

In this thesis, methods for solving higher-order two-point boundary value problems (BVPs) directly are developed. These methods are known as one- and two-step explicit Runge-Kutta type methods. Conventionally, higher-order BVPs are solved by converting them to a system of first-order BVPs. However, it is more efficient in terms of accuracy, the number of function evaluations as well as computational time, if these problems can be solved directly by using the proposed methods with constant step length via shooting technique.

In the first part of the thesis, one-step Runge-Kutta type methods are constructed to solve second- and third-order BVPs. An exponentially-fitted technique is implemented in four stages fourth-order Runge-Kutta-Nyström (EFMRKN4) for solving special second-order BVPs which possesses an exponential solution. Meanwhile, four-stage fourth-order general Runge-Kutta-Nyström (RKNG4) method is constructed for solving general second-order BVPs. Thereafter, two-stage third-order and three-stage fourth-order explicit Runge-Kutta type (RKT2s3) and (RKT3s4) methods are constructed respectively for solving special third-order BVPs. Besides, two-stage third-order exponentially-fitted modified Runge-Kutta type (EFMRKT2s3) method is derived in order to improve the efficiency of RKT2s3 method. The Local Truncation Error (LTE) of the fitted methods is computed, the absolute stability of the EFMRKN method is discussed. The numerical results obtained show that the developed methods are more efficient in terms of accuracy and number of function evaluations in comparison with the existing methods in the literature for the same order.

In the second part of the thesis, two-step Runge-Kutta-Nyström (TSRKN) method is

derived for the direct solution of special second-order BVPs. The two-step method has an advantage that it can estimate the solution with fewer function evaluations compared to the one-step method. The order conditions are provided and three stages fourth-order two-step Runge-Kutta-Nyström (TSRKN3s4) method is derived. The stability of TSRKN method is analyzed and the numerical results show a clear advantage of the TSRKN method as compared with the existing methods in terms of number of function evaluations per step and time.

In conclusion, the developed methods are able to solve the second- and special third-order BVPs directly.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Master Sains

KAEDAH JENIS RUNGE-KUTTA SECARA LANGSUNG BAGI MENYELESAIKAN MASALAH NILAI SEMPADAN PERINGKAT DUA DAN TIGA

Oleh

ATHRAA ABDULSALAM JASIM AL-ITHAWI

Mei 2019

Pengerusi: Profesor Madya Norazak Senu, PhD
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Di dalam tesis ini, kaedah yang dapat menyelesaikan masalah nilai sempadan (MNS) dua titik peringkat tinggi secara langsung dibangunkan. Kaedah ini dikenali sebagai kaedah jenis Runge-Kutta tak tersirat satu- dan dua-langkah. Secara konvensional, MNS peringkat tinggi diselesaikan dengan menukarkannya kepada sistem MNS peringkat pertama. Walau bagaimanapun, ia lebih cekap dari segi kejituan, bilangan fungsi penilaian serta masa pengiraan, jika ia dapat diselesaikan secara langsung dengan menggunakan kaedah yang dicadangkan dengan panjang langkah tetap melalui teknik penembakan.

Di dalam bahagian pertama tesis ini, kaedah jenis Runge-Kutta satu langkah dibentuk untuk menyelesaikan MNS peringkat kedua dan ketiga. Teknik suai secara eksponen Runge-Kutta-Nyström dilaksanakan dalam tahap-empat peringkat-keempat (EFMRKN4) untuk menyelesaikan MNS khas peringkat kedua yang mempunyai penyelesaian berbentuk eksponen. Sementara itu, kaedah Runge-Kutta-Nyström umum tahap-empat peringkat-keempat (RKNG4) dibentuk untuk menyelesaikan MNS umum peringkat kedua. Selepas itu, kaedah tak tersirat Runge-Kutta tahap-dua peringkat-ketiga (RKT2s3) dan tahap-tiga peringkat-keempat (RKT3s4) masing-masing dibentuk untuk menyelesaikan MNS khas peringkat ketiga. Di samping itu, kaedah Runge-Kutta tahap-dua peringkat-ketiga di ubah suai secara eksponen yang diubah suai (EFMRKT2s3) diterbitkan untuk meningkatkan kecekapan kaedah RKT2s3. Ralat Pangkasan Tempatan (RPT) bagi setiap kaedah dikira, selang kestabilan mutlak bagi kaedah EFMRKN turut dibincangkan. Keputusan berangka yang diperoleh menunjukkan bahawa kaedah yang dibangunkan lebih cekap dari segi kejituan dan bilangan fungsi penilaian berbanding dengan kaedah sedia ada di dalam kajian-kajian

lepas bagi peringkat yang sama.

Dalam bahagian kedua tesis ini, kaedah dua langkah Runge-Kutta-Nyström (TSRKN) diperolehi untuk penyelesaian langsung MNS khas peringkat kedua. Kaedah dua-langkah mempunyai kelebihan yang dapat menganggarkan penyelesaian dengan bilangan fungsi penilaian yang kurang berbanding dengan kaedah satu-langkah. Syarat-syarat peringkat disertakan dan kaedah Runge-Kutta-Nystrom tahap-tiga peringkat-keempat dua-langkah (TSRKN3s4) diterbitkan. Kestabilan kaedah TSRKN dianalisis dan keputusan berangka menunjukkan kelebihan yang jelas pada kaedah TSRKN berbanding kaedah sedia ada dari segi bilangan fungsi penilaian untuk setiap langkah dan masa.

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This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Master of Science. The members of the Supervisory Committee were as follows:

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LIST OF ABBREVIATIONS

ODEs	Ordinary Differential Equations
BVPs	Boundary Value Problems
h	Step size
N	Subinterval
LTE	Local Truncation Error
MAXE	Maximum Error
F.N	The number of function evaluations
RK	Runge-Kutta method
RKN	Runge-Kutta-Nyström method
EFMRKN	Exponentially-fitted Runge-Kutta-Nyström method
EFMRKN4	The new four-stage fourth-order EFMRKN method
RKN4	The four-stage fourth-order RKN method of Dormand et al. (1987)
EFRKN4F	The four-stage fourth-order EFRKN method of Franco (2004)
EFRK4S	The four-stage fourth-order EFRK method of Simos (2000)
EFRK4F	The five-stage fourth-order EFRK method of Franco (2002)
RKNG4s4	The new four-stage fourth-order RKNG
RK4s4M	The four-stage fourth-order RK method given in Dormand (1996)
RK5s4M	The five-stage fourth-order RK method given in Lambert (1991)
RK6s4E	The six-stage fourth-order RK method given in Lambert (1991)
Na	The adjoint operator method given in Na (1980)
RKT2s3	The new two-stage third-order RKT method
RK3Heun	The three-stage third-order RK method given in Butcher (2008)
RK3Kutta	The three-stage third-order RK method given in Butcher (2008)
QB-Saini	Quartic B-spline method used by Saini and Mishra (2015)
QB-Akram	Quartic B-spline method used by Akram (2012)
EFMRKT2s3	The new two-stage third-order EFMRKT method
RKT3s4	The new three-stage fourth-order RKT method
RKD4M	The three-stage fourth-order RKT method of Mechee et al. (2014)
RK5s4Z	The five-stage fourth-order RK method given in Dormand (1996)
TSRKN3s4	The new three-stage fourth-order two-step RKN method
RKN3s4H	The three-stage fourth-order RKN method given in Hairer et al. (2010)
RKN3s4G	The three-stage fourth-order RKN method given in Garcıa et al. (2002)
RK4s4	The four-stage fourth-order RK method given in Lambert (1991)

CHAPTER 1

INTRODUCTION

Differential Equations are considerd the fundamental tools in modeling various problems which are stand out dramatically in the applications of physics and chemistry, and mathematical models of biological, social and economic processes that involve the change of some variables with respect to another. Most of these problems can be classified into two types of equations, Ordinary Differential Equations (ODEs) and Partial Differential Equations (PDEs) based on the number of independent variables in differential equations.

For several years, differential equations have been studied analytically and numerically. Sometimes in common real-life situations, the differential equation that models the problem is complicated or impossible to solve exactly which makes recourse to numerical methods is the only choice. The approximate numerical solutions of these differential equations are important for us to understand the behavior of their solutions.

Early work on the numerical solutions of differential equations appeared at the end of the eighteenth century, through the research paper of Bashforth and Adams in 1883 and the research paper of Runge in 1895. They have offered the initial ideas that lead to developing the modern software on numerical methods (Butcher (2001)). Since then, ideas with the proper techniques have been proposed for solving ODEs by several authors.

1.1 Ordinary Differential Equation

Ordinary Differential Equation is an equation that includes an unknown function with an independent variable and one or more of its derivatives.
Consider the function f of t, y, and the mth derivative of y, consequently, the following equation

$$f(t,y,...,y^{(m)}) = 0, \quad m = 1,2,3,\ldots \tag{1.1}$$

Is called mth-order ODE. In (1.1), the quantity being differentiated, y is named as the dependent variable, while the quantity with respect to which y is differentiated, t is named as an independent variable. ODEs can be categorized into Initial Value Problems (IVPs) and Boundary Value Problems (BVPs).

Initial value problem is an ordinary differential equation whose boundary conditions are specified at a single point. A boundary value problem differs from an initial value problem in that the boundary conditions are specified at more than one point and in that solutions of the differential equation over an interval, satisfying the boundary conditions at the endpoints, are required.

1.1.1 Boundary Value Problems

Consider the boundary value problem:

$$f(t, Y'(t)) = 0, \tag{1.2}$$

with the boundary condition:

$$g(Y'(t_0), Y'(t_1), \ldots, Y'(t_k)) = 0, \tag{1.3}$$

where

$$\begin{aligned} t &\in \Re, \quad y(t) \in \Re^m, \\ Y'(t) &= (y(t), y'(t), \ldots, y^{(n-1)}(t), y^{(n)}(t)) \in \Re^{m\times(n+1)}, \\ f &: \Re^{m\times(n+1)+1} \to \Re^m, \\ g &: \Re^{m\times(k+1)} \to \Re^{m\times n}. \end{aligned}$$

If $k = 0$, the problem is an IVP, and is a two-point BVP if $k = 1$, and a multipoint boundary value problem if $k > 1$. For the two-point boundary value problem, the condition (1.3) of the type $g(Y'(t_0)) = 0$, $h(Y'(t_\eta)) = 0$ is called a separated boundary condition, and a non-separate type if the conditions are not of a separate type (see Chompuvised and Dhamacharoen (2011); Dhamacharoen and Chompuvised (2013)).

In this thesis, we are interested in studying the two-point boundary value problems involving the second and third-order differential equation together with separated boundary conditions.

1.1.2 Boundary Value Problem for Second-Order ODE

This research deals with boundary value problems of the second-order in both their special and general form

$$y'' = f(t, y), \tag{1.4}$$

$$y'' = f(t, y, y'), \quad \text{for} \quad a \le t \le b, \tag{1.5}$$

with boundary conditions:
(a) Type I

$$y(a) = \alpha, \quad y(b) = \beta_1. \tag{1.6}$$

(b) Type II

$$y(a) = \alpha, \quad y'(b) = \beta_2. \tag{1.7}$$

where a, b, α, β_1, β_2 are constants.

1.1.3 Boundary Value Problem for Third-Order ODE

A special third-order boundary value problem can be defined as follows:

$$y''' = f(t, y), \quad \text{for} \quad a \leq t \leq b, \tag{1.8}$$

with boundary conditions:
(a) Type I

$$y(a) = \alpha, \quad y'(a) = \beta, \quad y(b) = \beta_1. \tag{1.9}$$

(b) Type II

$$y(a) = \alpha, \quad y'(a) = \beta, \quad y'(b) = \beta_2. \tag{1.10}$$

(c) Type III

$$y(a) = \alpha, \quad y''(a) = \beta_3, \quad y(b) = \beta_1. \tag{1.11}$$

where a, b, α, β, β_1, β_2, and β_3 are constants.

The following standard theorems give the general conditions that assert the existence and uniqueness of the solution of the initial and boundary value problem. In this research, we supposed that the hypotheses of these theorems are fulfilled.

1.1.4 Existence and Uniqueness of Solution

Definition 1.1 (see Burden and Faires (2010))
A function $f(t, y)$ is said to satisfy a Lipschitz condition in the variable y on a set $D \subset \mathbb{R}^2$ if a constant $L > 0$ exists with

$$| f(t, y_1) - f(t, y_2) | \leq L \mid y_1 - y_2 \mid,$$

whenever (t, y_1) and (t, y_2) are in D. The constant L is called a Lipschitz constant for f.

Theorem 1.1 (Existence and Uniqueness of Solution of IVP)
Let $f(t, y)$ be defined and continuous for all points (t, y) in the domain D defined by $t \in [a, b]$, $y \in (-\infty, \infty)$, a and b are finite, and that $f(t, y)$ satisfies Lipschitz condition. Then if $\xi \in R$ is any number, there exists a unique solution $y(t)$ of the IVP, where $y(t)$

is continuous and differentiable for all $(t,y) \in D$.

Theorem 1.2 (Existence and Uniqueness of Solution of BVP)
Suppose the function f in the BVPs is continuous on the set,

$$D = \{(t,y,y') \mid for\, a \leq t \leq b,\, with\, -\infty \leq y \leq \infty and\, -\infty \leq y' \leq \infty\},$$

and that $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial y'}$ are also continuous on D, and suppose that f satisfied the Lipschitz condition on D

$$| f(t,y_1,y') - f(t,y_2,y') | \leq L \mid y_1 - y_2 |,$$
$$| f(t,y,y_1') - f(t,y,y_2') | \leq L \mid y_1' - y_2' |,$$

for all points $(t,y_i,y'),(t,y,y_i')$, $i = 1,2$ in the region D. The BVPs having a unique solution solution when fulfill the following properties:
(i) $\frac{\partial f(t,y,y')}{\partial y} > 0$ for all $(t,y,y') \in D$, and
(ii) a constant M exists, with $\mid \frac{\partial f(t,y,y')}{\partial y} \mid \leq M$, for all $(t,y,y') \in D$.

then the boundary value problems have unique solutions.
For the proof for Theorem 1.1 and Theorem 1.2, see Henrici (1962).
The differential equations

$$y'' = f(t,y,y'),$$
$$y''' = f(t,y,y',y''),$$

are linear problems when functions $g(t)$, $p(t)$, $q(t)$, and $r(t)$ exist with

$$f(t,y,y') = p(t)y' + q(t)y + r(t),$$
$$f(t,y,y',y'') = g(t)y'' + p(t)y' + q(t)y + r(t).$$

Corollary 1.1 (Linear Boundary Value Problems)
Suppose the linear BVPs

$$y'' = p(t)y' + q(t)y + r(t),$$
$$y''' = g(t)y'' + p(t)y' + q(t)y + r(t).$$

for $a \leq t \leq b$ satisfy
(i) $g(t)$, $p(t)$, $q(t)$, and $r(t)$ are continuous on $[a,b]$,
(ii) $q(t) > 0$ on $[a,b]$.

Then the boundary value problem has a unique solution. (see Burden and Faires (2010))

1.2 Numerical Methods for Solving ODE

The approximate solution of BVPs in this thesis relies heavily on the approximate numerical integration of IVPs. The numerical procedures for the solution of the IVP can be classified into two major groups: one-step methods and multi-step methods. Consider the *m*th-order initial value problem

$$y^{(m)} = f(t, y, y', y'', \ldots, y^{(m-1)}),$$

with given initial conditions

$$y^{(\nu)}(a) = \eta_i, \quad 0 < \nu < m-1$$

where $y^{(m)} = \frac{d^m y}{dt^m}$ and divide the interval $[a,b]$ over which the independent variable t is divided into N subintervals. The mesh or step size h is given by $h = \frac{(b-a)}{N}$ where $t_{i+1} = t_i + h$. Solving an IVP means to find approximate values of the dependent variable y and its derivatives $y^{(\nu)}$ at the step size points h of the interval $[a,b]$ on which the solution is sought. In one-step methods, the approximation of the solution is computed using the information of only one previous point. In other words, $y_{i+1}^{(\nu)}$ can be calculated with only the knowledge of $y_i^{(\nu)}$. Otherwise, for multi-step methods, the approximation of the solution is computed using the information of n previous points. Both one-step and multi-step methods have their advantages and disadvantages.
Examples of the one-step are

- Taylor's method
- Runge-Kutta (RK) method

The numerical algorithm of the RK method is considered the most widely used scheme, due to its low truncation error (see Na (1980)).

In this thesis, we are concerned with Runge-Kutta method via shooting technique for solving two-point linear BVPs for second and third-order ODE.

1.2.1 Shooting Technique

Shooting technique is used to convert the BVP to IVP The idea of shooting technique is to obtain the missing initial value until the boundary condition at the other end converges to its correct value.
Suppose we want to solve a BVP

$$y'' = f(t, y), \quad a < t < b, \tag{1.12}$$
$$y(a) = \alpha, \quad y(b) = \beta. \tag{1.13}$$

The BVP (1.12) will turn into an IVP by replacing the boundary condition at $t = b$ with the condition

$$y'(a) = \theta_1, \tag{1.14}$$

where θ_1 is any number. Then the resulting IVP can be solved by any method used to solve IVP, and obtain the value of its solution $y(b)$ at $t = b$. If $y(b) = \beta$, then the BVP have been solved. Most likely, after the first attempt, $y(b) \neq \beta$. Then we should choose another value for θ_1 and try again. There is actually a strategy of how the values of θ_1 need to be chosen. This strategy is uncomplicated for linear BVPs, so this is the case we consider in this thesis.

1.2.2 Runge-Kutta Method

The general μ-stage Runge-Kutta method for solving first-order ODEs can be defined as follows:

$$y_{n+1} = y_n + h\sum_{i=1}^{\mu} b_i \kappa_i, \tag{1.15}$$

where

$$\kappa_1 = f(t_n, y_n), \tag{1.16}$$

$$\kappa_i = f(t_n + c_i h, y_n + h\sum_{j=1}^{\mu} a_{ij}\kappa_j), \quad i = 1, 2, \ldots, \mu, \tag{1.17}$$

and the following row-sum assumption holds

$$c_i = \sum_{j=1}^{\mu} a_{ij}, \quad i = 1, 2, \ldots, \mu, \tag{1.18}$$

with the idea of Butcher it became customary to symbolize method (1.15)-(1.17) by the tableau (see Table 1.1)

Table 1.1: Butcher tableau for μ-stage RK method.

$$\begin{array}{c|ccc} c_1 & a_{11} & \cdots & a_{1\mu} \\ \vdots & \vdots & & \vdots \\ c_\mu & a_{\mu 1} & \cdots & a_{\mu\mu} \\ \hline & b_1 & \cdots & b_\mu \end{array}$$

The μ-dimension vectors b and c and the $\mu \times \mu$ matrix A can be defined as follows

$$\begin{aligned} c &= [c_1, c_2, \ldots, c_\mu]^T, \\ b &= [b_1, b_2, \ldots, b_\mu]^T, \\ A &= [a_{ij}]. \end{aligned}$$

The RK method is said to be explicit if $a_{ij} = 0$ for $i \leq j$, $i = 1, 2, \ldots, \mu$, and implicit otherwise.

1.2.3 Runge-Kutta-Nyström (RKN) Method

RKN method is introduced by E.J.Nyström in 1925 to solve the special and general second-order ODEs of the form (1.4) and (1.5).
The general μ-stage explicit RKN method for solving special second-order ODE can be written as follows:

$$y_{n+1} = y_n + hy'_n + h^2 \sum_{i=1}^{\mu} b_i f(t_n + c_i h, Y_i), \tag{1.19}$$

$$y'_{n+1} = y'_n + h \sum_{i=1}^{\mu} \hat{b}_i f(t_n + c_i h, Y_i), \tag{1.20}$$

where

$$Y_i = y_n + c_i h y'_n + h^2 \sum_{j=1}^{\mu} a_{ij} f(t_n + c_i h, Y_j), \quad i = 2, 3, \ldots, \mu, \tag{1.21}$$

or

$$y_{n+1} = y_n + hy'_n + h^2 \sum_{i=1}^{\mu} b_i \kappa_i, \tag{1.22}$$

$$y'_{n+1} = y'_n + h \sum_{i=1}^{\mu} \hat{b}_i \kappa_i, \tag{1.23}$$

where

$$\kappa_1 = f(t_n, y_n), \tag{1.24}$$

$$\kappa_i = f(t_n + c_i h, y_n + c_i h y'_n + h^2 \sum_{j=1}^{\mu} a_{ij} \kappa_j), \quad i = 2, 3, \ldots, \mu, \tag{1.25}$$

where c_i, a_{ij}, b_i, and $\hat{b}_i$, for $i = 1, 2, \ldots, \mu$ and $j = 1, 2, \ldots, \mu$ are the parameters of the RKN method and they supposed to be real. The μ-dimension vectors c, b_i, and $\hat{b}_i$ and the $\mu \times \mu$ matrix A can be expressed as follows

$$\begin{aligned} c &= [c_1, c_2, \ldots, c_\mu]^T, \\ b &= [b_1, b_2, \ldots, b_\mu]^T, \\ \hat{b} &= [\hat{b}_1, \hat{b}_2, \ldots, \hat{b}_\mu]^T, \\ A &= [a_{ij}]. \end{aligned}$$

and the following Nyström row condition holds

$$\frac{1}{2} c_i^2 = \sum_{j=1}^{\mu} a_{ij}, \quad i = 1, 2, \ldots, \mu, \tag{1.26}$$

RKN method (1.19)-(1.21) can be written in Butcher tableau as represented in Table 1.2

Table 1.2: Butcher tableau for μ-stage RKN method.

$$\begin{array}{c|ccc} c_1 & a_{11} & \ldots & a_{1\mu} \\ \vdots & \vdots & & \vdots \\ c_\mu & a_{\mu 1} & \ldots & a_{\mu\mu} \\ \hline & b_1 & \ldots & b_\mu \\ & \hat{b}_1 & \ldots & \hat{b}_\mu \end{array}$$

while the general μ-stage general Runge-Kutta-Nyström (RKNG) method for solving general second-order ODE can be expressed as follows:

$$y_{n+1} = y_n + h y'_n + h^2 \sum_{i=1}^{\mu} b_i \kappa_i, \tag{1.27}$$

$$y'_{n+1} = y'_n + h \sum_{i=1}^{\mu} \hat{b}_i \kappa_i, \tag{1.28}$$

$$\kappa_1 = f(t_n, y_n, y'_n), \tag{1.29}$$

$$\kappa_i = f(t_n + c_i h, y_n + c_i h y'_n + h^2 \sum_{j=1}^{\mu} a_{ij} \kappa_j, y'_n + h \sum_{j=1}^{\mu} \hat{a}_{ij} \kappa_j), \tag{1.30}$$

where

$$a_{ij} = \sum_{k=1}^{\mu} \hat{a}_{ik} \hat{a}_{kj}, \quad b_i = \sum_{j=1}^{\mu} \hat{b}_j \hat{a}_{ji}, \quad i = 2, 3, \ldots, \mu.$$

RKNG method (1.27)-(1.30) can be represented in Butcher tableau as illustrated in Table 1.3

Table 1.3: Butcher tableau for μ-stage RKNG method.

$$\begin{array}{c|ccc|ccc} c_1 & a_{11} & \dots & a_{1\mu} & \hat{a}_{11} & \dots & \hat{a}_{1\mu} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ c_\mu & a_{\mu 1} & \dots & a_{\mu\mu} & \hat{a}_{\mu 1} & \dots & \hat{a}_{\mu\mu} \\ \hline & b_1 & \dots & b_\mu & \hat{b}_1 & \dots & \hat{b}_\mu \end{array}$$

1.2.4 Runge-Kutta Type (RKT) Method

Mechee et al. (2013) has been derived an explicit RKT method for the solution of special third-order ODE (1.8). The general μ-stage of the explicit RKT method can be defined as follows:

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2}y''_n + h^3\sum_{i=1}^{\mu} b_i\kappa_i, \tag{1.31}$$

$$y'_{n+1} = y'_n + hy''_n + h^2\sum_{i=1}^{\mu} \hat{b}_i\kappa_i, \tag{1.32}$$

$$y''_{n+1} = y''_n + h\sum_{i=1}^{\mu} \hat{\hat{b}}_i\kappa_i, \tag{1.33}$$

where

$$\kappa_1 = f(t_n, y_n), \tag{1.34}$$

$$\kappa_i = f(t_n + c_ih, y_n + c_ihy'_n + \frac{h^2}{2}c_i^2y''_n + h^3\sum_{j=1}^{\mu} a_{ij}\kappa_j), \quad i = 2,3,\dots,\mu. \tag{1.35}$$

The μ-dimension vectors c, b_i, $\hat{b}_i$, and $\hat{\hat{b}}_i$ and the $\mu \times \mu$ matrix A can be expressed as follows

$$\begin{aligned} c &= [c_1, c_2, \dots, c_\mu]^T, \\ b &= [b_1, b_2, \dots, b_\mu]^T, \\ \hat{b} &= [\hat{b}_1, \hat{b}_2, \dots, \hat{b}_\mu]^T, \\ \hat{\hat{b}} &= [\hat{\hat{b}}_1, \hat{\hat{b}}_2, \dots, \hat{\hat{b}}_\mu]^T, \\ A &= [a_{ij}]. \end{aligned}$$

RKT method (1.31)-(1.35) can be written in Butcher tableau as follows: (see Table 1.4)

Table 1.4: Butcher tableau for μ-stage RKT method.

$$
\begin{array}{c|ccc}
c_1 & a_{11} & \cdots & a_{1\mu} \\
\vdots & \vdots & & \vdots \\
c_\mu & a_{\mu 1} & \cdots & a_{\mu\mu} \\
\hline
 & b_1 & \cdots & b_\mu \\
 & \hat{b}_1 & \cdots & \hat{b}_\mu \\
 & \hat{\hat{b}}_1 & \cdots & \hat{\hat{b}}_\mu
\end{array}
$$

1.2.5 Explicit Modified Runge-Kutta-Nyström (MRKN) Method

The general μ-stage explicit MRKN method for the special second-order differential equation (1.4) can be expressed as follows:

$$
y_{n+1} = y_n + hy'_n + h^2 \sum_{i=1}^{\mu} b_i f(t_n + c_i h, Y_i), \tag{1.36}
$$

$$
y'_{n+1} = y'_n + h \sum_{i=1}^{\mu} \hat{b}_i f(t_n + c_i h, Y_i), \tag{1.37}
$$

where

$$
Y_1 = y_n + c_1 h \gamma_1 y'_n, \tag{1.38}
$$

$$
Y_i = y_n + c_i h \gamma_i y'_n + h^2 \sum_{j=1}^{i-1} a_{ij} f(t_n + c_j h, Y_j), \quad i = 2, 3, \ldots, \mu. \tag{1.39}
$$

The MRKN method (1.36)-(1.39) is equivalent with Butcher table given in Table 1.5.

Table 1.5: μ-stage explicit MRKN method.

$$
\begin{array}{c|c|ccccc}
c_1 & 1 & 0 & & & & \\
c_2 & \gamma_2 & a_{21} & & & & \\
c_3 & \gamma_3 & a_{31} & a_{32} & & & \\
\vdots & \vdots & \vdots & \vdots & & & \\
c_\mu & \gamma_\mu & a_{\mu,1} & a_{\mu,2} & \cdots & a_{\mu,\mu-1} & \\
\hline
 & & b_1 & b_2 & \cdots & b_{\mu-1} & b_\mu \\
 & & \hat{b}_1 & \hat{b}_2 & \cdots & \hat{b}_{\mu-1} & \hat{b}_\mu
\end{array}
$$

1.2.6 Exponentially-fitted Explicit Modified RKN Method

MRKN method (1.36)-(1.39) is said to be exponential-fitted if it integrates exactly the set of functions $\{\exp(\lambda t), \exp(-\lambda t)\}$ when $\lambda \in \mathbb{R}$ the principal frequency of the problem. By imposing that MRKN method (1.36)-(1.39) is exact for differential systems whose solutions are $y(t)=\exp(\pm\lambda t)$, and taking into consideration the meaning of the stages Y_i, it is obvious to consider that $Y_i=y(t_n+c_ih)=\exp(\pm\lambda(t_n+c_ih))$ and $f(t_n+c_ih,Y_i)=y''(t_n+c_ih)=\lambda^2\exp(\pm\lambda(t_n+c_ih))$. This leads to the following equations for the coefficients of the method:

$$
\begin{aligned}
\mathrm{e}^{\pm c_1\omega} &= 1\pm c_1\omega\gamma_1, && (1.40)\\
\mathrm{e}^{\pm c_i\omega} &= 1\pm c_i\omega\gamma_i+\omega^2\sum_{j=1}^{\mu}a_{ij}\,\mathrm{e}^{\pm c_j\omega}, \qquad i=2,\ldots,\mu, && (1.41)\\
\mathrm{e}^{\pm\omega} &= 1\pm\omega+\omega^2\sum_{j=1}^{\mu}b_j\,\mathrm{e}^{\pm c_j\omega}, && (1.42)\\
\mathrm{e}^{\pm\omega} &= 1\pm\omega\sum_{j=1}^{\mu}\hat{b}_j\,\mathrm{e}^{\pm c_j\omega}. && (1.43)
\end{aligned}
$$

where $\omega=\lambda h$.
Taking into account the relations $\cosh(\omega)=\frac{\mathrm{e}^{\omega}+\mathrm{e}^{-\omega}}{2}$ and $\sinh(\omega)=\frac{\mathrm{e}^{\omega}-\mathrm{e}^{-\omega}}{2}$.
In Eq (1.40) suppose that $c_1=0$ and $\gamma_1=1$, then the system of equations (1.41)-(1.43) can be written as follow:

$$
\begin{aligned}
\sum_{j=1}^{i-1}a_{ij}\cosh\left(c_j\omega\right) &= \frac{\cosh(c_i\omega)-1}{\omega^2}, \qquad i=2,\ldots,\mu, && (1.44)\\
\sum_{j=1}^{i-1}a_{ij}\sinh\left(c_j\omega\right) &= \frac{\sinh(c_i\omega)-c_i\omega\gamma_i}{\omega^2}, && (1.45)\\
\sum_{j=1}^{\mu}b_j\cosh\left(c_j\omega\right) &= \frac{\cosh(\omega)-1}{\omega^2}, && (1.46)\\
\sum_{j=1}^{\mu}b_j\sinh\left(c_j\omega\right) &= \frac{\sinh(\omega)-\omega}{\omega^2}, && (1.47)\\
\sum_{j=1}^{\mu}\hat{b}_j\sinh\left(c_j\omega\right) &= \frac{\cosh(\omega)-1}{\omega}, && (1.48)\\
\sum_{j=1}^{\mu}\hat{b}_j\cosh\left(c_j\omega\right) &= \frac{\sinh(\omega)}{\omega}. && (1.49)
\end{aligned}
$$

1.2.7 Two-Step Explicit Runge-Kutta-Nyström (TSRKN) Method

Paternoster (2002) has introduced a two-step explicit Runge-Kutta-Nyström method for the direct solution of special second-order ODE (1.4), which is defined as follows:

$$\begin{aligned} y_{n+2} &= (1-\Theta)y_{n+1} + \Theta y_n + h\sum_{i=1}^{\mu} v_i y'_n + h\sum_{i=1}^{\mu} w_i y'_{n+1} \\ &+ h^2\sum_{i=1}^{\mu} \bar{v}_i \kappa_n^i + \bar{w}_i \kappa_{n+1}^i, \end{aligned} \tag{1.50}$$

$$y'_{n+2} = (1-\Theta)y'_{n+1} + \Theta y'_n + h\sum_{i=1}^{\mu} v_i \kappa_n^i + w_i \kappa_{n+1}^i, \tag{1.51}$$

where

$$\kappa_n^i = f(x_n + c_i h, y_n + c_i h y'_n + h^2\sum_{j=1}^{\mu} a_{ij}\kappa_n^j), \quad i = 1,\ldots,\mu, \tag{1.52}$$

$$\kappa_{n+1}^i = f(x_{n+1} + c_i h, y_{n+1} + c_i h y'_{n+1} + h^2\sum_{j=1}^{\mu} a_{ij}\kappa_{n+1}^j), \quad i = 1,\ldots,\mu \tag{1.53}$$

where Θ, v_i, w_i, $\bar{v}_i$, $\bar{w}_i$, a_{ij} are the coefficients of the method and can be represented by the following array (see Table 1.6)

Table 1.6: μ-stage explicit TSRKN method of Paternoster (2002).

$$\begin{array}{c|ccccc} c_1 & 0 & & & & \\ c_2 & a_{21} & & & & \\ c_3 & a_{31} & a_{32} & & & \\ \vdots & \vdots & \vdots & & & \\ c_\mu & a_{\mu,1} & a_{\mu,2} & \cdots & a_{\mu,\mu-1} & \\ \hline \Theta & v_1 & v_2 & \cdots & v_{\mu-1} & v_\mu \\ & w_1 & w_2 & \cdots & w_{\mu-1} & w_\mu \\ & \bar{v}_1 & \bar{v}_2 & \cdots & \bar{v}_{\mu-1} & \bar{v}_\mu \\ & \bar{w}_1 & \bar{w}_2 & \cdots & \bar{w}_{\mu-1} & \bar{w}_\mu \end{array}$$

1.3 Algebraic Order Conditions for RKNG Method

The order conditions of a μ-stage RKNG method up to fifth-order have been proposed in Fehlberg (1974) as follows:

The order conditions for y:
Order 2:

$$\sum b_i = \frac{1}{2}, \tag{1.54}$$

Order 3:

$$\sum b_i c_i = \frac{1}{6}, \tag{1.55}$$

Order 4:

$$\sum b_i c_i^2 = \frac{1}{12}, \quad \sum b_i \hat{a}_{ij} c_j = \frac{1}{24}, \tag{1.56}$$

Order 5:

$$\sum b_i c_i^3 = \frac{1}{20}, \quad \sum b_i c_i \hat{a}_{ij} c_j = \frac{1}{40}, \quad \sum b_i a_{ij} c_j = \frac{1}{120},$$
$$\sum b_i \hat{a}_{ij} c_j^2 = \frac{1}{60}, \quad \sum b_i \hat{a}_{ij} \hat{a}_{jk} c_k = \frac{1}{120}.$$

The order conditions for y':
Order 1:

$$\sum \hat{b}_i = 1, \tag{1.57}$$

Order 2:

$$\sum \hat{b}_i c_i = \frac{1}{2}, \tag{1.58}$$

Order 3:

$$\sum \hat{b}_i c_i^2 = \frac{1}{3}, \quad \sum \hat{b}_i \hat{a}_{ij} c_j = \frac{1}{6}, \tag{1.59}$$

Order 4:

$$\sum \hat{b}_i c_i^3 = \frac{1}{4}, \quad \sum \hat{b}_i c_i \hat{a}_{ij} c_j = \frac{1}{8}, \quad \sum \hat{b}_i a_{ij} c_j = \frac{1}{24},$$
$$\sum \hat{b}_i \hat{a}_{ij} c_j^2 = \frac{1}{12}, \quad \sum \hat{b}_i \hat{a}_{ij} \hat{a}_{jk} c_k = \frac{1}{24}, \tag{1.60}$$

Order 5:

$$\sum \hat{b}_i c_i^4 = \frac{1}{5}, \quad \sum \hat{b}_i c_i^2 \hat{a}_{ij} c_j = \frac{1}{10}, \quad \sum \hat{b}_i c_i a_{ij} c_j = \frac{1}{30},$$
$$\sum \hat{b}_i c_i \hat{a}_{ij} c_j^2 = \frac{1}{15}, \quad \sum \hat{b}_i a_{ij} c_j^2 = \frac{1}{60}, \quad \sum \hat{b}_i \hat{a}_{ij} c_j^3 = \frac{1}{20},$$
$$\sum \hat{b}_i (\hat{a}_{ij} c_j)^2 = \frac{1}{20}, \quad \sum \hat{b}_i c_i \hat{a}_{ij} \hat{a}_{jk} c_k = \frac{1}{30}, \quad \sum \hat{b}_i a_{ij} \hat{a}_{jk} c_k = \frac{1}{120},$$
$$\sum \hat{b}_i \hat{a}_{ij} c_j \hat{a}_{jk} c_k = \frac{1}{40}, \quad \sum \hat{b}_i \hat{a}_{ij} a_{jk} c_k = \frac{1}{120},$$
$$\sum \hat{b}_i \hat{a}_{ij} \hat{a}_{jk} c_k^2 = \frac{1}{60}, \quad \sum \hat{b}_i \hat{a}_{ij} \hat{a}_{jk} \hat{a}_{km} c_m = \frac{1}{120}. \tag{1.61}$$

1.4 Algebraic Order Conditions for RKT Method

The order conditions of a μ-stage RKT method up to fifth-order as set in Mechee et al. (2013) as follows:
The order conditions for y:
Order 3:

$$\sum b_i = \frac{1}{6}, \tag{1.62}$$

Order 4:

$$\sum b_i c_i = \frac{1}{24}, \tag{1.63}$$

Order 5:

$$\sum b_i c_i^2 = \frac{1}{60}. \tag{1.64}$$

The order conditions for y':
Order 2:

$$\sum \hat{b}_i = \frac{1}{2}, \tag{1.65}$$

Order 3:

$$\sum \hat{b}_i c_i = \frac{1}{6}, \tag{1.66}$$

Order 4:

$$\sum \hat{b}_i c_i^2 = \frac{1}{12}, \tag{1.67}$$

Order 5:

$$\sum \hat{b}_i c_i^3 = \frac{1}{20}, \quad \sum \hat{b}_i a_{ij} = \frac{1}{120}. \tag{1.68}$$

The order conditions for y'':
Order 1:

$$\sum \hat{\hat{b}}_i = 1, \tag{1.69}$$

Order 2:

$$\sum \hat{\hat{b}}_i c_i = \frac{1}{2}, \tag{1.70}$$

Order 3:

$$\sum \hat{\hat{b}}_i c_i^2 = \frac{1}{3}, \tag{1.71}$$

Order 4:

$$\sum \hat{\hat{b}}_i c_i^3 = \frac{1}{4}, \quad \sum \hat{\hat{b}}_i a_{ij} = \frac{1}{24}, \tag{1.72}$$

Order 5:

$$\sum \hat{\hat{b}}_i c_i^4 = \frac{1}{5}, \quad \sum \hat{\hat{b}}_i a_{ij} c_j = \frac{1}{120}, \quad \sum \hat{\hat{b}}_i a_{ij} c_i = \frac{1}{30}. \tag{1.73}$$

All indices are from 1 to μ. To obtain the higher order RKT method, the following simplifying assumption is used in order to reduce the number of equations to be solved:

$$\hat{b}_i \quad = \hat{\hat{b}}_i (1 - c_i), \quad i = 1, 2, \ldots, \mu. \tag{1.74}$$

1.5 Stability Properties of Runge-Kutta Type Methods

Runge-Kutta type methods have their own stability polynomial. However, all methods have the same properties as given by the following definitions and theorems:

Definition 1.2 *The method is said to satisfy the root condition if all the roots of characteristic polynomial have modulus less than or equal to unity (within or on the unit circle), and those of modulus unity are simple.*

Theorem 1.3 (Lambert (1991))
The necessary and sufficient condition for the method to be zero-stable is that it satisfies the root condition.

Theorem 1.4 (Watt (1967))
The necessary conditions for the method to be convergent are that it must be both consistent and zero-stable. The method is consistent if it has at least order 1.

Definition 1.3 *The method is said to be absolutely stable for a given roots if all the roots lies within the unit circle.*

Definition 1.4 (Jackiewicz et al. (1991))
The two-step method is zero stable if it satisfied $-1 < \Theta \leq 1$.

Definition 1.5 (Absolute stability interval)
An interval $(-H_a, 0)$ *is called the interval of absolute stability of the method if, for all* $H \in (-H_a, 0)$, $|\xi_{1,2}| < 1$, *where* $\xi_{1,2}$ *are the roots of the stability polynomial.*

Definition 1.6 (Dormand (1996))
The one step method is said to have order p if p is the largest positive integer such that

$$y(t+h) - y(t) - h\Phi(t; y; h) = O(h^{p+1}) \tag{1.75}$$

where $y(t)$ *is the analytical solution*

1.6 Local Truncation Error (LTE)

Dormand (1996) suggested that having achieved a certain order of accuracy, the best strategy for practical purposes is to minimize the error norms. The quantities of the norms of the local truncation error coefficients are defined by:

$$\| \tau^{(p+1)} \|_2 = \sqrt{\sum_{i=1}^{n_{p+1}} (\tau_i^{(p+1)})^2}, \quad \text{for } y_n, \tag{1.76}$$

$$\| \tau'^{(p+1)} \|_2 = \sqrt{\sum_{i=1}^{n_{p+1}} (\tau_i'^{(p+1)})^2}, \quad \text{for } y_n', \tag{1.77}$$

$$\| \tau''^{(p+1)} \|_2 = \sqrt{\sum_{i=1}^{n_{p+1}} (\tau_i''^{(p+1)})^2}, \quad \text{for } y_n''. \tag{1.78}$$

Below is the y error coefficients up to sixth-order RKN processes:
Order 2:

$$\tau_1^{(2)} = \sum b_i - \frac{1}{2}, \tag{1.79}$$

Order 3:

$$\tau_1^{(3)} = \sum b_i c_i - \frac{1}{6}, \tag{1.80}$$

Order 4:

$$\tau_1^{(4)} = \frac{1}{2}\sum b_i c_i^2 - \frac{1}{24}, \tag{1.81}$$

Order 5:

$$\tau_1^{(5)} = \frac{1}{6}\sum b_i c_i^3 - \frac{1}{120}, \tag{1.82}$$

$$\tau_2^{(5)} = \sum b_i a_{ij} c_j - \frac{1}{120}, \tag{1.83}$$

Order 6:

$$\tau_1^{(6)} = \frac{1}{24}\sum b_i c_i^4 - \frac{1}{720}, \tag{1.84}$$

$$\tau_2^{(6)} = \frac{1}{4}\sum b_i c_i a_{ij} c_j - \frac{1}{720}, \tag{1.85}$$

$$\tau_3^{(6)} = \frac{1}{2}\sum b_i a_{ij} c_j^2 - \frac{1}{720}. \tag{1.86}$$

The y' error coefficients up to sixth-order for RKN:
Order 1:

$$\tau_1'^{(1)} = \sum \hat{b}_i - 1, \tag{1.87}$$

Order 2:

$$\tau_1'^{(2)} = \sum \hat{b}_i c_i - \frac{1}{2}, \tag{1.88}$$

Order 3:

$$\tau_1'^{(3)} = \frac{1}{2}\sum \hat{b}_i c_i^2 - \frac{1}{6}, \tag{1.89}$$

Order 4:

$$\tau_1'^{(4)} = \frac{1}{6}\sum \hat{b}_i c_i^3 - \frac{1}{24}, \tag{1.90}$$

$$\tau_2'^{(4)} = \sum \hat{b}_i a_{ij} c_j - \frac{1}{24}, \tag{1.91}$$

Order 5:

$$\tau_1'^{(5)} = \frac{1}{24}\sum \hat{b}_i c_i^4 - \frac{1}{120}, \tag{1.92}$$

$$\tau_2'^{(5)} = \frac{1}{4}\sum \hat{b}_i c_i a_{ij} c_j - \frac{1}{120}, \tag{1.93}$$

$$\tau_3'^{(5)} = \frac{1}{2}\sum \hat{b}_i a_{ij} c_j^2 - \frac{1}{120}, \tag{1.94}$$

Order 6:

$$\tau_1'^{(6)} = \frac{1}{120}\sum \hat{b}_i c_i^5 - \frac{1}{720}, \tag{1.95}$$

$$\tau_2'^{(6)} = \frac{1}{20}\sum \hat{b}_i c_i^2 a_{ij} c_j - \frac{1}{720}, \tag{1.96}$$

$$\tau_3'^{(6)} = \frac{1}{10}\sum \hat{b}_i c_i a_{ij} c_j^2 - \frac{1}{720}, \tag{1.97}$$

$$\tau_4'^{(6)} = \frac{1}{6}\sum \hat{b}_i a_{ij} c_j^3 - \frac{1}{720}, \tag{1.98}$$

$$\tau_5'^{(6)} = \sum \hat{b}_i a_{ij} a_{jk} c_k - \frac{1}{720}. \tag{1.99}$$

and for the y error coefficients up to fifth-order RKT processes, see the following

Order 3:

$$\tau_1^{(3)} = \sum b_i - \frac{1}{6}, \tag{1.100}$$

Order 4:

$$\tau_1^{(4)} = \sum b_i c_i - \frac{1}{24}, \tag{1.101}$$

Order 5:

$$\tau_1^{(5)} = \sum b_i c_i^2 - \frac{1}{60}. \tag{1.102}$$

The y' error coefficients up to fifth-order for RKT:

Order 2:

$$\tau_1'^{(2)} = \sum \hat{b}_i - \frac{1}{2}, \tag{1.103}$$

Order 3:

$$\tau_1'^{(3)} = \sum \hat{b}_i c_i - \frac{1}{6}, \tag{1.104}$$

Order 4:

$$\tau_1^{\prime(4)} = \sum \hat{b}_i c_i^2 - \frac{1}{12}, \tag{1.105}$$

Order 5:

$$\tau_1^{\prime(5)} = \sum \hat{b}_i c_i^3 - \frac{1}{20}, \tag{1.106}$$

$$\tau_2^{\prime(5)} = \sum \hat{b}_i a_{ij} - \frac{1}{120}. \tag{1.107}$$

The y'' error coefficients up to fifth-order for RKT:

Order 1:

$$\tau_1^{\prime\prime(1)} = \sum \hat{\hat{b}}_i - 1, \tag{1.108}$$

Order 2:

$$\tau_1^{\prime\prime(2)} = \sum \hat{\hat{b}}_i c_i - \frac{1}{2}, \tag{1.109}$$

Order 3:

$$\tau_1^{\prime\prime(3)} = \sum \hat{\hat{b}}_i c_i^2 - \frac{1}{3}, \tag{1.110}$$

Order 4:

$$\tau_1^{\prime\prime(4)} = \sum \hat{\hat{b}}_i c_i^3 - \frac{1}{4}, \tag{1.111}$$

$$\tau_2^{\prime\prime(4)} = \sum \hat{\hat{b}}_i a_{ij} - \frac{1}{24}, \tag{1.112}$$

Order 5:

$$\tau_1^{\prime\prime(5)} = \sum \hat{\hat{b}}_i c_i^4 - \frac{1}{5}, \tag{1.113}$$

$$\tau_2^{\prime\prime(5)} = \sum \hat{\hat{b}}_i a_{ij} c_j - \frac{1}{120}, \tag{1.114}$$

$$\tau_3^{\prime\prime(5)} = \sum \hat{\hat{b}}_i a_{ij} c_i - \frac{1}{30}. \tag{1.115}$$

1.7 Problem Statement

A direct numerical method to solve higher-order ODEs is considered essential in the field of numerical analysis. While the common technique in the literature to solve the high-order ODEs is by converting them first into a system of first-order, then solving them using an appropriate numerical method. The advantage of the direct technique is superior and more efficient since it does not require increasing the number of equations and calculating more function evaluations which lead to a time-consuming process and more human effort as in classical method.

In this research, new direct methods are constructed to solve the second-order and special third-order BVPs. Furthermore, the formulation is covered and the implementation is considered in detail. It is possible to solve the second and third-order BVPs by applying various multistep methods in the literature, because the multistep methods require the subroutine to provide the starting values, which leads to complicated computational work. In this study, Runge-Kutta type methods are proposed which can be implemented as self-starting methods for solving directly higher-order BVPs.

Before this work began, there was no study has been conducted on the method of explicit Runge-Kutta type to solve BVPs of second-order and special third-order directly. Therefore, the following problems were taken into account in this research to deal with the gap in the scientific literature on numerical solutions of second-order and special third-order BVPs.

1.8 Scope of the Study

This study focuses on special and general second-order and special third-order BVPs. The aim and scope of this study are:

1. The development of efficient methods such as RKN, TSRKN, and RKT which can be implemented for solving special second and third-order BVPs directly with minimal complexity and lowest number of function evaluation and number of coefficients compared to the general methods of the same order. Add to that, the special methods are implemented to obtain the approximation solutions for specific real-life problems efficiently. Hence, the methods derived for special problems are limited to solve the special form of the second and third-order BVPs.
2. The development of efficient method such as RKNG which can be implemented for directly solving the special and general second-order BVPs in which the special methods cannot solve it. In addition to that, the general method is implemented to obtain the approximation solutions for specific real-life problems efficiently.

1.9 Objectives of the Study

The main objective of this thesis is to drive explicit Runge-Kutta type direct methods with constant step size for solving second and third-order two-point BVPs. The objectives of the thesis can be accomplished by:

1. To construct Runge-Kutta-Nyström methods for solving special and general second-order BVPs directly.
2. To construct Runge-Kutta type methods for solving special third-order BVPs directly.

3. To develop the order conditions of two-step Runge-Kutta-Nyström method by using Taylor series for solving special second-order BVPs directly.

4. To investigate the stability and convergence of the derived methods.

1.10 Outline of The Study

A brief description of the organization of the thesis will be provided here. Chapter 1 discusses the brief overview of ODEs. The theories and definitions that are related to the proposed methods are provided.

Chapter 2 reviews some of the previous works on the numerical solutions of the BVPs and previous works of Runge-Kutta type methods. In Chapter 3 the implementation of exponentially-fitted explicit modified RKN method of order four for solving special second-order BVPs directly with exponential solutions and explicit RKNG method for solving general second-order BVPs directly is presented.

Chapter 4 is concerned with the construction of third and fourth-order explicit RKT methods for solving special third-order BVPs and exponentially-fitted explicit modified RKT method of order three for solving directly special third-order BVPs with exponential solutions. Chapter 5 deals with the two-step explicit RKN method for the direct solution of special second-order BVPs.

Finally, Chapter 6 summarizes the thesis. Future work is also recommended.

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LIST OF PUBLICATIONS

A. A. Salam, N. Senu and Z. A. Majid. (2018). An exponentially-fitted direct Runge-Kutta method for the numerical solution of boundary value problem with exponential solutions. *International Fundamental Science Congress* (IFSC2018). 23–24 October 2018, Putrajaya, Malaysia. (Submitted)

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