UNIVERSITI PUTRA MALAYSIA

FIXED POINT FOR DERIVATIVE AND DIFFERENTIATION OF SINGLE-VALUED AND SET-VALUED FUNCTIONS ON METRIC SPACES

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FIXED POINT FOR DERIVATIVE AND DIFFERENTIATION OF SINGLE-VALUED AND SET-VALUED FUNCTIONS ON METRIC SPACES

By

MOHAMAD MUSLIKH

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Doctor of Philosophy

March 2019

DEDICATION

To my wife, Leni Martini, and our children Vani Laila Fitriani, Fikri Nusantara, Alfi Naba and Yasfiy Andzar Arianiy with love

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Doctor of Philosophy

FIXED POINT FOR DERIVATIVE AND DIFFERENTIATION OF SINGLE-VALUED AND SET-VALUED FUNCTIONS ON METRIC SPACES

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March 2019

Chair : Professor Adem Kilicman, PhD
Faculty : Science

Study of the fixed point for derivative functions is an effort to expand the knowledge of fixed point for functions. This study represents original research on the existence of the fixed point for derivative functions which has been not studied before. Therefore this study attempts to explore the existence of fixed point for derivative functions. The research found that the derivative function defined on a closed unit interval into itself has a fixed point. In addition, this study attempts to extend those results for the derivative function defined on the whole real number line. By the concepts of commutativity and compatibility between the function and its derivatives show that the derivative function of the real-valued function has a fixed point. Meanwhile, in the case of set-valued function, we use the definition of the generalizations of the Hukuhara derivative. By using hybrid composite mapping compatible with Hausdorff metric, this study shows that derivative of the interval-valued function has a fixed point. Furthermore, based on the absolute derivative notion on metric spaces in the study of differentiation for single-valued functions, we introduce the new notions of the "Straddle Lemma" and the class of the "Darboux function". Other results in this study are the absolute derivative and the metric derivative of the set-valued functions. This expansion adds the literature on differentiability references for set-valued functions, among others the continuity of the set-valued function, absolute derivative of the constant set-valued function, and comparisons with the Hukuhara derivative and generalization of the Hukuhara derivative. The metric derivative concept introduced for the set-valued function generates the generalization of the famous Rademacher's theorems.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

TITIK TETAP UNTUK TERBITAN DAN PEMBEZAAN FUNGSI BERNILAI TUNGGAL DAN BERNILAI SET PADA RUANG METRIK

Oleh

MOHAMAD MUSLIKH

March 2019

Pengerusi : Professor Adem Kilicman, PhD
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Kajian ini adalah usaha untuk mengembangkan pengetahuan berkenaan titik tetap untuk fungsi. Kajian ini merupakan penyelidikan baharu berkenaan kewujudan titik tetap bagi fungsi terbitan yang belum pernah dikaji sebelum ini. Kajian ini cuba meneroka kewujudan titik tetap bagi fungsi terbitan. Kajian ini mendapati bahawa fungsi terbitan yang ditakrifkan pada sela unit tertutup itu sendiri mempunyai titik tetap. Di samping itu, kajian ini cuba untuk mengembangkan dapatan tersebut kepada fungsi terbitan untuk keseluruhan jajaran nombor nyata. Menggunakan konsep kalis tukar tertib dan keserasian antara fungsi dan terbitannya menunjukkan bahawa fungsi terbitan bernilai sebenar mempunyai titik tetap. Sementara itu, dalam kes fungsi bernilai set, kajian ini menggunakan takrif pengitlakan terbitan Hukuhara. Menggunakan pemetaan hibrid gubahan metrik Hausdorff yang bersesuaian, kajian ini menemui bahawa fungsi terbitan dari fungsi bernilai sela mempunyai titik tetap. Tambahan pula, berdasarkan idea terbitan mutlak pada ruang metrik dalam kajian pembezaan bagi fungsi bernilai tunggal, kajian ini memperkenalkan idea baharu berkenaan "lemma Straddle" dan kelas "fungsi "Darboux". Hasil lain dalam kajian ini adalah terbitan mutlak dan terbitan metrik untuk fungsi bernilai set. Dapatan daripada kajian ini memperkaya literatur rujukan untuk terbitan fungsi bernilai set, antaranya kesinambungan fungsi bernilai set, terbitan mutlak fungsi bernilai set tetap, perbandingan dengan terbitan Hukuhara dan pengitlakan terbitan Hukuhara. Konsep terbitan metrik yang diperkenalkan bagi fungsi bernilai set menghasilkan pengitlakan teorem Rademacher yang terkenal.

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I certify that a Thesis Examination Committee has met on 13 March 2019 to conduct the final examination of Mohamad Muslikh on his thesis entitled "Fixed Point for Derivative and Differentiation of Single-Valued and Set-Valued Functions on Metric Spaces" in accordance with the Universities and University Colleges Act 1971 and the Constitution of the Universiti Putra Malaysia [P.U.(A) 106] 15 March 1998. The Committee recommends that the student be awarded the Doctor of Philosophy.

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This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Doctor of Philosophy. The members of the Supervisory Committee were as follows:

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Declaration by graduate student

I hereby confirm that:

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This is to confirm that:
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TABLE OF CONTENTS

LIST OF FIGURES

LIST OF ABBREVIATIONS

$\mathbb{R}$	Real line numbers
$\mathbb{Q}$	Rational numbers
$\mathbb{N}$	Natural numbers
$\mathbb{R}^k$	Vector k-space over the real field
$B_1^k(\mathbf{x})$	Closed unit ball of the vector $\mathbf{x}$ in vector k-spaces
(X, d)	Metric spaces with metric d
$\overline{A}$	Closure of subset A of the metric spaces
$N_r(p)$	Neighbourhood of the point p with radius $r > 0$
$\mathcal{U}_r(A)$	Neighbourhood of the set A with radius $r > 0$
$B_r(A)$	Ball of radius $r > 0$ around a subset A
$\overline{B_r}(A)$	Closed ball of radius $r > 0$ around a subset A
$\ell(x, y)$	Straight line segment joining points x and y
$d(x, A)$	Distance the point x to subsets A
$d(A, B)$	Distance between of subsets A and B
$H(A, B)$	Hausdorff distance between of subsets A and B
$\|a\|$	Absolute value of a real number a
$\|[a, b]\|$	Absolute value of a closed interval $[a, b]$
$\\|\mathbf{x}\\|$	Norm of the vector $\mathbf{x}$
$\mathbf{co}(U)$	Convex hull of subset U
$diam(U)$	Diameter of subset U
$U \overset{h}{-} V$	Hukuhara difference of subset U and V
$U \overset{gh}{-} V$	Generalization Hukuhara difference of subset U and V
$\mathrm{Lip}(f)$	Lipschitz constant of the function f

$\mathcal{I}(\mathbb{R})$	Collection of all closed intervals of $\mathbb{R}$
$\mathcal{P}_0(X)$	Collection of all non-empty subsets of X
$\mathcal{B}(X)$	Collection of all non-empty bounded subsets of X
$\mathcal{CL}(X)$	Collection of all non-empty closed subsets of X
$\mathcal{CB}(X)$	Collection of all non-empty closed bounded subsets of X
$\mathcal{K}(X)$	Collection of all non-empty compact subsets of X
$\mathcal{KC}(X)$	Collection of all non-empty compact convex subsets of X
h-difference	Hukuhara difference
gh-difference	Generalization Hukuhara difference
$\limsup A_n$	Limit superior of the sequences of subset A_n
$\liminf A_n$	Limit inferior of the sequences of subset A_n
$f : X \longrightarrow Y$	Single-valued function
f'	Derivative of real valued function f
f'_{abs}	Absolute derivative of single-valued function f
f'_{md}	Metric derivative of single-valued function f
$F : X \longrightarrow \mathcal{P}_0(Y)$	Set-valued function
$Gr(F)$	Graph of set-valued function F
$F^-(A)$	Lower image of subset A under the set-valued function F
$F^+(A)$	Upper image of subet A under the set-valued function F
F'_h	Hukuhara derivative of set-valued function F
F'_{gh}	Generalization Hukuhara derivative of set-valued function F
F'_{abs}	Absolute derivative of set-valued function F
F'_{hm}	Metric derivative of set-valued function F

usc	Upper semi continuous
lsc	Lower semi continuous

CHAPTER 1

INTRODUCTION

This Chapter begins with the background of the study that presents an overview of the progress of the fixed point theory and the current fixed point theorems. The importance of fixed point for derivative of single-valued or set-valued functions is also highlighted. Furthermore, the problem statement addressing fixed points, derivatives, and gaps are also identified in the study. This section is followed by the research objectives and research questions of the study. The significance and scope of the study are also presented. The last section is the organization of the thesis.

1.1 Background of the Study

The study of fixed point theory began in 1912 with a theorem given by a Dutch mathematician Brouwer. This is the most famous and important theorem on the topological fixed point property. It can be formulated as; A closed unit ball $B^k \subseteq \mathbb{R}^k$ has the topological fixed point property or for every continuous self maps on a closed unit ball in $\mathbb{R}^k$ has a fixed point. Moreover, he also proved fixed point theorems for a square, a sphere and their k-dimensional counterparts. Brouwer's theorem has many applications in analysis and differential equation. Its discovery had a tremendous influence in the development of several branches of mathematics, in particular, algebraic topology.

An important generalization of Brouwer's theorem into Banach spaces was discovered in 1930 by Schauder stating "every continuous map on the compact convex subspace of the Banach spaces has a fixed point". Whereas in 1935 Tychonoff modified Brouwer result with stated: "every continuous map on the compact convex subspace of the locally convex topological vector spaces has a fixed point".

The study of fixed point problems for set-valued mappings was initiated by Kakutani (1941) in the finite dimensional spaces by generalizing Brouwer's fixed point theorem. This was the beginning of the fixed point theory for set-valued mappings having a vital connection with the minimax theory in game theories.

In Mathematics, the existence and uniqueness of a problem's solution are essential and that is the main purpose to solve the problems. Therefore, researchers have made various efforts to study the fixed point theory for continuous mappings. Stefan Banach, in that year, introduced a concept of mapping called contraction mapping and he showed that a contraction self-mapping on a complete metric space has a unique fixed point. In 1969, Meir and Keller have weakened the contraction mapping that they called weakly uniformly strict contraction so that the requirements are more general than in the Banach principle.

The contraction mapping is continuous, due to this a natural question arises: does there exist a contractive mapping, which does not enforce a mapping to be continuous? In 1969, Kannan proved the existence of a fixed point for "contraction" mapping that are not continuous. Since Kannan (1969) introduced the field of "contraction" mapping, many researchers have expanded the study in this field (Reich, 1971; Chatterjea, 1972).

In 1976, Caristi introduced the fixed point theorem which was one of the generalizations of the Banach's fixed point theorem but the method used is different from other generalizations. Namely, there is no "contraction" impression of its mapping. This class of mapping introduced by Caristi is larger and covers all types of contraction mapping developed before.

The study of fixed point of set-valued mappings on a metric space was initiated by Nadler (1969). By using the Hausdorff metric he proved that every contraction of the set-valued mapping in the sense of the Hausdorff metric has a fixed point on a complete metric space.

The celebrated Banach contraction principles is one of the main tools for both the theoretical and computational aspects in mathematical sciences. In this case, Jungck (1976) obtained an important generalization of Banach contraction principles in the form of common fixed point theorems for the pairs of commuting maps. Subsequently, Sessa (1982) obtained the same thing but by using the weaker concepts of commuting maps. This concept was further improved by Jungck (1986) which introduced compatible map to observe common fixed point. On the other hand, Singh and Mishra (1994) studied coincidence and fixed points of reciprocally continuous and compatible hybrid maps.

Ciesielski asked in the article (Gibson and Natkaniec, 1998/1999) whether the composition of two derivative mappings from a closed unit interval into itself always has a fixed point? An affirmative answer is given by Elekes et al. (2001/2002). Similarly, answer also has been given by Cs*ö*rnyei et al. (2001/2002) as an alternative proof. This result is very interesting to be developed in this thesis.

1.2 Problem Statement

The concept of fixed point comprises a triplet (X, x, f) where X is the set, x is an element of X, and f is a self-mapping of X such that $x = fx$. In other words, the point x remains invariant under the mapping f.

Theorems concerning the existence and properties of fixed point are known as fixed point theorems. By a fixed point theorem, one will understand a statement which asserts that under what conditions the mapping f and / or the set X, the self-mapping f on the set X has one or more fixed points (Brouwer, 1912; Banach, 1922; Caristi, 1976). The condition also holds in the set-valued mapping (Kakutani, 1941; Nadler, 1969). Thus the mapping f and the set X plays an important role in determining the existence of fixed point.

In calculus, the derivative of a function is an important theme that supports the development of other areas especially in identifying the behaviour of function on its domains. Study of the fixed point for derivative is relatively new and undeveloped. Recently, Elekes et al. (2001/2002) and Csörnyei et al. (2001/2002) proved with different ways that the composition mapping of two derivative functions on the closed unit interval into itself has a fixed point. Csornyei et al. (2001/2002) proved that composition of two Darboux Baire-1 functions[1] on the closed unit interval into itself has a fixed point since derivative functions are examples of Darboux Baire-1 functions. Whereas Elekes et al. (2001/2002) proved directly of the composition of two derivative functions.

Implicitly the above problems are related to function with intermediate value property [2]. This property was believed, by some 19th-century mathematicians, to be equivalent to the property of continuity (Bruckner, 1978). In 1875, Darboux showed that this belief is not justified. He proved that every derivative has the intermediate value property and he gave examples of some rather badly discontinuous derivatives (see (Gordon, 1994)). Because of Darboux's work on the subject, one now usually calls a function having the intermediate value property as Darboux function (Bruckner, 1978).

It is clear that every continuous function is a Darboux function. A more interesting result is the fact that every derivative is a Darboux function. In fact the derivative may not be continuous. Thus the class of Darboux function is a generalization of continuous functions class (Gordon, 1994).

Obviously, there is still a need to further investigate on the existence of fixed point for derivatives. Among the issue to be investigated are the extent of the derivative functions and their behaviour at their fixed point. However, not many research has

1. A function $f : [a, b] \longrightarrow \mathbb{R}$ is a Darboux Baire-1 function if f is the pointwise limit of a sequence of continuous functions.

2. A function $f : [a, b] \longrightarrow \mathbb{R}$ is said to have the intermediate value property if whenever $x.$ and y are in $[a, b]$, and c is any number between $f(x)$ and $f(y)$, there is a number z between x and y, such that $f(z) = c$.

explored the fixed point for composition mapping of two derivatives on the closed unit interval (Elekes et al., 2001/2002). Their result is only in response to questions related to the Darboux function class.

The researcher did not focus on investigating areas that support the existence of a fixed point for derivatives. Therefore, the present study focuses on those neglected areas. To fill the gap, the present study will study on the existence of a fixed point for the derivative of a real-valued function and derivative of an interval-valued function. This study seeks to expand discourse on fixed point in the context of derivative on real number and differentiation in metric spaces.

1.3 Research Objectives

The research on the fixed point for the derivative and the differentiation of mapping in metric spaces may open a new path to produce a fixed point of a derivative and help researchers to develop the derivative concepts in abstract metric spaces. Therefore, the major objective formulated for this study is to investigate the existence of a fixed point for derivative (single-valued, set-valued functions) and to develop differentiation (single-valued and set-valued) in metric spaces.

This study has five major objectives:

1. To determine the existence of a fixed point of the derivative function on closed unit interval $[0, 1]$.
2. To determine the existence of a fixed point of the derivative function on real numbers $\mathbb{R}$.
3. To determine the existence of a fixed point on the derivative of the interval-valued functions.
4. To identify the relationship between existence of a fixed point of the derivative functions and a fixed point of the original functions.
5. To develop the differentiation concepts of single-valued and set-valued mappings on metric spaces.

1.4 Research Questions

Based on the objectives, the following research questions are developed.

Q1 : What is the necessary conditions for the derivative a function to have a fixed point? In this case, is the fixed point unique or not?

Q2 : What is the necessary conditions for the derivative a set-valued function to have a fixed point? In this case, is the fixed point unique or not?

Q3 : What is the relation between the fixed point of a function and the fixed point of its derivative?

Q4 : How to define the derivative of a set-valued mapping defined on metric spaces?

1.5 Significance of the Study

This study adds to the existing frame of Mathematical rules by observing the character of the function in the context of determining the existence of the fixed point for its derivative. Such insight can be used to explain many questions pertaining to the additional or simplification requirements, and proofing strategy. More importantly, these insights may develop a greater understanding of how to determine the fixed point for derivative, where the existence of the derivative depends on the function given to the problem being studied. This study also introduces the new notions of derivative on the abstract metric spaces.

1.6 Scope of the Study

Since the scope of the study is focused on extending the existing fixed point for the derivative function model, the main conceptual framework of this study is built from that model. The main scope of this study is to integrate the existing fixed point of the function into its derivative function. The investigation in the study is focused on the real-valued functions with real number domain. The scope of the study will cover the existence and uniqueness of the fixed point for single-valued functions (real-valued function) and set-valued functions (interval-valued function).

1.7 Organization of the Thesis

This subsection will describe the organization of the whole thesis and content of every chapter. Chapter 1 presents an overview of the fixed point as the phenomenon which preserves a point by mapping. This chapter includes background of the study; problem statement; research objectives; research questions; significance of the study; scope of the study; and organization of the thesis. Chapter 2 presents the literature review for all ingredients used for the design of the theoretical framework. There are eight sections in this chapter: metric spaces, Euclidean space, single-valued mappings, set-valued mappings, differentiation, fixed points, common fixed points and fixed point of derivatives. In line with the literature study, Chapter 3 explains hypothesis development and research result in fixed point theorems form for the derivative of single-valued function on a closed unit interval and real number lines.

The result of the fixed point for the derivative of the set-valued functions is presented in Chapter 4. This chapter covers the fixed point for the derivative of the interval-valued functions. The fifth chapter presents the development process and discussion on differentiation and the characterization of Caristi's type mappings. The last chapter discusses on general conclusion, the contributions of the study from three perspectives, that is theoretical, conceptual, and simplicity contributions. Limitations of the study and suggestions for future research are presented in the final chapter.

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