UNIVERSITI PUTRA MALAYSIA

NEW CLASSES OF BLOCK BACKWARD DIFFERENTIATION FORMULA FOR SOLVING STIFF INITIAL VALUE PROBLEMS

HAMISU MUSA

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By

HAMISU MUSA

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Doctor of Philosophy

June 2013

DEDICATIONS

To my parents

To my wife and children

"May ALLAH bestow His mercy upon them and make Al-Jannatul Firdaus their final aboard"

...

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Doctor of Philosophy

NEW CLASSES OF BLOCK BACKWARD DIFFERENTIATION FORMULA FOR SOLVING STIFF INITIAL VALUE PROBLEMS

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June 2013

Chair: Professor Dato' Mohamed Bin Suleiman, PhD

Faculty: Science

Implicit numerical methods for solving stiff Initial Value Problems (IVPs) are known to perform better than explicit ones. There has been a great deal of interest to develop implicit block and non-block numerical methods for solving stiff IVPs. One of the most popular methods is the Backward Differentiation Formula (BDF). The BDF still remain a foundation for most widely used algorithms.

In this thesis, new classes of block methods are developed for the solution of stiff initial value problems. The methods are based on the BDF and produce more than one solution value per step. The first class is a super class of the Block Backward Differentiation Formula (BBDF) and contains the BBDF as a subclass. This class has the advantage of generating different set of formulae with A-stability properties by simply varying a value of a parameter within the interval $(-1, 1)$. $2-$point and $3-$point block methods of constant step size belonging to this class are developed and codes are designed to implement the methods. The stability analysis of the

methods shows that they are A−stable. The performance of the methods in terms of accuracy is seen to outperform the non−block BDF and the BBDF methods of the same order. In addition, a 2−point variable step size superclass of BBDF method is formulated. The strategy for controlling the step size ratio is described. The problems tested indicate the method's suitability for solving stiff IVPs.

The second class of formulae developed involved the addition of an extra future point in the BBDF method to produce new formula called Block Extended Backward Differentiation Formula (BEBDF). 2−point and 3−point formulae of this class are also developed and their codes implemented. Using the same number of points, this class has the advantage of obtaining higher order A-stable methods than the BBDF. In addition, the accuracy is seen to be better than the BBDF. The stability of the methods developed is analyzed and the methods are found to possess A-stability properties.

Both the codes developed proved to be efficient for solving stiff initial value problems.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

KELAS BAHARU RUMUS PEMBEZAAN KEBELAKANG BLOK UNTUK MENYELESAIKAN MASALAH NILAI AWAL KAKU

Oleh

HAMISU MUSA

Jun 2013

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Kaedah berangka tersirat untuk menyelesaikan Masalah Nilai Awal (MNA) adalah diketahui berkelakuan lebih baik berbanding kaedah tak tersirat. Terdapat kepentingan yang besar untuk menerbitkan kaedah berangka blok tersirat dan tak-tersirat untuk menyelesaikan MNA kaku. Satu daripada kaedah paling popular adalah Rumus Pembezaan Ke Belakang (RPK). RPK masih kekal sebagai asas kepada penggunaan algoritma yang meluas.

Dalam tesis ini, kelas baharu kaedah blok dibangunkan untuk penyelesaian kepada masalah nilai awal kaku. Kedua-dua kaedah adalah berdasarkan kepada RPK dan menghasilkan lebih daripada satu nilai penyelesaian bagi setiap langkah. Kelas pertama adalah kelas terbaik bagi Rumus Pembezaan Kebelakang Blok (RPKB) dan mengandungi RPKB sebagai subkelas. Kelas ini mempunyai kelebihan untuk menjanakan set rumus yang berbeza dengan sifat A-kestabilan dengan secara mudah mempelbagaikan nilai parameter di dalam selang (-1,1). Kaedah blok 2-titik dan 3-titik bagi saiz langkah tetap bagi kelas ini dibangunkan dan kod dibina un-

tuk menggunakan kaedah terhasil. Analisis kestabilan bagi kaedah menunjukkan ianya adalah A-stabil. Prestasi rumus dari segi kejituan menunjukkan yang terbaik berbanding bukan blok RPK dan kaedah RPKB berperingkat sama. Seterusnya, kaedah kelas terbaik RPKB 2-titik bagi saiz langkah boleh ubah diterbitkan. Strategi dalam mengawal nisbah saiz langkah turut diberikan. Ujian masalah menunjukkan kecekapan kaedah dalam menyelesaikan MNA kaku.

Rumus kelas kedua dibangunkan melibatkan tambahan titik hadapan di dalam kaedah RPKB untuk menghasilkan rumus baharu dipanggil Rumus Pembezaan Kebelakang Lanjutan Blok (RPKLB). Rumus 2-titik dan 3-titik dalam kelas ini turut dibangunkan berserta kod implimentsinya. Dengan menggunakan bilangan titik yang sama, kelas ini mempunyai kelebihan dalam mendapatkan kaedah peringkat tinggi A-stabil berbanding RPKB. Malahan kejituan diperolehi adalah lebih baik berbanding RPKB. Kestabilan kaedah dianalisa dan kaedah diperolehi mempunyai sifat A-kestabilan.

Kedua-dua kod yang diterbitkan terbukti cekap untuk menyelesaikan masalah nilai awal kaku.

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Finally, I pray to Almighty Allah to bless the knowledge acquired.

This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Doctor of Philosophy. The members of the Supervisory Committee were as follows:

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DECLARATION

I declare that the thesis is my original work except for quotations and citations which have been duly acknowledged. I also declare that it has not been previously, and is not concurrently, submitted for any other degree at Universiti Putra Malaysia or at any other institution.

HAMISU MUSA

Date: 25/6/2013

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LIST OF ABBREVIATIONS

ODEs	Ordinary Differential Equations
BDF	Backward Differentiation Formula
IVPs	Initial Value Problems
LMM	Linear Multistep Method
L	Linear Operator
det	Determinant
LTE	Local Truncation Error
GMM	Generalized Multistep Method
EBDF	Extended Backward Differentiation Formula
BBDF	Block Backward Differentiation Formula
BEBDF	Block Extended Backward Differentiation Formula
1BDF	1 point BDF method
NBDF	1 point Non block variable step BDF method
2BBDF	2 point BBDF method
2SBBDF	2 point superclass of BBDF method
3SBBDF	3 point superclass of BBDF method
VSBBDF	2 point variable step size BBDF
VSSBBDF	2 point new variable step size superclass of BBDF
NS	Total number of integration steps
TS	Total steps taken
MAXE	Maximum Error
TOL	The tolerance used
IST	Number of successful steps
FS	Number of failed steps
2BEBDF	2 point Block Extended Backward Differentiation Formula
3BEBDF	3 point Block Extended Backward Differentiation Formula

CHAPTER 1

INTRODUCTION

Ordinary differential equation (ODE) is an equation containing a function, its derivatives and the variable it depends on. The general form of an ODE is

$$F\left(x, y, y', y'', ..., y^{(m-1)}\right) = y^{(m)} \tag{1.1}$$

where F is a given function of x, y and the derivatives of y. m is the order of the differential equation.

Many physical problems in science and engineering are formulated as ordinary differential equations (ODEs). For instance, problems in mechanics, electrical circuits, vibrations, chemical reactions, kinetics, population growth, economic growth e.t.c. can be modelled by differential equations. Such differential equations can be categorized into stiff and non stiff. Majority of both categories cannot be solved analytically and hence the use of suitable numerical scheme is advocated. Stiff differential equations describe equations where different physical phenomena acting on different time scales occur simultaneously. Solution to such differential equations is thus characterized by components with small and rapidly decay rate. Implicit methods proved to be more efficient for such systems and their development remained an interest in the study of stiff ODEs.

Considerable effort in dealing with stiffness has led to the development of many implicit methods. One of the most popular methods is the Backward Differentiation Formula (BDF). The method generates a sequence of values for the independent variable, $x_0, x_1, ...$, and a sequence of values for the dependent variable, $y_0, y_1, ...$, so that each y_n approximates the solution at x_n, $n = 0, 1, ...$ at one point per step.

The BDF is still a foundation for most widely used algorithms.

There has been a shift from solving (1.1) at one point per step, to two or more points (block methods, Hall (1976)) per step. Although a variety of block methods exist, the ones developed for stiff Initial Value Problems (IVPs) remain relatively small. One of the few algorithms available for stiff systems is based on the Block Backward Differentiation Formula (BBDF) developed in Ibrahim et al. (2007b). This work will focus on developing two classes of new block methods that are based on the BBDF for the integration of stiff system of IVPs. The first class will contain the BBDF as a subclass and the second class will be based on addition of a super future point to formulate higher order, but yet stable methods. The former will be called the Superclass of the BBDF while the later will be referred Block Extended BDF. In what follows, we give definitions of some related terms.

1.1 Basic Definitions

Definition 1.1 (ODE)

Ordinary differential equation (ODE) is an equation containing a function, its derivatives and the variable which they depend. The general form of an ODE is

$$F\left(x, y, y', y'', ..., y^{(n-1)}\right) = y^n \tag{1.2}$$

where n represents the highest derivative; y and its derivatives are functions of x.

Definition 1.2 (Order of an ODE)

The order of the differential equation (1.2) is the order of its highest derivative n.

Definition 1.3

Equation (1.2) is said to be linear if no product of the dependent variable $y(x)$ (with itself or any of its derivatives) occur. Otherwise, it is called non-linear.

Definition 1.4 (IVPs)

The system of IVPs of first order differential equation is defined by

$$\mathbf{y}' = \mathbf{f}(x, \mathbf{y}), \qquad \mathbf{y}(x_0) = \eta \qquad \eta = [\eta_1, ..., \eta_m] \tag{1.3}$$

Definition 1.5 (Stiff)

The definition of stiff has not been precise due to the fact that stiffness occurs in several applications of different nature. According to Shampine and Thompson (2007), no universally accepted definition of stiffness exists. Brugnano et al. (2011) compiled the following definitions of stiffness with respect to IVPs:

(1) *"Systems containing very fast components as well as very slow components"* Dahlquist (1973).

(2) *"Differential equations that represent coupled physical systems having components varying with very different times scales. i.e. they are systems having some components varying much more rapidly then the others"* Liniger (1972).

(3) *"A stiff system is one for which λ_{max} is enormous so that either the stability or the error bound or both can only be assured by unreasonable restriction on h (the step size)... Enormous means enormous relative to the scale which here is $\bar{x}$ (the integration interval)..."* Miranker (1975).

Definitions $1 - 3$ agree on a crucial point: *"the relation among stiffness and the appearance of different time-scales in the solutions"* Brugnano et al. (2011).

Brugnano et al. (2011) also argued that *"the most successful definitions seems to be the one based on particular effects of the phenomenon (stiff) rather than on the phenomenon itself"* and give the following examples:

(4) *"Stiff equations are equations where certain implicit methods ... perform bet-*

ter, usually tremendously better, than explicit ones" Curtiss and Hirschfelder (1952).

(5) *"Stiff equations are problems for which explicit methods don't work"* Hairer and Wanner (2004).

(6) *"If a numerical method with a finite region of absolute stability, applied to a system with any initial condition, is forced to use a certain interval of integration a step length which is excessively small in relation to the smoothness of the exact solution in that interval, then the system is said to be stiff in that interval"* Lambert (1991).

Lambert (1973) also defined stiff system of IVPs as follows:

(7) The linear system $\mathbf{y'} = \mathbf{Ay} + \rho(x)$ is said to be stiff if

(i) $Re(\lambda_t) < 0, t = 1, 2, \ldots, m$ and

(ii) $max_{t=1,2,\ldots,m} Re|\lambda_t| >> min_{t=1,2,\ldots,m} Re|\lambda_t|$

where λ_t, $t = 1, 2, \ldots, m$ are the eigen values of A.

The ratio

$$max_{t=1,2,\ldots,m} Re|\lambda_t| : min_{t=1,2,\ldots,m} Re|\lambda_t|$$

is called the stiffness ratio.

Theorem 1.1

Let $f(x, y)$ be defined and continuous for all points (x, y) in the region D defined by $a \leq x \leq b, -\infty < y < \infty$, a and b finite, and let there exist a constant L such that, for every x, y, y^* such that (x, y) and (x, y^*) are both in D,

$$|f(x, y) - f(x, y^*)| \leq L\,|y - y^*| \tag{1.4}$$

Then, if η is any given number, there exists a unique solution $y(x)$ of the IVP (1.3), where $y(x)$ is continuous and differentiable for all (x, y) in D. see Henrici

(1962)

The condition (1.4) is known as Lipschitz condition and the constant L is known as Lipschitz constant. Throughout this work, we shall assume that the above theorem is satisfied and establishes the existence of the unique solution of (1.3). The detailed proof of Theorem 1.1 can be found in Henrici (1962).

Definition 1.6 (Linear Multistep Method)

A general linear multistep method (LMM) has the form

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} \tag{1.5}$$

where α_j and β_j are constants and $\alpha_k \neq 0$. α_0 and β_0 cannot both be zero at the same time. For any k step method, α_k is normalised to 1.

The method (1.5) is said to be explicit if $\beta_k = 0$ and implicit if $\beta_k \neq 0$. In this research, we are going to deal with the implicit class of methods.

Definition 1.7

The Taylor's series expansion of $y(x_n + h)$ about x_n is defined by

$$y(x_n + h) = y(x_n) + hy^{(1)}(x_n) + \frac{h^2}{2!} y^{(2)}(x_n) + \ldots \tag{1.6}$$

where

$$y^{(q)}(x_n) = \tfrac{d^q y}{dx^q} \mid_{x=x_n}, \qquad q = 1, 2, \ldots.$$

Definition 1.8

The linear difference operator L associated with the LMM (1.5) is defined by

$$L\left[y(x),h\right]=\sum_{j=0}^{k}\left[\alpha_j y(x+jh)-h\beta_j y'(x+jh)\right] \qquad (1.7)$$

where $y(x)$ is an arbitrary test function and it is continuously differentiable on [a,b].

Expanding $y(x+jh)$ and $y'(x+jh)$ as Taylor series about x, and collecting common terms yields

$$L[y(x);h]=C_0 y(x_n)+C_1 h y'(x_n)+\ldots+C_q h^q y^{(q)}(x)+\ldots \qquad (1.8)$$

where C_q are constants given by

$$\begin{aligned} C_0 &= \alpha_0+\alpha_1+\alpha_2+\ldots+\alpha_k \\ C_1 &= \alpha_1+2\alpha_2+\ldots+k\alpha_k-(\beta_0+\beta_1+\beta_2+\ldots+\beta_k) \\ &\vdots \\ C_q &= \tfrac{1}{q!}(\alpha_1+2^q\alpha_2+\ldots+k^q\alpha_k)-\tfrac{1}{(q-1)!}(\beta_1+2^{q-1}\beta_2+\ldots+k^{q-1}\beta_k), \\ & \qquad q=2,3,\ldots \end{aligned} \qquad . \ (1.9)$$

Definition 1.9 (Order of LMM)

The linear operator (1.8) and the associated LLM (1.5) are said to be of order p if, in (1.8), $C_0=C_1=\ldots=C_q=0$; but $C_{p+1}\neq 0$.

Definition 1.10 (BDF)

When $\beta_0=\beta_1=\ldots=\beta_{k-1}=0$ and $\beta_k\neq 0$ in the LMM (1.5), the method becomes the BDF method and it takes the form

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\beta_k f_{n+k} \tag{1.10}$$

Definition 1.11 (Local Truncation error)

The local truncation error (LTE) at x_{n+k} is defined to be the expression $L[y(x_n); h]$ given by (1.7), when $y(x)$ is the theoretical solution of the given IVP.

Definition 1.12

The first and the second characteristic polynomial of the LMM (1.5) are defined by

$$\begin{aligned} \rho(\xi) &= \textstyle\sum_{j=0}^{k} \alpha_j \xi^j, \\ &. \\ \sigma(\xi) &= \textstyle\sum_{j=0}^{k} \beta_j \xi^j, \end{aligned} \tag{1.11}$$

respectively.

Definition 1.13 (Consistency)

The LMM (1.5) is said to be consistent if its order p≥ 1.

It follows from (1.9) that the LMM (1.5) is consistent if and only if

$$\begin{aligned} &\textstyle\sum_{j=0}^{k} \alpha_j = 0 \\ &\textstyle\sum_{j=0}^{k} j\alpha_j = \sum_{j=0}^{k} \beta_j \end{aligned} . \tag{1.12}$$

It also follows from (1.12) that the LMM (1.5) is consistent if and only if

$$\begin{aligned} &\rho(1) = 0 \\ &\rho'(1) = \sigma(1) \end{aligned} . \tag{1.13}$$

Definition 1.14

The characteristic polynomial of the method (1.5) is defined as

$$\pi(r,\bar{h}) = \rho(r) - \bar{h}\sigma(r) = 0 \tag{1.14}$$

where $\bar{h} = h\lambda$ and $\lambda = \frac{\delta f}{\delta y}$ is complex.

Definition 1.15 (Zero Stability)

The LMM (1.5) is said to be zero stable if no root of the first characteristic polynomial $\rho(\xi)$(defined by (1.11)) has modulus greater than one, and that every root with modulus one is simple.

Definition 1.16

The LMM (1.5) is said to be absolutely stable if in (1.14), $|r_s| \leq 1$, $s = 1, 2, \ldots, k$. The region of absolute stability is the set of the points in the $h\lambda-$plane for which the method is absolutely stable.

Definition 1.17

The LMM (1.5) when applied to the differential equation $y' = \lambda y$, where λ is a (complex) constant with negative real part, is said to be A$-$stable if all solutions of (1.14) tend to zero, as $j \to \infty$.

This implies that when λ is complex, the region of absolute stability is the entire left half of the $h\lambda-$plane. When λ is real, this implies that the method is absolutely stable in $(-\infty, 0)$. Jain (2003).

Definition 1.18 (Convergence)

The LMM (1.5) is said to be convergent if for all IVPs satisfying the conditions stated in Theorem 1.1 above, the following holds for all $x \in [a, b]$, and for all solutions y_n of the difference equation (1.5) satisfying the starting conditions $y_\mu = \eta_\mu(h)$ for which $\lim_{h\to 0} \eta_\mu(h) = \eta$, $\mu = 0, 1, 2, \ldots, k-1$:

$$\lim_{h\to 0, n\to\infty} y_n = y(x_n) \tag{1.15}$$

See Lambert (1973) for details.

A method is convergent if, *"as more grid points are taken or step size is decreased, the numerical solution converges to the exact solution, in the absence of round-off errors"* Jain (2003).

Theorem 1.2 (Convergence of LMM)

Henrici (1962) gave the following theorems on convergence of LMM

(1) A necessary condition for convergence of the LMM (1.5) is that the modulus of no root of the associated polynomial $\rho(\xi)$ (given in (1.11)) exceeds 1, and that the roots of modulus 1 be simple.

The condition thus imposed on $\rho(\xi)$ is called the condition of zero stability.

(2) A necessary condition for convergence of the LMM defined by (1.5) is that the order of the associated difference operator be at least 1.

The condition that the order $p \geq 1$ is called the condition of consistency.

Proof

For the proof of these theorems, refer to Henrici (1962).

Definition 1.19 (Block Method)

According to Hall (1976), block method is about *"the idea of simultaneously producing a block of approximations y_{n+1}, y_{n+2}, ..., y_{n+N} where N refers to the $N-$point block formula"*.

Definition 1.20

Vijitha-Kumara (1985) defined the following fixed step formula (in non−block form) of the BDF given in (1.10) by considering $\beta_0 = \beta_1 = \ldots = \beta_{k-2} = 0$, $\beta_{k-1} \neq 0$ and $\beta_k \neq 0$.

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\beta_k \left(f_{n+k} - \rho f_{n+k-1}\right) \qquad (1.16)$$

where $\beta_{k-1} = \rho\beta_k$. The value ρ is a free parameter which is restricted to $[-1, 1)$

Theorem 1.3

The method (1.16) is A$-$stable for $\rho \in [-1, 1)$.

Proof

The proof can be found in Vijitha-Kumara (1985).

Definition 1.21 ($L-$stability)

A method is $L-$stable if in addition to being A$-$stable, $\mid \phi(z) \mid \to 0$ as $z \to \infty$; where $\phi(z)$ is the stability function.

1.2 Problem to be Considered

Throughout the thesis, a system of first order IVPs of the form

$$\begin{array}{ll} \mathbf{y}' = \mathbf{f}(x, \mathbf{y}), & \mathbf{y}(x_0) = \boldsymbol{\eta} \\ \boldsymbol{\eta} = [\eta_1, ..., \eta_m], & x \in [a, b] \end{array} . \tag{1.17}$$

will be considered, where $\mathbf{y} = (y_1, y_2, \ldots, y_m)$. It is assumed that $\mathbf{f}(x, \mathbf{y})$ satisfies the following Lispchitz conditions:

(1) $\mathbf{f}(x, \mathbf{y})$ is defined and continuous for all points in the interval $a \leq x \leq b$, $-\infty < \mathbf{y} < \infty$ where a and b are finite.

(2) There exist a Lipschitz constant L such that for any $x \in [a, b]$ and any $\mathbf{y}$ and $\mathbf{y}^*$, $|\mathbf{f}(x, \mathbf{y}) - \mathbf{f}(x, \mathbf{y}^*)| \leq L|\mathbf{y} - \mathbf{y}^*|$

Theorem 1.4

If the system (1.17) satisfied the conditions 1 and 2 above, then there exists a unique solution $y(x)$ with the following properties:

(1) $y(x)$ is continuous and differentiable for $x \in [a, b]$,

(2) $y(x) = f(x, y)$ for $x \in [a, b]$,

(3) $y(a) = \bar{\eta}$

Proof

A detailed proof can be found in Coppel (1965).

1.3 Motivation of the Study

The BDF remains one of the renowned numerical method for integration of stiff IVPs. However, a famous result due to Dahlquist (1963) has shown that no A−stable LMM can have order greater than two. According to Bickart and Rubin (1974), *"the Dahlquist bound of two on the order of A−stable multistep methods was the imperative to propound ... weaker stability properties, ... An alternative approach for circumventing Dahlquist's bound is to modify the class of methods, rather than the property"* (quoting from Hairer and Wanner (2004)). Hairer and Wanner (2004) suggested the following ways of obtaining higher order A−stable methods:

(1) Using higher derivatives of the solutions;

(2) Throwing in additional stages, off−step points, super−points and the like, which leads into the large field of general linear methods.

In an attempt to circumvent the Dalhquist's barrier, Cash (1980) developed a class of Generalized Multistep Methods (GMM) called the Extended Backward Differentiation Formula (EBDF) which is based on the use of an additional future point in the BDF method, as suggested in Hairer and Wanner (2004). The method, which is L−stable up to order 4 and $L(\alpha)$−stable up to order 9, computes one solution value per step (non−block). The scheme showed an improvement in terms of accuracy over the conventional BDF. With the emergence of block methods, Ibrahim (2006); Ibrahim et al. (2007b) developed a formula called the Block Backward Differentiation Formula (BBDF) which produces r points simultaneously. The formulae posses A$-$stability properties.

The number of implicit block methods developed for the solution of stiff IVPs is quite small and the need arises for the development of more methods in that regard. In this thesis, we adopt the idea in Hairer and Wanner (2004) by throwing in additional future point to the conventional BBDF to come up with a new class of formulae in the sense of Cash (1980); called the Block Extended Backward Differentiation Formula (BEBDF); which are higher order A$-$stable and more accurate than the BBDF. In addition, Vijitha-Kumara (1985) developed a class of BDF formula of the form (1.10), but with the coefficient $\beta_{k-1} \neq 0$. The method is non-block and is found to be competitive with the BDF method. It is efficient for solving stiff IVPs. Therefore, another concern of this research is the development of a new class of formulae in the form of the BBDF, but with $\beta_{k-1,i} \neq 0$; suitable for solving stiff initial value problems. The formulae compute the solution in block and contain the BBDF method as a subclass.

1.4 Objectives of the Thesis

The main objective of the thesis is to develop implicit block methods suitable for the integration of stiff initial value problems. Specifically, the work is aimed at

(1) Introducing a non-zero coefficient ($\beta_{k-1,i}$) in the 2$-$point BBDF method in order to develop a new 2$-$point superclass of formulae that will contain the BBDF as a subclass and at the same time, compete with the BBDF in solving stiff IVPs.

(2) Introducing a non-zero coefficient ($\beta_{k-1,i}$) in the 3$-$point BBDF method

in order to develop a new 3−point superclass of formulae that will contain the BBDF as a subclass and at the same time, compete with the BBDF in solving stiff IVPs.

(3) Developing a new 2−point variable step size BBDF with non-zero $\beta_{k-1,i}$ coefficient.

(4) Adding a super "future point" to the 2−point BBDF to come up with a new 2−point block extended BDF, which is of higher order and A−stable.

(5) Adding a super "future point" to the 3−point BBDF to come up with a new 3−point block extended BDF, suitable for solving stiff IVPs..

(6) Developing codes based on the methods developed to solve stiff IVPs.

1.5 Layout of the Thesis

The thesis is arranged in 8 chapters as follows:

Chapter 1 presents an overview of stiff differential equations and some basic definitions that are related to the research. A general theory on convergence and stability analysis of multistep methods is given. The motivation of the research and its objectives are also stated.

A review of related literature that forms the basis for the research is given in chapter 2.

In chapter 3, a 2-point superclass of block backward differentiation formula is formulated. The order of the method is derived and the stability analysis presented. Consistency and zero stability conditions are established to show convergence of the method. Numerical results that compare the performance of the method with some existing methods for solving stiff problems are given.

A 3-point superclass of BBDF that computes 3-points simultaneously is derived in chapter 4. The method uses constant step size. The order of the method is determined. The method is analysed in terms of stability. The necessary and sufficient conditions to ensure convergence of the method are established. Some stiff IVPs are solved with the method and its performance is compared with some existing methods.

The variable step size derivation of the 2−point superclass of the BBDF is presented in chapter 5. The implementation of the method is described. The step size and ratio selection strategies are explained and a detailed stability analysis of the methods derived is given. Numerical results obtained are also presented and compared with some known results.

A class of new method called the block extended backward differentiation formula (BEBDF) is introduced in chapter 6. In this chapter, a detailed derivation of the 2-point BEBDF method is given. The order is determined and the method is analysed in terms of stability. Consistency and zero stability conditions are also established to give the convergence of the method. Numerical results obtained by solving stiff initial value problems are presented and the efficiency of the method is compared with existing results in the literature.

In chapter 7, the 3-point BEBDF is derived and the order of the method determined. The method is analysed in terms of stability. It is also shown that the method converges. Numerical results and their comparison with some existing methods are presented.

Finally, in chapter 8, we conclude the work with a brief discussion of the research carried out and the results obtained. An outline of a number of possible ways of extending this work for future research is given.

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BIODATA OF STUDENT

Educational Background

Hamisu Musa was born on 7^{th} January 1973 in Katsina, a state in the northern part of Nigeria. He started his primary education in the year 1979 at Rafindadi primary school Katsina. He obtained both junior and senior secondary school certificates at Arabic Teachers College Katsina. Hamisu received his national certificate in education from Federal College of Education Katsina in the year 1993. He later enrolled for a Bachelor of science and Master of science degrees at Bayero University Kano. He joined Universiti Putra Malaysia for a PhD programme in numerical analysis in the year 2009. He is married, with children.

Experience

In the year 2000, Hamisu joined the services of Isa Kaita College of Education Dutsin-ma as a lecturer and in January 2007, he joined Umar Musa Yar'adua University, Katsina State, Nigeria.

Publications

Hamisu has participated at both national and international conferences. He also attended many workshops and seminars during his candidature.
He has written thirteen (13) articles from this thesis, out of which 8 are journal articles and the remaining 5 are papers presented at various conferences. Currently, some of the journal articles have been published in indexed journals and the remaining are under review; also in indexed journals. Three of the conference papers are also published in indexed conference proceedings.

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