UNIVERSITI PUTRA MALAYSIA

MODIFIED HOMOTOPY PERTURBATION METHOD FOR INTEGRO-DIFFERENTIAL AND HYPERSINGULAR INTEGRAL EQUATIONS

FATIMAH SAMIHAH BINTI ZULKARNAIN

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By

FATIMAH SAMIHAH BINTI ZULKARNAIN

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Doctor of Philosophy

April 2018

DEDICATION

I dedicate this work of mine to my beloved father Zulkarnain bin Zainal and my beloved mother Khairu-n-nisa-i Binti Hashim. May the mercy of Allah be upon us all. Your support and courage is what made me what I am today. Thank you.

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Doctor of Philosophy

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April 2018

Chairman : **Assoc. Prof. Nik Mohd Asri Nik Long, PhD**
Faculty : **Science**

Homotopy perturbation method (HPM) is implemented to solve the mathematical problems. The solution is obtained by taking the summation of infinite series. This thesis present a modification of HPM by equating the second series as zero. Convergence and error estimation of HPM and modified HPM (MHPM) are obtained in the class of $C[a,b]$ for Fredholm-Volterra integral equation (FVIE) problem and $C^k(D)$ where D is a closed subspace of $\mathfrak{R}^2$ for higher order FVIDE problem.

Many researchers solved singular integral equation with kernel equal to one. This study describes the implementation of HPM and MHPM on HSIE of the first kind with kernel is a constant on a diagonal. Convergence and error estimation are obtained in the class of $\mathscr{L}_\rho[-1,1]$. MHPM is also used to solve HSIE of the second kind.

For all cases, numerical examples are provided to exhibit the efficiency of the methods. The results obtained are more accurate than the previous works.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

PENGUBAHSUAIAN KAEDAH PERUSIKAN HOMOTOPI PADA PERSAMAAN KAMIRAN-PEMBEZAAN FREDHOLM-VOLTERRA DAN PERSAMAAN KAMIRAN HIPERSINGULAR

Oleh

FATIMAH SAMIHAH BINTI ZULKARNAIN

April 2018

Pengerusi : **Prof. Madya Nik Mohd Asri Nik Long, PhD**
Fakulti : **Sains**

Kaedah perusikan homotopi (KPH) digunakan untuk menyelesaikan masalah matematik. Penyelesaian diperolehi dengan mengambil penjumlahan siri tak terhingga. Tesis ini membentangkan pegubahusaian KPH dengan menyamakan siri kedua dengan kosong. Penumpuan dan anggaran ralat bagi KPH dan pengubahsuaian KPH (PKPH) diperolehi dalam kelas $C[a,b]$ untuk masalah persamaan kamiran Fredholm-Volterra dan kelas $C^k(D)$ dengan D ialah subruang tertutup dalam $\Re^2$ untuk masalah persamaan kamiran-pembezaan Fredholm-Volterra.

Ramai penyelidik telah menyelesaikan masalah persamaan kamiran singular dengan kernel bersamaan dengan satu. Kajian ini menerangkan penggunaan KPH dan PKPH pada persamaan kamiran hipersingular jenis pertama dengan kernel adalah pemalar pada pepenjuru. Penumpuan dan anggaran ralat diperolehi dalam kelas $\mathscr{L}_2[-1,1]$. PKPH digunakan untuk menyelesaikan HSIE jenis kedua.

Bagi semua kes, contoh berangka disediakan untuk mempamerkan keberkesanan kaedah-kaedah tersebut. Keputusan yang diperoleh adalah lebih tepat berbanding kajian terdahulu.

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This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Doctor of Philosophy.

The members of the Supervisory Committee were as follows:

Nik Mohd Asri Nik Long, PhD
Assosiate Professor
Faculty of Science
Universiti Putra Malaysia
(Chairperson)

Zainidin K. Eshkuvatov, PhD
Assosiate Professor
Faculty of Science and Technology
Universiti Sains Islam Malaysia
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Zahridin Muminov, PhD
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Faculty of Science and Engineering
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Faculty of Science
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(Member)

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Name and Matric No: Fatimah Samihah Binti Zulkarnain, GS34522

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LIST OF ABBREVIATIONS

HPM	Homotopy Perturbation Method
FVIDE	Fredholm-Volterra Integro-Differential Equation
FVIE	Fredholm-Volterra Integral Equation
HSIE	Hypersingular Integral Equation
IDE	Integro-Differential Equation
FIDE	Fredholm Integro-Differential Equation
VIDE	Volterra Integro-Differential Equation
MDM	Modified Decomposition Method
NVFIE	Nonlinear Volterra Fredholm Integral Equation
LP	Lagrange Polynomial
RK	Reproducing Kernel
BP	Bernstein Polynomials

CHAPTER 1

INTRODUCTION

1.1 Integral Equations

Integral equations appear in many different areas of mathematics, mainly on differential equations and operator theory. Numerous problems in ordinary and partial differential equations are recasted as integral equations. Problems of mathematical physics are the examples of application and important role of integral equations. Subject of integral equations are view as extension of linear algebra and a precursor of modern functional analysis. Particularly, the fundamental of linear vector spaces, eigenvalues and eigenfunctions play a significant role in dealing with linear integral equations (Hochstadt, 1989).

The integral equation are classified by various characteristics which are listed as follows:

1. Fredholm integral: In this type of integral, the limits of integration are known. Moreover, the unknown function may appear only inside the integral equation. The following equation are most frequently studied:

$$f(x) = \int_a^b K(x,t)\, u(t)\, dt, \tag{1.1}$$

$$u(x) = f(x) + \lambda \int_a^b K(x,t)\, u(t)\, dt, \tag{1.2}$$

$$s(x)\, u(x) = f(x) + \lambda \int_a^b K(x,t)\, u(t)\, dt. \tag{1.3}$$

   Eqs. (1.1)-(1.3) are known as Fredholm integral equations of the first, second and third kind, respectively. Functions $f(x)$ and $s(x)$ are known, $K(x,t)$ is a kernel, λ is the given parameter, $u(x)$ is the unknown function and the interval (a,b) are finite. If $s(x) = 1$, Eq. (1.3) reduces to Eq. (1.2) and if $s(x) = 0$, Eq. (1.3) reduces into Eq. (1.1).

2. Volterra integral: Volterra (Tricomi, 1985) investigated the solutions of Eq. (1.1)-(1.3) which the kernel satisfies the condition

$$K(x,t) \equiv 0 \qquad \text{if } t > x. \tag{1.4}$$

The later equation reduce to the corresponding Volterra integral equations of the form

$$f(x) = \int_a^x K(x,t)\, u(t)\, dt, \tag{1.5}$$

$$u(x) = f(x) + \lambda \int_a^x K(x,t)\, u(t)\, dt, \tag{1.6}$$

$$s(x)\, u(x) = f(x) + \lambda \int_a^x K(x,t)\, u(t)\, dt. \tag{1.7}$$

Volterra equations have many interesting properties that do not emerge from the general theory of Fredholm equations, therefore a separate study is warranted.

1.2 Linear integro-differential equations

General linear integro-differential with its corresponding initial conditions take the forms:

$$\sum_{k=0}^{m} s_k(x)\, u^{(k)}(x) = f(x) + \lambda \int_{\alpha(x)}^{\beta(x)} \sum_{j=0}^{n} K_j(x,t)\, u^{(j)}(t)\, dt, \tag{1.8}$$

$$u^{(k)}(a) = d_k,\ 0 \le k \le m-1,\ 0 \le j \le n-1, \tag{1.9}$$

where $K(x,t)$ is called the kernel of integral, $f(x)$ and $s(x)$ are the given functions, $\alpha(x) \le \beta(x)$ are limits of the integration, λ is a parameter, m and n are the order of differentation such that $n \le m$, $u(x)$ is the unknown function need to be determined and d_k are initial values for IDEs. Our goal is to determine $u(x)$ satisfies Eqs. (1.8) and (1.9) and this may be achieved by some methods. The initial conditions are given for IDEs to obtain the particular solutions.

Fredohlm integro-differential (FIDE) equation occured when limits of integration are fixed of the form

$$\sum_{k=0}^{m} s_k(x)\, u^{(k)}(x) = f(x) + \lambda \int_a^b \sum_{j=0}^{n} K_j(x,t)\, u^{(j)}(t)\, dt, \tag{1.10}$$

$$u^{(k)}(a) = d_k,\ 0 \le k \le m-1,\ 0 \le j \le n-1,$$

where a and b are fixed numbers. Volterra integro-differential equation (VIDE) occured when one of the limits of integration is a variable of the form

$$\sum_{k=0}^{m} s_k(x)\, u^{(k)}(x) = f(x) + \lambda \int_a^x \sum_{j=0}^{n} K_j(x,t)\, u^{(j)}(t)\, dt, \tag{1.11}$$

$$u^{(k)}(a) = d_k,\ 0 \le k \le m-1,\ 0 \le j \le n-1.$$

Combination of FIDE and VIDE is called Fredholm-Volterra integro-differential equation (FVIDE) and can be written as

$$\sum_{k=0}^{m} s_k(x)\,u^{(k)}(x) = f(x) + \lambda_1 \int_a^b \sum_{j_1=0}^{n_1} K_{1j_1}(x,t)\,u^{(j_1)}(t)\,dt,$$

$$+\lambda_2 \int_a^x \sum_{j_2=0}^{n_2} K_{2j_2}(x,t)\,u^{(j_2)}(t)\,dt, \tag{1.12}$$

$$u^{(k)}(a) = d_k,\ 0 \le k \le m-1,\ 0 \le j_1 \le n_1 - 1,\ 0 \le j_2 \le n_2 - 1, \tag{1.13}$$

where $K_{1j_1}(x,t)$ and $K_{2j_2}(x,t)$ are kernels, λ_1, λ_2 are parameters and n_1 and n_2 are less than or equal to m. Nonlinear integro-differential equations of order m take the form

$$\sum_{k=0}^{m} s_k(x)\,u^{(k)}(x) = f(x) + \lambda \int_{\alpha(x)}^{\beta(x)} \sum_{j=0}^{n} K_j(x,t)\,F\left(u^{(j)}(t)\right)\,dt, \tag{1.14}$$

$$u^{(k)}(a) = d_k,\ 0 \le k \le m-1,\ 0 \le j \le n-1, \tag{1.15}$$

where F is a nonlinear function. However, solution for nonlinear equation may not be unique.

Solution of integral and integro-differential equations (IDEs) play a prime role in science and engineering studies. A physical system is modeled under a differential equation, an integral equation or an IDE. The IDEs contain both integral and differential operators. The derivatives of the unknown functions may appear to any order (Wazwaz, 2009).

1.3 Singular and Hypersingular Integral Equation

Singular integral equation is an improper integral occured if either the limits of integration become infinite or the kernel has singularities within the range of integration. Singular integral equation frequently arises in mathematical physics. For one dimensional equations, basic integral is of the form (Martin and Rizzo, 1996)

$$I_n(x) = \int_A^B \frac{F(t)}{(t-x)^n}, \qquad n = 1, 2, \tag{1.16}$$

where $A < x < B$ and $F(t)$ is called density function. When $n = 1$, Eq. (1.16) is called Cauchy singular integral and when $n = 2$, Eq. (1.16) is called hypersingular integral. $I_n(x)$ exist when $F(t)$ have certain smoothness or continuity properties that are usually expressed in terms of Hölder continuity and function spaces $C^{m,\alpha}$ (Evans, 1994).

Functions with varies smoothness properties are listed in the following. A particularly useful example is the simple discontinuous function

$$f(t) = \begin{cases} f_L, & t < 0, \\ f_R, & t > 0. \end{cases} \tag{1.17}$$

Another useful example is $f(t) = |t|$. Assume that f is a given bounded function, defined on an interval $a < t < b$,

1. f is picewise. Such functions are continuous except for finite discontinuities. Hence, for a discontinuity at $t = x$, the left hand limit, $f(x^-)$ and the right hand limit, $f(x^+)$ both exist with $f(x^-) \neq f(x^+)$. For example, Eq. (1.17) has $f\left(0^-\right) = f_L$ and $f\left(0^+\right) = f_R$ and f is not defined at $t = 0$.

2. f is continuous, $f \in C$. Since continuous functions are defined for all t with $a < t < b$; thus particularly we have $f(x^-) = f(x^+) = f(x)$.

3. f is Hölder continuous, $f \in C^{0,\alpha}$. Therefore, positive contants A and α can be obtained so that

$$|f(t_1) - f(t_2)| < A|t_1 - t_2|^{\alpha} \quad \text{with } 0 < \alpha \leqslant 1,$$

for all t_1 and t_2 in $a < t < b$. In particular case, when f is Hölder continuous at $t = 0$, then we have

$$|f(t) - f(0)| < A|t|^{\alpha} \quad \text{with } 0 < \alpha \leqslant 1,$$

for all t in some inverval containing $t = 0$. In general, functions in $C^{0,\alpha}$ are smoother than continuous functions but not differentiable.

4. f is Hölder-continuous first derivative, $f' \in C^{0,\alpha}$ or $f \in C^{1,\alpha}$. In general, functions in $C^{1,\alpha}$ are smoother than differentiable functions but not continuous differentiable at second derivative.

Thus, it is well known that (Martin and Rizzo, 1989)

$$\text{if} \quad F \in C^{n-1,\alpha} \quad \text{then } I_n \text{ exists.} \tag{1.18}$$

This condition is local; if it holds in a neighbourhood of x, then $I_n(x)$ exists. The integral in Eq. (1.16) is not a proper integral. Thus it is regularalized by Cauchy Principal Value integrals (Kanwal, 1997; Martin and Rizzo, 1989)

$$\oint_A^B \frac{F(t)}{t-x}\,dt = \lim_{\varepsilon \to 0^+} \left\{ \int_A^{x-\varepsilon} \frac{F(t)}{t-x}\,dt + \int_{x+\varepsilon}^B \frac{F(t)}{t-x}\,dt \right\}. \tag{1.19}$$

In general discussion, Ang (2013) assume that $F(t)$ represented by Taylor series about $t = x$ for $a < x < b$, that is $F(t)$ may written as

$$F(t) = F(x) + \sum_{m=1}^{\infty} \frac{F^{(m)}(x)}{m!} (t-x)^m, \tag{1.20}$$

where $F^{(m)}$ is the derivative of $F(x)$. Both Eq. (1.20) and definition for Cauchy principal integrals in Eq. (1.19) are shown to be equivalent. Consider the divergent integral

$$\int_A^B \frac{F(t)}{(t-x)^2}\,dt = \lim_{\varepsilon \to 0^+} \left\{ \int_A^{x-\varepsilon} \frac{F(t)}{(t-x)^2}\,dt + \int_{x+\varepsilon}^B \frac{F(t)}{(t-x)^2}\,dt \right\} \tag{1.21}$$

where $A < x < B$. From Eqs. (1.19) and (1.20), the limit in Eq. (1.21) may written as

$$\begin{aligned}
&\lim_{\varepsilon \to 0^+} \left\{ \int_A^{x-\varepsilon} \frac{F(t)}{(t-x)^2}\,dt + \int_{x+\varepsilon}^B \frac{F(t)}{(t-x)^2}\,dt \right\} \\
&= \lim_{\varepsilon \to 0^+} \frac{F(x)}{2\varepsilon} + F(x)\left\{ \frac{1}{A-x} - \frac{1}{B-x} \right\} + F'(x)\left\{ \ln|B-x| - \ln|A-x| \right\} \\
&\quad + \sum_{m=1}^{\infty} \frac{F^{(m+1)}(x)}{(m+1)!\,m} \left\{ (B-x)^m - (A-x)^m \right\}.
\end{aligned} \tag{1.22}$$

The bounded terms on the right hand side of Eq. (1.22) are the "finite part" of the divergent integral in Eq. (1.21). There exists a close relations between hypersingular boundary integral equations and finite part integrals in the sense of Hadamard finite part. Therefore, Ang (2013) define Hadamard finite-part integral of the form

$$\fint_A^B \frac{F(t)}{(t-x)^2}\,dt = \lim_{\varepsilon \to 0^+} \left\{ \int_A^{x-\varepsilon} \frac{F(t)}{t-x}\,dt + \int_{x+\varepsilon}^B \frac{F(t)}{t-x}\,dt - \frac{2f(x)}{\varepsilon} \right\}, \quad A < x < B.$$

It also can be written as

$$\begin{aligned}
\frac{d}{dx}\left[\int_A^B \frac{F(t)}{t-x}\,dt \right] &= \fint_A^B \frac{F(t)}{(t-x)^2}\,dt, \\
&= \fint_A^B \frac{\partial}{\partial x}\left[\frac{F(t)}{(t-x)^2}\,dt \right],
\end{aligned} \tag{1.23}$$

whereas the right hand side of Eq. (1.23) is hypersingular integral equation.

Hypersingular integral equation (HSIE) is reviewed as an important tool to solve a large class of mixed boundary value problems in mathematical physics. Many problems of fluid mechanics, elasticity, and wave dynamics (acoustics) with mixed boundary conditions can be reduced to hypersingular integral equations (Iovane et al., 2003). Davydov et al. (2003) stated that hypersingular integrals are integrals with strong singularities. A general hypersingular integral equation of the first kind

(Mandal and Bera, 2006), over a finite interval, can be represented of the form

$$\frac{1}{\pi}\fint_{-1}^{1}\frac{K(x,t)}{(t-x)^2}\varphi(t)\,dt+\frac{1}{\pi}\int_{-1}^{1}L(x,t)\varphi(t)dt=f(x), \tag{1.24}$$

where $K(x,t)$ and $L(x,t)$ are the square integrable kernels on $D=\{(x,t)\in\mathbb{R}^2|-1\leq x,t\leq 1\}$. Let $K(x,t)=1$, the hypersingular integral denoted by $\fint_{-1}^{1}\frac{\varphi(t)}{(t-x)^2}dt$ is defined as

$$\begin{aligned}\frac{1}{\pi}\fint_{-1}^{1}\frac{\varphi(t)}{(t-x)^2}dt = \lim_{\varepsilon\to 0^+}\Bigg[&\frac{1}{\pi}\int_{-1}^{x-\varepsilon}\frac{\varphi(t)}{(t-x)^2}dt\\ &+\frac{1}{\pi}\int_{x+\varepsilon}^{1}\frac{\varphi(t)}{(t-x)^2}dt-\frac{\varphi(x+\varepsilon)+\varphi(x-\varepsilon)}{\varepsilon}\Bigg],\end{aligned} \tag{1.25}$$

and understood as Hadamard finite part with interval $-1\leq x\leq 1$. Eq. (1.24) appear in mathematical physics problems such as water wave scattering (Kanoria and Mandal, 2002) and radiation problems involving thin submerged plates (Mandal et al., 1995; Parsons and Martin, 1994), and fracture mechanics (Chan et al., 2003; Nik Long and Eshkuvatov, 2009).

1.4 Homotopy Perturbation Method and Its Modification

Homotopy perturbation method (HPM) was proposed by He (1999). In this method, the solution is considered as the summation of an infinite series, which usually converges rapidly to the exact solution. HPM has been used for a wide range of problems; to find the exact and approximate solutions of the Volterra-Fredholm integral equations (Ghasemi et al., 2007), nonlinear ordinary differential equations(ODEs) (Ramos, 2008), the integro-differential equations (Golbaba and Javidi, 2007; Dehghan and Shakeri, 2008), linear and nonlinear integral equations (Jafari et al., 2010) and one-phase inverse Stefan problem (Słota, 2010).

HPM is the combination of two methods: the homotopy and the perturbation methods. The homotopy technique also knowns as the continuous mapping technique, embeds a parameter p that ranges from zero to one. When the embedding parameter is zero, the equation is one of a linear system, when it is one, the equation is the same as the original one. Then, the embedded parameter $p\in[0,1]$ is considered as a small parameter. To illustrate the basic ideas of the method, He (1999) consider the following operator of the form

$$Lu+Nu=f, \tag{1.26}$$

where L is linear differential operator and N is nonlinear integral operator. Homotopy function $v(r,p):\Omega\times[0,1]\to\mathbb{R}$ is constructed to satisfies

$$\mathscr{H}(v,p)=(1-p)\,(L(v)-L(u_0))+p\,(L(v)+N(v)-f)=0,\ p\in[0,1], \tag{1.27}$$

where p is embedding parameter and u_0 is an initial approximation of Eq. (1.26). It is easy to see that

$$\mathscr{H}(v,0) = L(v) - L(u_0) = 0, \tag{1.28}$$

$$\mathscr{H}(v,1) = L(v) + N(v) - f = 0. \tag{1.29}$$

Deformation occurs in the changing process of p from zero to unity, $\mathscr{H}(v,p)$ from $\mathscr{H}(v,0)$ to $\mathscr{H}(v,1)$. Meanwhile, $L(v) - L(u_0)$ and $L(v) + N(v) - f$ are called homotopic. Applying perturbation technique, the solution of Eq. (1.27) can be expressed as series in p of the form

$$v = \sum_{j=0}^{\infty} p^j v_j(x). \tag{1.30}$$

Substituting (1.30) into (1.27) and equate the terms with identical powers of p yields

$$\begin{aligned} L(v_0) &= L(u_0), \\ L(v_1) &= -L(u_0) - N(v_0) - f, \\ L(v_j) &= -N(v_{j-1}), j = 2,3,\ldots. \end{aligned} \tag{1.31}$$

Approximate solution of (1.26) can be readily obtained by

$$u(x) = \lim_{p\to\infty} v(x) = v_0(x) + v_1(x) + v_2(x)... = \sum_{j=0}^{\infty} v_j(x). \tag{1.32}$$

Eq. (1.32) should be converge under a few conditions.

Turkyilmazoglu (2011) showed that HPM converges under certain circumstances without knowing a prior knowledge of the exact solution. He also obtained the error estimate for the aproximate solution and provided information on interval of convergence of homotopy series. He considered the nonlinear boundary value problem

$$N(u(r)) = 0; \quad r \in \Omega, \quad B\left(u(r), \frac{du}{dn}\right) = 0; \quad r \in \Gamma, \tag{1.33}$$

where $u(r)$ is the function to be solved under the boundary constraints in B. Other analysis of convergences of HPM and some related theorems were developed by Ayati and Biazar (2015) and Jafari et al. (2010).

Recently, HPM has been modified in different approach. For example, Javidi and Golbabai (2009) added an accelerating parameter for solving nonlinear Fredholm integral equation. Ghorbani and Saberi-Nadjafi (2008) adding a series of parameter and selective functions so-called improved homotopy perturbation method to find solutions of nonlinear Fredholm and Volterra equations. On the other hand, homotopy function was developed by using De Casteljau algorithms (Mohamad Nor et al., 2013). We focused on the improved HPM by Ghorbani and Saberi-Nadjafi (2008). In this method, the exactness is improved, and it is possible to find all possible exact and approximate solutions. HPM in Eq. (1.27) of Eq. (1.26) is reconstructed as

$$\mathscr{H}(v,\alpha,p)=(1-p)\left(L(v)-\sum_{r=0}^{N}\alpha_r g_r(x)\right)+p\left(L(v)+N(v)-f\right)=0,\ \ p\in[0,1], \tag{1.34}$$

where $\alpha=[\alpha_r],\ \ r=0,1,2...,N$ are called the accelarating parameters, and $g(x)=[g_r(x)],\ \ r=0,1,2...,N$ are the selective functions. Applying series (1.30) into Eq. (1.34), thus the solution of (1.26) is obtained in the form

$$L\left(\sum_{j=0}^{\infty}p^j v_j\right)=\sum_{r=0}^{N}\alpha_r g_r(x)+p\left(-N\left(\sum_{j=0}^{\infty}p^j v_j\right)-\sum_{r=0}^{N}\alpha_r g_r(x)+f\right),\ \ p\in[0,1]. \tag{1.35}$$

Comparing the similar power of parameter p, leads to the following iterations

$$\begin{aligned} L(v_0)&=\sum_{r=0}^{N}\alpha_r g_r(x),\\ L(v_1)&=-\sum_{r=0}^{N}\alpha_r g_r(x)-N(v_0)+f\\ L(v_k)&=-N(v_{j-1}),\ \ k=2,3,\ldots \end{aligned} \tag{1.36}$$

Most of the cases in the modified HPM (MHPM), the unknown coefficients α_r of v_0 in Eq. (1.36) are obtained by equating $v_1=0$ leads to $v_j=0,\ \ j\geq 2$ which implies two step method. So, the iteration method can be use to find the approximate solution by choosing selective functions g_r and unknown coefficients α_r.

1.5 Research Scopes and Objectives

1.5.1 Objectives

Objectives of thesis are:

1. To find the approximate solution and analyze convergence of linear FVIDE when $m=0$, $m=1$ and $m\leq 1$ in Eq. (1.12) by using HPM and MHPM.

2. Solving HSIE of the first kind in Eq. (1.24) with $K(x,t)=1$ and $K(x,t)\neq 1$. Analyzing the convergence of the approximate solution.

3. Solving HSIE of the second kind by using MHPM.

4. Find approximate solution for nonlinear Volterra Fredholm integral equation

by using MDM

1.5.2 Motivation

Improvement of HPM (Ghorbani and Saberi-Nadjafi, 2008) has been implemented to solve nonlinear integral equation of the form Eq. (1.14). Therefore, we applied MHPM (Ghorbani and Saberi-Nadjafi, 2008) to solve combination of Fredholm and Volterra of the form in Eq. (1.12).

Previous research focus on solving singular integral equation in Eq. (1.24) with $K(x,t) = 1$. In our studies, we consider $K(x,t)$ as a diagonal in a square region.

1.5.3 Scopes

Scope of the study is to find the error of approximate solution and the convergence of methods use. This study has been carried out within following scope:

1. Types of methods use and problems solve accordingly:

   (a) Use HPM to solve linear FVIDE when $m = 0$ and $m = 1$ and linear HSIE of the first kind when $K(x,t) = 1$ and $K(x,t) \neq 1$.

   (b) Use MHPM for FVIDE when $m \geq 1$, HSIE of the first kind when $K(x,t) = 1$ and $K(x,t) \neq 1$ and HSIE of the second kind.

   (c) Use MDM for NVFIE.

2. Convergence and error estimation are analyzed under following class:

   (a) FVIDE when $m = 0$ and $m = 1$ in clas $C[a,b]$

   (b) FVIDE when $m \geq 1$ in class $C^k(D)$,

   (c) linear HSIE of the first kind when $K(x,t) = 1$ and $K(x,t) \neq 1$ in class $\mathscr{L}_\rho[-1,1]$

1.6 Thesis Outline

The thesis is structured as follows: Chapter 1 provides the basic information on integro-differential and hypersingular integral equations. In addition, we describe a brief introduction on HPM and the improvement of HPM by the previous researchers. Chapter 2 presents all the related literature reviews. Application of HPM and modified HPM for solving Fredholm Volterra integro-differential equation in general case are discussed in Chapter 3. Convergence and error estimation are obtained and numerical examples are provided to prove the efficiency and accuracy of the method. In Chapter 4, HPM and modified HPM are used to solve HSIE with two conditions of kernels, $K(x,t) = 1$ and $K(x,t) \neq 1$. Convergence and error estimation are established along with numerical examples with comparison with past methods. Chapter 5 explains the used of modified HPM to solve HSIE of the second kind together with numerical examples. Modified decomposition method is implemented to obtain an approximate solutions of nonlinear Fredholm-Volterra equation of the second kind in Chapter 6 with providing some numerical results. Chapter 7 gives conclusion and suggestion for future works.

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BIODATA OF STUDENT

Fatimah Samihah Binti Zulkarnain born on 16th December 1988 in Manchester, UK. She who lives in Selangor, Malaysia has first education in Sekolah Kebangsaan Seri Serdang, Malaysia from 1995 until 2000. She continued her education for a secondary school in Sekolah Kiblah, Bangi, Malaysia for three years and Sekolah Menengah Kebangsaan Seri Serdang, Malaysia for two years later on. For pre-university education, she enrolled in Kolej Matrikulasi Negeri Sembilan.

She received B. Sc (Degree) in Applied Mathematics from Universiti Sains Malaysia, Penang, Malaysia in 2010. A year later, she gained M. Sc (Degree) in Science of Mathematics from the same university. She furthering her studies at Universiti Putra Malaysia, Serdang, Selangor, Malaysia as a PhD students in Applied Mathematics. Her research studies area cover computational mathematics, numerical methods and integral equations.

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