***NEW DEVELOPMENTS IN CONVERGENCE OF WAVELET EXPANSION OF FUNCTIONS $L^p(R^2)$, SOBOLEV SPACE $H^s(R^2)$ AND $L^p(S^2)$***

**RAGHAD SAHIB ABBAS SHAMSAH**

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**By**

**RAGHAD SAHIB ABBAS SHAMSAH**

**Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Doctor of Philosophy**

**March 2019**

**DEDICATIONS**

**To spirit of my dear Dad; Sahib Shamsah,**
**To my dear Mum; Fatam Abdulameer,**
**For all my loved ones; my dear husband; Akram Hassoon,**
**my kids; Shahad Mohammed Anareez.**

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment **of the requirement for the degree of Doctor of Philosophy**

# NEW DEVELOPMENTS IN CONVERGENCE OF WAVELET EXPANSION OF FUNCTIONS $L^p(R^2)$, SOBOLEV SPACE $H^s(R^2)$ AND $L^p(S^2)$

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**March 2019**

**Chairman : Hishamuddin Zainuddin, PhD**
**Faculty : Institute for Mathematical Research**

**In this work, we highlight to some methods that can develop the convergence of wavelet expansions under some new forms of partial sums operators. We improve some requirements on classical wavelet expansions on $R^2$ domain. In addition, it is** interesting to consider the spherical wavelets which are defined by polar coordinates **on $R^3$ domain, and establish a convergence of unique expansions called spherical wavelet expansions.**

**We introduce a generalization of wavelet expansions principle in two dimensions with new conditions under some associated operators which are Wavelet Projection Operator, Hard Sampling Operator and Soft Sampling Operator. The expansions can be generalized to expand functions for different types of functional spaces such as $L^p(R^2)$, Sobolev space $H^s(R^2)$ and $L^p(S^2)$. The wavelet expansions are analyzed by two methods of analysis which are classical multi-resolution analysis and spher-**ical multi-resolution analysis. We investigate the sufficient conditions for a wavelet **function and its expansions to achieve the convergence of wavelet expansions of the function under its related operators. For instance, after imposing a minimal regularity on the wavelet functions we can establish the rapidly decreasing property in two and four dimensions, that is, the expansion of any wavelet function is dependent on four integer parameters ( $j_1; j_2; k_1; k_2$) in analyzing the wavelet. It is important as well to take the boundedness property of wavelet expansions of functions**

**into consideration. A special technique is established to achieve the convergence of wavelet expansions of $L^p(R^2)$ and $L^p(S^2)$ functions by limiting the operator s magnitude with another bounded operator such as Hardy-Littlewood maximal operator and spherical Hardy-Littlewood maximal operator. While other techniques like use the boundedness condition of Zak transform and the structure of Meyer wavelet, are used to prove the convergence of wavelet expansions of Sobolev spaces functions with using high-regularity wavelet function. Some basic properties of wavelet functions as well as sharp examples are also given.**

**The performance of some partial sums operators developed by improving the conditions of their wavelet expansion. The two dimensional wavelet expansions of functions for some functional spaces converged in the two cases of classical wavelet and spherical wavelet expansions. Depending on some required properties for wavelet and its expansions, the convergence appeared almost everywhere along the Lebesgue set points of $L^p(R^2)$ and $L^p(S^2)$ functions. On the other hand, new type of convergence produced by making equivalent between the wavelet expansion and Fourier expansion of Sobolev space functions $H^s(R^2)$. By this, the partial sums operator behaved likes a truncated parts of inverse Fourier transformation, such that the convergence appeared uniformly at the singularity points of the partial sums operator kernel.**

**Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah**

# PERKEMBANGAN BAHARU DALAM PENUMPUAN PENGEMBANGAN GELOMBANG-KECIL BAGI FUNGSI DALAM RUANG $L^p(R^2)$, RUANG SOBOLEV $H^s(R^2)$ DAN $L^p(S^2)$

**Oleh**

**RAGHAD SAHIB ABBAS SHAMSAH**

**Marc 2019**

**Pengerusi : Hishamuddin Bin Zainuddin, PhD**
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**Dalam penyelidikan ini, kami ingin menekankan beberapa kaedah yang boleh membangunkan pengembangan gelombang-kecil di bawah beberapa bentuk baharu operator penjumlahan separa. Kami menambahbaik beberapa keperluan ke atas pengembangan gelombang-kecil klasik pada domain $R^2$. Sebagai tambahan, adalah juga menarik untuk pertimbangkan gelombang-kecil sfera yang dapat ditakrif dengan koordinat kutub dalam domain $R^3$ dan menentukan penumpuan pengembangan unik yang dipanggil pengembangan gelombang-kecil sfera.**

**Kami perkenalkan pengitlakan prinsip pengembangan gelombang-kecil dalam dua dimensi dengan syarat-syarat baharu di bawah beberapa operator berkaitan iaitu Operator Unjuran Gelombang-Kecil, Operator Pensampelan Keras, dan Operator Pensampelan Lembut. Pengembangan ini dapat ditlakkan untuk mengambang fungsi bagi ruang fungsian berbeza seperti $L^p(R^2)$, ruang Sobolev $H^s(R^2)$ dan $L^p(S^2)$. Pengembangan gelombang-kecil dianalisis dengan dua kaedah analisis iaitu analisis multi-resolusi klasik dan analisis multi-resolusi sfera. Kami mengkaji syarat-syarat cukup untuk fungsi gelombang-kecil dan pengembangannya bagi mencapai penumpuan pengembangan gelombang-kecil di bawah operator-operator berkaitan. Sebagai contoh, setelah mengenakan kenalaran minimum fungsi gelombang-kecil, kita akan perolehi ciri penurunan deras dalam dua dan empat dimensi iaitu pengembangan sebarang fungsi gelombang-kecilakan bergantung kepada empat parame-**

ter integer $(j_1; j_2; k_1; k_2)$ dalam menganlisis gelombang-kecil. Adalah juga penting untuk mengambilkira ciri keterbatasan pengembangan gelombang-kecil yang terlibat. Satu Teknik khas ditentukan untuk mencapai penumpuan pengembangan gelombang-kecil bagi fungsi-fungsi $L^p(R^2)$ dan $L^p(S^2)$ dengan menghadkan magnitude operator dengan operator terbatas lain seperti operator maksimum Hardy-Littlewood dan operator maksimum Hardy-Littlewood sfera. Sementara itu teknik selainnya menggunakan syarat keterbatasan jelmaan Zak dan struktur gelombang-kecil Meyer bagi membuktikan penumpuan pengembangan gelombang-kecil fungsi ruang Sobolev dengan menggunakan gelombang-kecil kenalaran tinggi. Beberapa sifat asas bagi gelombang-kecil serta contoh-contoh jelas ada diberikan.

Kelakuan sebahagian operator penjumlahan separa dibangunkan dengan menambahbaik syarat-syarat pengembangan gelombang-kecil mereka. Pengembangan gelombang-kecil dua dimensi fungsi bagi sebhagian ruang fungsian menumpu bagi dua kes pengembangan gelombang-kecil klasik dan gelombang-kecil sfera. Bergantung kepada beberapa syarat yang diperlukan untuk gelombang-kecil dan pengembangannya, penumpuan hamper menyeluruh muncul sepanjang titik-titik set Lebesgue bagi fungsi $L^p(R^2)$ dan $L^p(S^2)$. Selain itu, satu jenis penumpuan baharu dibangunkan dengan mengambil kesetaraan antara pengembangan gelombang-kecil dengan pengembangan Fourier bagi fungsi-fungsi ruang Sobolev $H^s(R^2)$. Dengan cara ini, operator penjumlahan separa akan berkelakuan seperti bahagian terpenggal transformasi Fourier songsang supaya penumpuan muncul sekata pada titik-titik singular bagi inti operator penjumlahan separa.

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I certify that a Thesis Examination Committee has met on 28 March 2019 to conduct the final examination of Raghad Sahib Abbas on her thesis entitled "New Developments in Convergence of Wavelet Expansion of Functions $L^P(\Re^2)$, Sobolev Space $H^S(\Re^2)$ and $L^P(S^2)$" in accordance with the Universities and University Colleges Act 1971 and the Constitution of the Universiti Putra Malaysia [P.U.(A) 106] 15 March 1998. The Committee recommends that the student be awarded the Doctor of Philosophy.

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**Declaration by graduate student**

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## LIST OF ABBREVIATIONS AND SYMBOLS

| | |
|---|---|
| y | Appropriate mother wavelet |
| $f_{j;k}$ | Scaling basis function |
| $y_{j;k}$ | Wavelet basis function |
| N | Natural numbers begin with zero |
| $R^2$ | The Cartesian coordinate system in two-dimensions space |
| $R^3$ | The Cartesian coordinate system for a three-dimensional space |
| $S^2$ | The two-dimensional sphere space |
| $L^2$ | Hilbert space |
| $L^p$ | Banach space |
| $H^s$ | Sobolev space |
| $C_c$ | The space of infinitely differentiable functions with compact support |
| $f(x)$ | Space s function |
| y | Fourier transform of wavelet |
| $a_{j;k}$ | Wavelet coefficient |
| $P_j$ | Projection operator on subspase $V_j$ |
| $_j$ | Projection operator on subspace $W_j$ |
| MRA | Multi-resolution analysis |
| SMRA | Spherical multi-resolution analysis |
| SMRA | The dual spherical multi-resolution analysis |
| d | Kronecker delta function |
| $h_{j;k;l}$ | Scaling filter coefficient |
| $g_{j;l;m}$ | Wavelet filter coefficient |
| M | Hardy-Littlewood maximal operator |
| $A_r$ | Maximal function |
| $B(x;r)$ | One dimensional ball with center point x and radius r |
| W | Lebesgue measure |
| m | Spherical Hardy-Littlewood maximal operator |
| C | Function space |
| | Set of numbers |
| | The closure set of the set |
| | Schwartz class s function |
| | Dual space |
| $L^1_{loc}$ | Locally integrable function |
| $I_{j;k}$ | Dyadic interval |
| $K(2^{J_1}x;2^{J_2}y;2^{J_1}c;2^{J_2}z)$ | Two dimensional kernel function |
| P | Wavelet projection operator |
| $T_l$ | Hard sampling operator |
| $T_l$ | Soft sampling operator |
| $S_{n_1;n_2}$ | Two dimensional partial summation operator |

# LIST OF ABBREVIATIONS AND SYMBOLS

| | |
|---|---|
| $T_{j;l}^{k}$ | **Triangular partition of the sphere** |
| $t_{j;k}$ | **Characteristic function** |
| $h_{j;k}$ | **Normalization constant** |
| $a_{j;k}$ | **The area of triangle $T_{j;k}$** |
| $l_{j;l}^{k}$ | Scaling coefficient on sphere |
| $g_{j;l}^{m}$ | Wavelet coefficient on sphere |
| ■ | **End of the proof** |

# CHAPTER 1

# INTRODUCTION

## 1.1 Introduction

This chapter will discuss the terminology and the notation utilized for the rest of the thesis. The reader can read through this chapter for the main fundamental concepts and ideas, and refer back to this chapter if needed to support the results and provided them.

## 1.2 Function Representation

There are several different ways to represent a mathematical function, often through a formula, a graph, or an algorithm. In the subsequent chapters, a set of basis functions will be proposed as representative of the functions on two-dimensional model of certain function spaces. This representation method is similar to the manner by which a vector can be expressed as a linear combination of basis vectors. Any space function spanned by the basis functions can be constructed as a distinct sequence of basis function coefficients. Projection refers to the process of converting a mathematical function to a basis representation, while reconstruction refers to the reverse process of creating a function using the basis function coefficients. Naturally, the amount of contribution that a basis function makes to the basis representation is determined by a basis function coefficient. Furthermore, the similarity between the mathematical function and a basis function is measured by a basis function coefficient. Thus, a basis properties and limitations are determined by the characteristics of its basis functions and its span. The next section will discuss important basis functions that are useful in this research.

## 1.3 Wavelet Functions

Sweldens (1995) exemplifies wavelets as:

> . . . building blocks that can quickly de-correlate data.

The description given by Sweldens integrates the three major features of wavelets. First, wavelets are considered to be analogous to building blocks. They serve as the basis for which one can represent functions via a linear combination of wavelet basis functions. Furthermore, the basis functions of wavelets possess structures that are very similar and therefore look like blocks. In fact, a first generation wavelet's basis functions are shifts and dilations of a mother wavelet function.

Second, the use of the word quickly refers to the existence of fast algorithms that can be used for function synthesis and analysis. These algorithms are able of transforming a function between its wavelet representation and its original representation in time that is linearly dependent on the function s size. It complemented this speed efficiency by considering the wavelets. Then, one can obtain the basis function coefficients using inner products between the function and the dual basis functions.

Finally, wavelets are able to decorrelate data in the function s space for that the wavelet representation to be more compact in comparison to the original representation of function. The wavelet basis function coefficients mostly are known to be small in magnitude because they centrate most of the energy in a few coefficients. Natural signal functions, unlike random noise, are correlated in frequency and space. For example, two neighboring pixels in an image are typically similar relative to compared to those pixels that are spaced farther apart. Similarly, in the frequencies domains, natural signal functions have a tendency to explain a correlation structure that shows localization in space, and decay towards the two ends of the frequency spectrum. This structure can be used so that compact representations can be obtained, where the greater number of the wavelet coefficients are nearer to zero in terms of magnitude. The concept of wavelets is to formulate basis functions that are able to capture local differences within a signal function. Thus, a large wavelet coefficient is indicative of a large degree of difference between the wavelet basis function and the mathematical function. Compact representations are more ideal because accurate approximations can be obtained by ignoring those with small coefficients. Thus, the approximation preserves its distinguished features while it discards the insignificant details.

### 1.3.1 First Generation of Wavelet

The classical or first generation wavelet can be characterized by three simple operators that are implemented within an appropriate mother wavelet $y \; 2 \; L^2(R)$: translation $t_k$ by $k$ defined by $(t_k y)(x) \quad y(x-k)$, dilation $r_{2^j}$ by $2^j$ defined by $(r_{2^j} y)(x) \quad y(2^j x)$ and multiplication by $2^{\frac{j}{2}}$. The wavelet basis functions are considered to be dyadic scales and translates of a mother wavelet y so that

$$y_{j;k}(x) \quad 2^{\frac{j}{2}} y\,(2^{j}x - k); \tag{1.1}$$

serves as an orthonormal basis of $L^2(R)$, where $j\ 2\ Z$ is used to define the basis functions scale or dilation and $k\,2\ Z$ is used to determine the translation on level $j$. If y represents a 2-dimensional classical wavelet function, then $y_{j,k}(x;y)$ can take on the following forms:

$$y_{j,k}(x;y) \quad y_{j_1;k_1}(x);y_{j_2;k_2}(y); \tag{1.2}$$

$$y_{j,k}(x;y) \quad 2^{(j_1 \quad j_2) \ 2} y\,(2^{j_1}x - k_1)y\,(2^{j_2}y - k_2); \tag{1.3}$$

such that $j \quad (j_1;j_2)$, $k \quad (k_1;k_2)$ and $j;k\ 2\ Z^2$.

**Proposition 1.1** One can normalize the classical wavelet basis functions $f\,y_{j;k}g$ such that

$$ky_{j;k}k_2 \quad ky\,k_2 \quad 1;$$

for all $j;k\,2\ Z$.

**Proof:**
The proof can be directly obtained from the wavelet basis function s structure ■

**Example 1.1** If

$$y\,(x) \quad \begin{cases} 1; & \text{if} \quad 0 \le x \quad \frac{1}{2}; \\ -1; & \text{if} \quad \frac{1}{2} \le x \quad 1; \\ 0 & \text{elsewhere}; \end{cases}$$

then y represents an orthonormal wavelet for $L^2(R)$. This is known as the Haar wavelet. Proving that $f\,y_{j;k} : j;k\,2\ Zg$ is an orthonormal system in $L^2(R)$ is easy.

**Example 1.2** Let y be a wavelet function such that

$$y\,(x) \quad c_I(x);$$

where y represents the Fourier transform of y and $c_I$ is its characteristic function defined on $I \quad [-2p;-p][\ [p;2p]$: To demonstrate that y has an orthonormal wavelet basis for $L^2(R)$, a simple calculation can be conducted:

$$y_{j;k}(x) \quad 2^{-\frac{j}{2}} y\,(2^{-j}x)e^{i2^{-j}kxx}:$$

For j 6 l this equality illustrates that the intersection of supports of $\left(y_{j;k}\right)$ and $\left(y_{l;m}\right)$ measures zero; hence,

$$\langle y_{j;k}; y_{l;m}\rangle \quad \frac{1}{2p}\langle (y_{j;k}); (y_{l;m})\rangle \quad 0;$$

since,

$$\langle y_{j;k}; y_{l;m}\rangle \quad \int_R y_{j;k}\overline{y_{l;m}}dx \quad \int_R \left(p\frac{1}{2p}\int_R y_{j;k}(x)e^{ixx}dx\right)\overline{\left(p\frac{1}{2p}\int_R y_{l;m}(x)e^{ixx}dx\right)}dx;$$

$$\frac{1}{2p}\int_R y_{j;k}(x)\overline{y_{l;m}(x)}dx \quad \frac{1}{2p}\langle y_{j;k}; y_{l;m}\rangle :$$

Hence,

$$\frac{1}{2p}\langle y_{j;k}; y_{l;m}\rangle \quad \frac{1}{2p}\int_R y_{j;k}(x)\overline{y_{l;m}(x)}dx;$$

$$\frac{1}{2p}\int_R \left(2^{-j}2 y(2^{-j}x)e^{i2^{-j}kxx}\right)\overline{\left(2^{-l}2 y(2^{-l}x)e^{i2^{-l}mxx}\right)}dx;$$

$$\frac{1}{2p}\int_R \left(2^{-j}2\int_R y(kx)e^{-i2^{-j}kxx}e^{i2^{-j}kxx}dx\right)\overline{\left(2^{-l}2\int_R y(mx)e^{-i2^{-l}mxx}e^{i2^{-l}mxx}dx\right)}dx;$$

$$\frac{2^{-j}2 2^{-l}2}{2p}\int_R\left(\int_R y(kx)dx\right)\overline{\left(\int_R y(mx)dx\right)}dx \quad 0:$$

Let j  l, we can write

$$\frac{1}{2p}\langle y_{j;k}; y_{j;m}\rangle \quad \frac{1}{2p}\int_R y_{j;k}(x)\overline{y_{j;m}(x)}dx;$$

$$\frac{1}{2p}\int_R \left(2^{-j}2 y(2^{-j}x)e^{i2^{-j}kxx}\right)\overline{\left(2^{-j}2 y(2^{-j}x)e^{i2^{-j}mxx}\right)}dx;$$

thus,

$$\frac{1}{2p}\langle y_{j;k}; y_{j;m}\rangle \quad \frac{1}{2p}2^{-j}\int_R \left|y\left(2^{-j}x\right)\right|^2 e^{i2^{-j}(k-m)xx}dx;$$

since, $\left|y\left(2^{-j}x\right)\right|^2 \quad 1$, let $h \quad 2^{-j}x$, $x \quad 2^{j}h$ and $dx \quad 2^{j}dh$:

$$\frac{1}{2p}\langle y_{j;k}; y_{j;m}\rangle \quad \frac{1}{2p}\left\{\int_{-2p}^{-p} e^{i(k-m)xh}dh \quad \int_{p}^{2p} e^{i(k-m)xh}dh\right\} \quad d_{k;m};$$

where $d_{k;m}$ is a kronecker delta function.

### 1.3.2 Classical Wavelet Expansions

The function $f$ can be represented by using wavelets basis function as an infinite series expansions of scaled and shifted versions of wavelet $\psi \in L^2(R)$, each one multiplied with a suitable coefficient is known as wavelet coefficient. Obviously, one can analyze the functions $f(x)$, with an arbitrary high precision, using a linear combination of the wavelets $\psi_{j;k}(x)$, i.e.

$$f(x) \approx \sum_{j} \sum_{k} a_{j;k}\psi_{j;k}(x); \tag{1.4}$$

where the summation first presented is over scales (from small to large) and, at every scale, all translates are summed over. One can obtain the coefficients as

$$a_{j;k} = \langle f;\psi_{j;k}\rangle = \int f(y)\psi_{j;k}(y)dy.$$

The set of all wavelet basis functions $\psi_{j;k}$ for a fixed $j$ can generate the subspace $W_j \subset L^2(R)$ as in the following expression:

$$W_j = \left\{\psi_{j;k}(x) \in L^2(R) : \psi_{j;k}(x) = 2^{j/2}\psi(2^j x - k); j;k \in Z\right\}.$$

It is noticeable how wavelets provide a time and scale representation of the multi-resolution analysis, where indices k and j represent the time location and scale of wavelet function, respectively. The wavelet expansion presented above is close to a Fourier series with the following dissimilarities:

- The wavelet expansion series is double indexed, which is indicative of scale and location.

- The basis functions have the property of time-scale (time-frequency) localization. It gives the ability to study function's features with details identified by their scale. This property is important for functions that are:

  1. Non-stationary (functions with time-varying parameters and not periodic).

  2. Having features at different scales.

  3. Having singularities.

With the use of an intermediate scale $j_0$, one can break down Equation (1.4) as two sums

$$f(x) \approx \sum_{j \quad j_0} \sum_{k \, -} \langle f; y_{j;k} \rangle y_{j;k}(x) \quad \sum_{j \, -}^{j_0} \sum_{k \, -} \langle f; y_{j;k} \rangle y_{j;k}(x): \qquad (1.5)$$

It is well known that for each wavelet function $y(x)\ 2\ L^2(R)$ there exist an auxiliary function $f(x)\ 2\ L^2(R)$ be called scaling wavelet function and meeting its certain properties.

$$\int_R f(y)dy \quad 1;$$

$$kf(x)k^2 \quad \int_R jf(y)j^2 dy \quad 1:$$

The function $f_{j;k}(x)$ is referred to as a scaling basis function that are defined to be similar in work to wavelet basis function $y_{j;k}(x)$, i.e.

$$f_{j;k}(x) \quad 2^{j\,2} f(2^j x - k):$$

The set of all scaling basis functions at a fixed $j$ can generate a subspace $V_j \subset L^2(R)$ as a following definition:

$$V_j \quad \left\{ f_{j;k}(x)\ 2\ L^2(R) : f_{j;k}(x) \quad 2^{j\,2} f(2^j x - k); j;k\ 2\ Z \right\}:$$

The second sum for Equation (1.5) may be expressed as a linear combination of $f_{j_0;k}$, such that we impose here $j_0$ is the intermediate scale level of analysis and contains the other level before it i.e.

$$\sum_{k \, -} \left\langle f; f_{j_0;k} \right\rangle f_{j_0;k}(x) \quad \sum_{j \, -}^{j_0} \sum_{k \, -} \langle f; y_{j;k} \rangle y_{j;k}(x): \qquad (1.6)$$

Consequently,

$$f(x) \approx \sum_{k \, -} \left\langle f; f_{j_0;k} \right\rangle f_{j_0;k}(x) \quad \sum_{j \quad j_0} \sum_{k \, -} \langle f; y_{j;k} \rangle y_{j;k}(x): \qquad (1.7)$$

This process is known to as the wavelet multi-resolution analysis. According to Equation (1.6), it is possible to analyze all the features of $f(x)$ that are smaller than the scale $2^{j_0}$ using a linear combination of the scaling function s $f(x)$ translates (over k) at the fixed scale $2^{j_0}$. This analysis can be reformulated using the projection operator $P_{j_0} f(x)$ on subspace $V_{j_0}$, i.e.

$$P_{j_0} f(x) \quad \sum_{k\ -} \left\langle f;f_{j_0;k} \right\rangle f_{j_0;k}(x); \tag{1.8}$$

where $V_{j_0}$ is a subspace of $L^2(R)$ generated by scaling basis functions $f_{0;k}$ at $j \quad 0$ as:

$$V_{j_0} \quad \left\{ f_{0;k}(x)\ 2\ L^2(R) : f_{0;k}(x) \quad f\,(x-k);k\,2\ Z \right\}:$$

The projection operator $\ _j f(x)$ on subspace $W_j$ can be defined as

$$_j f(x) \quad \sum_{j\ \ j_0} \sum_{k\ -} \left\langle f;y_{j;k} \right\rangle y_{j;k}(x):$$

Since $j_0$ is arbitrary starting scale, we also analyze the function at level $j_0 \quad 1$ by using the projection operator

$$P_{j_0\ \ 1} f(x) \quad P_{j_0} f(x) \quad\ _{j_0} f(x):$$

Thus, Equation (1.7) becomes

$$P_j f(x) \quad P_{j_0} f(x) \quad\ _j f(x);$$

where $j_0$ is arbitrary starting scale. In general the projection operator on sub space $V_{j\ \ 1}$ is

$$P_{j\ \ 1} f(x) \quad P_j f(x) \quad\ _j f(x); \tag{1.9}$$

where $P_j f(x)$ is the projection of function on subspace $V_j$ and $\ _j f(x)$ is the projection of function on subspace $W_j$ such that $L^2(R) \quad \oplus_{j\ -} V_j \oplus W_j$. This equation can be used to characterize the basic structure of orthogonal wavelet decomposition of Equation (1.4). It was previously mentioned that $P_j f(x)$ on subspace $V_j$ has all information about features of $f(x)$ that are considered larger than the scales $2^j$. Equation (1.9) makes this evident when one moves from the scale $2^j$ in subspace $V_j$ to the next scale that is bigger $2^{j\ \ 1}$ in the next subspace $V_{j\ \ 1}$. Then, some details are added to $P_j f(x)$, which is given by $\ _j f(x)$ in the subspace $W_j$ i.e. $W_j \quad V_{j\ \ 1} \ominus V_j$. Thus, one can say that $\ _j f(x)$, or consequently the functions wavelet expansion at any scale $2^j$, can characterize the difference observed between the projection at two varying scales $2^j$ and $2^{j\ \ 1}$, or equivalently at two varying resolutions. The following Figures 1.1 and 1.2 illustrate this process:

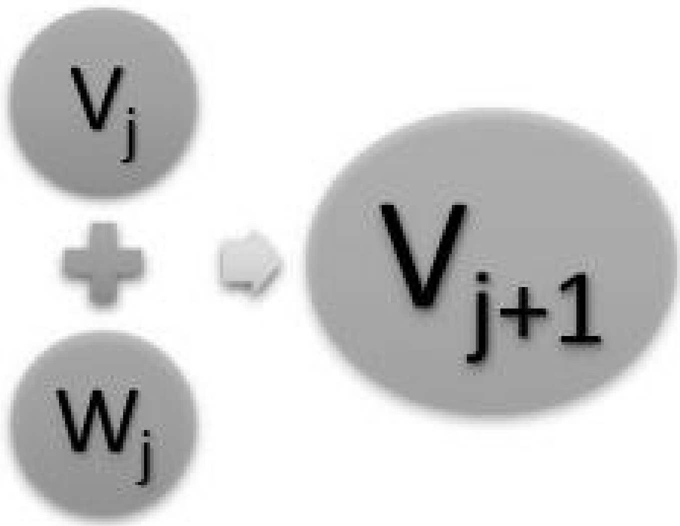


**Figure 1.1: Sketch Illustrates the Decomposition of Subspace $V_{j\;\;1}$**

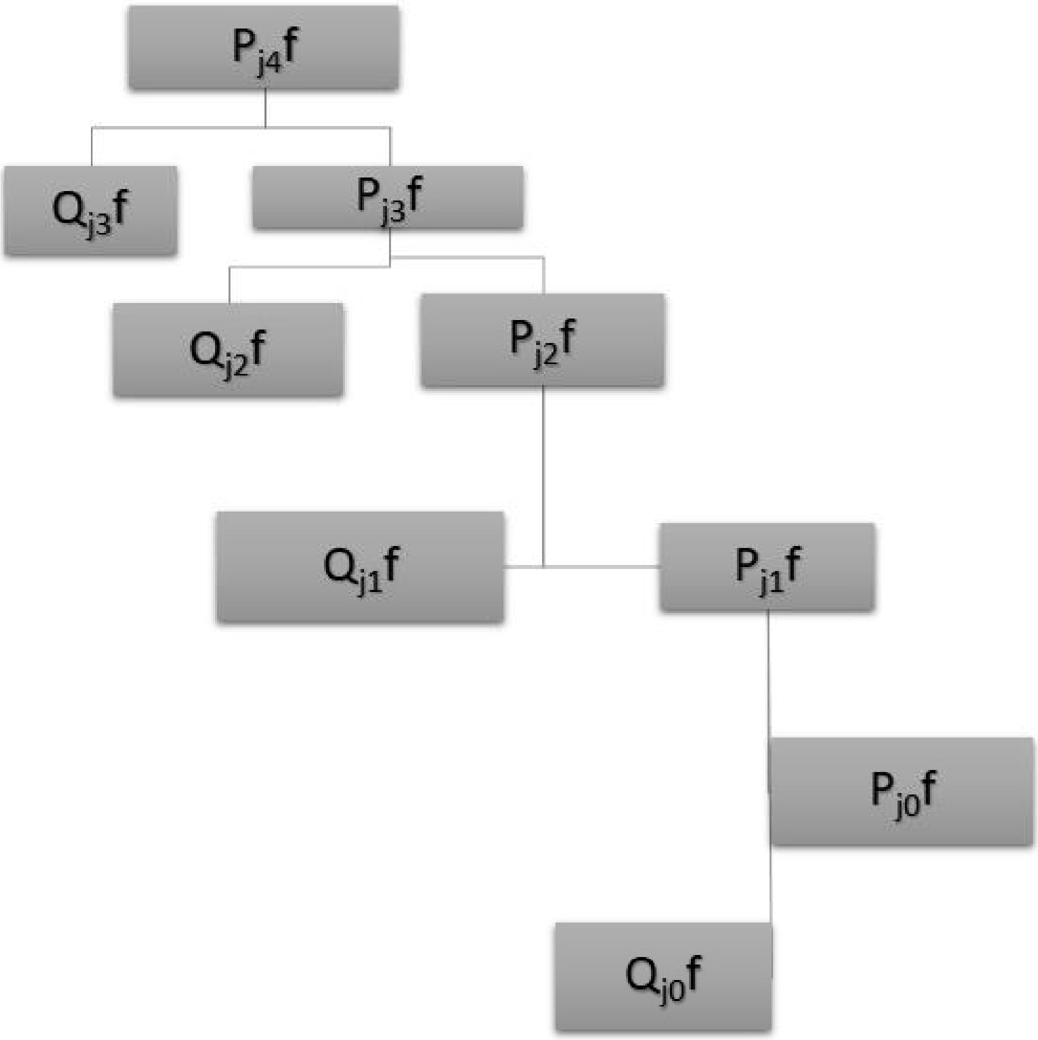


**Figure 1.2: Sketch Illustrates the Linear Combination of Projection Operator $P_j\, f(x)$ at 4-Levels of Multi-Resolution Analysis**

The following definition can be used to summarize the above process:

**Definition 1.1 The wavelet multi-resolution analysis (MRA) refers to the representation of a function within the nested structure. It is made up of a sequence of closed sub-spaces $\{V_j\}_{j\in Z}$ of the Hilbert spaces $L^2(R)$. One can then use these subspaces to characterize a function s behavior at varying resolutions or scales if the properties** *listed below are satisfied:*

**a)** $V_j \subset V_{j-1}$; **for all** $j \in Z$.

**b)** $f(x) \in V_j$ **if and only if** $f(2x) \in V_{j-1}$ **for all** $j \in Z$.

**c)** $\overline{\bigcup_{j\in Z} V_j} = L^2(R)$, **(MRA is dense in** $L^2(R)$**).**

**d)** $\bigcap_{j\in Z} V_j = \{0\}$:

**e) The existence of a function** $\phi \in V_0$ **implies that** $\{\phi(x-k) : k \in Z\}$ **is an orthonormal basis for** $V_0$.

**Note: A is dense in the space X if** $\bar{A} = X$**, where** $\bar{A} = A \bigcup \{\lim_n a_n : \forall n \geq 0; a_n \in A\}$:

**Sometimes, the functions** $\{\phi(x-k) : k \in Z\}$ **under condition (e) are assumed to be a Riesz basis for** $V_0$.
**For more details, one can read Hernandez and Weiss (1996); Meyer (1990).**

**Definition 1.2 (Riesz basis functions): A countable set of elements** $\{\phi_{j,k}(x) : j,k \in Z\}$ **represents a sequence of Riesz basis functions for the multi-resolution analysis (MRA) of** $L^2(R)$**. In other words, for every** $f \in V_j$ *there are countable infinite sequences* $\{a_{j,k}\}_{j,k\in Z} \in l^2(R)$ **in such a way that**

$$f(x) = \sum_{j,k\in Z} a_{j,k}\phi_{j,k}(x);$$

**and the inequalities that are presented below hold true**

$$\sum_{j,k} |a_{j,k}|^2 < \infty;$$
$$A\sum_{j,k} |a_{j,k}|^2 \leq \|\sum_{j,k\in Z} a_{j,k}\phi_{j,k}(x)\|_2^2 \leq B\sum_{j,k} |a_{j,k}|^2; \quad (1.10)$$

with 0 A 6 B that are constants independent of f.

For more details, one can read Wojtaszczyk (1997).

### 1.3.3 Second Generation of Wavelet

Spherical wavelets which are second generation wavelets overcome the limitations of first generation wavelets and allowed for the representation of functions in $L^2$ within the space of functions having finite energy, in a very general $L^2(R^n)$ setting. Thus, first generation wavelets are actually a subset of second generation wavelets on $R^n$, and where one uses the Lebesgue measure.

The wavelet basis functions were designed on the sphere with the use of multi-resolution analysis (MRA) defined on $S^2$, in generating bi-orthogonal bases. The results of this analysis is known as spherical multi-resolution analysis (SMRA), which led to a subdivision of surface of $S^2$ into subsets having an unequal area through the successive partitioning of triangles. There is always an attempt to refine a partition and not coarsen it.

A given function $f(w)$ has undergone expansion at multiple detail levels on $L^p(S^2)$ when using two sets of biorthogonal functions $y^m_{j;l}$ and $f_{j;k}$ defined on the sphere $S^2$. $K(j) \subset K(j \quad 1)$ and $M(j) \subset K(j \quad 1)$ are able to form a general index sets that have been given definition over the scaling functions and wavelet functions on level j, in such a way that, for every $j\,2\;N;k\,2\;K(j);l\,2\;K(j \quad 1)$ and $m\,2\;M(j)$. These two sets of functions $f\,y^m_{j;l}g$ and $f\,f_{j;k}g$ are used to form a Riesz basis function for $L^2(S^2)$ defined in Definition 1.2. It is not necessary for $y^m_{j;l}$ to be dilations and translations of the original wavelet y and to form a basis in each $W_j$.

Therefore, the $V_j$ and $W_j$ are Banach spaces. Based on the definition of a dual Banach space, the following represents the dual spherical multi-resolution analysis

$$SMRA \quad f\,\widetilde{V}_j;0 \leq j \leq J \quad j \subset Ng;$$

that is produces the dual spaces $V_j$ and $W_j$. The dual scaling functions $f\,f_{j;k}g$ provide a basis of the spaces $V_j$. $f\,f_{j;k}g$ need to be on the same level and biorthogonal to the primary scaling functions $f\,f_{j;k}g$, for that every dual scaling function $f_{j;k}$ complements a scaling function $f_{j;k}$. By a similar manner, for a given wavelet function $y^m_{j;l}$, there exists a dual wavelet $\widetilde{y}^m_{j;l}$, such that the dual and primary wavelet basis

**functions are biorthogonal as following:**

$$\int_{S^2} y^{m}_{j;l} y^{m}_{j;l} dw \quad d_{l;l} d_{m;m}:$$

$$\int_{S^2} f_{j;k} f_{j;k} dw \quad d_{k;k}: \tag{1.11}$$

**Based on the nested structure of the spaces $V_j$, it follows that the scaling functions** $f_{j;k}$ satisfy the following refinement relationship. Every $f_{j;k}$ **can be written as linear combination of scaling functions** $f_{j \;\; 1;l}$ that are situated at the next finer level:

$$f_{j;k} \qquad \sum_{l 2 K(j \;\; 1)} h_{j;k;l} f_{j \;\; 1;l}; \tag{1.12}$$

**where** $f h_{j;k;l} g$ represents scaling function's scaling filter coefficients. The spaces of **functions** $f V_j g_{0 \leq j \leq J}$ are defined by Equation (1.12) as follows:

$$V_j \quad \text{span} f f_{j;k} : 0 \leq j \leq J; k 2 K(j) g: \tag{1.13}$$

**For** $f_{j;k}$, an analogous relationship with dual scaling function filter coefficients $h_{j;k;l}$ holds true. The following assumes that all filters are uniformly bounded and of finite **extent. This is also implies that at each level of analysis j** there exist infinite index **sets $L(j;k)$ and $K(j;l)$ with**

$$L(j;k) \quad f l 2 K(j \;\; 1) j h_{j;k;l} 6 \;\, 0 g;$$

$$K(j;l) \quad f k 2 K(j) j h_{j;k;l} 6 \;\, 0 g:$$

**For the dual scaling functions, these analogous index sets $L(j;k)$ and $K(j;l)$ exist.** Unless otherwise specified, the assumption is that **l runs over $L(j;k)$, l runs over $L(j;k)$, k runs over $K(j;l)$ and k runs over $K(j;l)$.**

**The wavelet basis functions $f y^{m}_{j;l} j 0 \quad j 6 \;\, J; m 2 \;\, M(j) g$ are known to span the various spaces $W_j$, with**

$$V_j \oplus W_j \quad V_{j \;\; 1};$$

**and form a basis for the spaces $V_j$ and $W_j$ in which**

$$(\oplus_{j \;\; 0} V_j) \oplus (\oplus_{j \;\; 1} W_j) \quad L^2(S^2): \tag{1.14}$$

Based on the definition of the $y^{m}_{j;l}$ **over the subspace $W_j \subset V_{j \;\; 1}$, wavelet basis func-**tion filter coefficients $g_{j;l;m}$ **are found with**

$$y^{m}_{j;l} \qquad \sum_{l 2 K(j \;\; 1)} g_{j;m;l} f_{j \;\; 1;l}: \tag{1.15}$$

**Dual wavelet basis functions $y^{m}_{j;l}$ exist analogous to the primary wavelet basis func-**

tions $\psi^m_{j;l}$. These also cover the difference spaces $W_j$, with $V_j \oplus W_j = V_{j+1}$. Dual wavelet basis function filter coefficients $\tilde{g}_{j;l;m}$ are used in defining the $\tilde{\psi}^m_{j;l}$ as linear combinations of dual scaling functions $\tilde{\phi}_{j+1;l}$. Like $K(j;l)$ and $\tilde{K}(j;l)$, the definition of index sets $M(j;l)$ and $\tilde{M}(j;l)$ is likewise given, and $L(j;m)$ and $\tilde{L}(j;m)$ represent index sets that run over the nonzero wavelet basis function filter coefficients. Unless otherwise specified, the following assumes that $l$ runs over $L(j;m)$ and $l$ runs over $\tilde{L}(j;m)$ and $m$ runs over $M(j;l)$ and $m$ runs over $\tilde{M}(j;l)$. For more details one may refer the reader to Nielson et al. (1997); Sweldens (1998); Rosca and Antoine (2010).

A multi-resolution analysis (SMRA) is used in the second generation setting to give a definition to the wavelet basis functions on $L^2(S^2)$. Therefore, one can define the spherical multi-resolution analysis (SMRA) of the space $L^2(S^2)$ as follows:

**Definition 1.3** A spherical wavelet multi-resolution analysis SMRA $\{V_j \subset L^2(S^2) | 0 \leq j \leq J \in N\}$ represents the function of space as a sequence of nested subspaces $\{V_j\}_{0 \leq j \leq J}$ of $L^2(S^2)$ space by employing wavelet functions, if it meets the following requisites:

a) $V_j \subset V_{j+1}$.

b) $\overline{\bigcup_{j \geq 0} V_j} = L^2\left(S^2\right)$, (SMRA is dense in $L^2\left(S^2\right)$).

c) There is a scaling function $\phi(\omega) \in V_0$ present, where the sequence $\{\phi_{j;k}; j \geq 0; k \in K(j)\}$ serve as a Riesz basis for $V_j$. For more details, one can refer to Rosca (2005b).

### 1.3.4 Regularity of Wavelet Functions

In this subsection we redefine the expression of the regularity for Daubechies (1992), pp 153-155 and Bownik and Dziedziul (2015) as:

**Definition 1.4** The set of wavelet basis functions $\{\psi_{j;k}\}(x) \in L^2(R^n)$ is considered as r-regular functions for $(r \in N)$, if $\psi$ is selected in such a way that:

$$\left|\psi_{j;k}(x)\right| \leq \frac{2^{j/2} c_n}{\left[1 + \left|2^j x - k\right|\right]^n}; \tag{1.16}$$

Furthermore, the derivatives $^{a}y$ are bounded by

$$B_{a;n} \int_{R^n} 2^{-j\ 2}\left(1 \quad \left|2^{j}x-k\right|\right)^{n} \left| \ ^{a}y_{j;k}(x)\right| dx \quad ; \qquad (1.17)$$

for $n\ 2\ N$, $a \quad f\, a_1; a_2; ::::; a_n g$ there is a multi-index satisfying $ja\,j\ 6\ \ r$, $(j;k)\ 2\ Z$ and $c_n; B_{a;n}$ are constants.

## 1.4 Functional Spaces

### 1.4.1 Vector spaces

This section will provide a definition of vector spaces. The exact definition is given **below:**

**Definition 1.5** *A vector space over a field* **K R or C is considered a set V together with two operations and ·, that meet the eight axioms listed below. Elements of V are often referred to as vectors and elements of K are commonly referred to as scalars.**

**Let u; v and w represent arbitrary vectors in V, and a and b represent scalars in K. They satisfy the following axioms:**

**Associativity of addition, u (v w) (u v) w:**

**Commutativity of addition, u v v u:**

**Identity element of addition, an element 0 2 V, called the zero vector, exists such that v 0 v for every v 2 V:**

**Inverse elements of addition, there exists an element −v 2 V, for every v 2 V. This is called the additive inverse of v, and it works in such a way that v (−v) 0:**

*Scalar multiplication's compatibility with field multiplication, a*(**bv**) **(ab)v:**

**Identity element of scalar multiplication, 1 · v v, where 1 represents the mul-**

**tiplicative identity in K.**

**Distributivity of scalar multiplication with respect to vector addition, a(u v) au av:**

*Distributivity of scalar multiplication based on field addition,* **(a b) v av bv:**

**A function space is one of vector spaces. Furthermore, its vectors are functions. It** *commonly uses those that are defined by an inner product or a norm, they also have* **a distance between two vectors. This is especially true in the case of Hilbert spaces and Banach spaces, which are essential in this study.**

**Example 1.3 Given the function space [x;y] and let x;y 2 R with x y. Consider [x;y] is made up of functions f : [x;y] K that are continuous on [x;y]. One can** *define the addition and scalar multiplication as follow.*
**If f;g 2 [x;y], then f g 2 [x;y] is the function given by**

**(f g) (u) f(u) g(u);**

**for u 2 [x;y]:**
**If a 2 K and f 2 [x;y], then a f 2 [x;y] is the function given by**

**(a f) (u) a f(u);**

**for u 2 [x;y]:**

### 1.4.2 Normed Spaces

One can define a normed space as:

**Definition 1.6 Let X represent a vector space over R or C with a norm. Furthermore, a norm on X is a function k:k : X [0; ) such that:**
**1. For all x 2 X, kxk 0. If x 2 X, then kxk 0 iff x 0:**
**2. For all a 2 R and for all x 2 X, ka xk ja jkxk:**
**3. For all x;y 2 X, kx yk 6 kxk kyk:**

**Example 1.4 $R^n$ is a vector space over R, let**

$\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}; \; x = \begin{bmatrix} x_1 \\ : \\ : \\ : \\ x_n \end{bmatrix} \in \mathbb{R}^n$: *This length function satisfies the a norm's* **required properties and is referred to as the Euclidean norm on $\mathbb{R}^n$. Therefore, $\mathbb{R}^n$ is** *a normed space. Moreover, this is not the only norm that is definable on* **$\mathbb{R}^n$. Thus, by selecting various norms for a particular vector space, different normed spaces can be obtained. For example, $(\mathbb{R}^n; \|.\|_2)$, $(\mathbb{R}^n; \|.\|_1)$ and $(\mathbb{R}^n; \|.\|_\infty)$ all represent various normed spaces.**

**Definition 1.7 (Convergence in A Normed Space): Given $(X; \|.\|)$ as a normed space. Consider a sequence $(f_n) \in X$, $f_n$ that will then converge to some $f \in X$, if for each $\varepsilon > 0$, there is a number n that exists such that for each $n \geq N$, we will have $\|f_n - f\| < \varepsilon$, i.e. $\lim_{n \to \infty} f_n = f$.**

### 1.4.3 Banach Spaces

**Definition 1.8 A normed space $(X; \|.\|)$ can be referred to as a Banach space if in that space, every Cauchy sequence $(x_n)_{n \in \mathbb{N}} \in X$ converges to an element of the space, i.e. consider $(x_n)_{n \in \mathbb{N}}$ as a sequence in Banach space X and consider $x \in X$. The sequence $(x_n)_{n \in \mathbb{N}}$ converges to x if $\forall \varepsilon > 0$, $\exists a \in \mathbb{N}$ in such a way that $\forall n \in \mathbb{N}$ and $n \geq a$**, *that satisfies* **$\|x_n - x\| < \varepsilon$:**

**Theorem 1.1 (Debnath and Mikusinski (2005)) Let $(x_n)_{n \in \mathbb{N}}$ be a sequence of Banach space and let $h_n = x_1 + \dots + x_n$. The series $\sum_{n=1}^{\infty} x_n$ converges, that is, there is a convergent sequence $(h_n)_{n \in \mathbb{N}}$ if $\sum_{n=1}^{\infty} \|x_n\| < \infty$. If $\lim_{n \to \infty} h_n$ is denoted by $\sum_{n=1}^{\infty} x_n$, then $\|\sum_{n=1}^{\infty} x_n\| \leq \sum_{n=1}^{\infty} \|x_n\|$:**

**Proof:**
**For $j > n$, we have**

$$h_j - h_n = \sum_{i=n+1}^{j} x_i$$

$$\|h_j - h_n\| \leq \sum_{i=n+1}^{j} \|x_i\| \leq \sum_{i=n+1}^{\infty} \|x_i\| < \varepsilon;$$

for $j \geq n \geq N$:

It follows that $(h_n)_{n \in N}$ is a Cauchy sequence, and hence it is convergent. If

$$\lim_{n \to \infty} h_n = h.$$

By applying the following triangle inequality

$$\|x - y\|_p^p = \sum_{i=1}^{n} |x_i - y_i||x_i - y_i|^{p-1} \leq \sum_{i=1}^{n} |x_i||x_i - y_i|^{p-1} + \sum_{i=1}^{n} |y_i||x_i - y_i|^{p-1};$$

where $\|\cdot\|_p^p$ is the p-th power of the p-norm function $\|\cdot\|_p$, we obtain

$$|\|h_n\| - \|h\|| \leq \|h_n - h\|;$$

so that

$$\lim_{n \to \infty} \|h_n\| = \|h\|.$$

Since

$$\|h_n\| \leq \sum_{i=1}^{n} \|x_i\| \leq \sum_{i=1}^{\infty} \|x_i\|;$$

since we denote $\lim_{n \to \infty} h_n$ by $\sum_{n=1}^{\infty} x_n$ and by taking the limit we have

$$\left\|\sum_{n=1}^{\infty} x_n\right\| \leq \sum_{n=1}^{\infty} \|x_n\|.$$

■

**Example 1.5** Let the sequence $(f_n)_{n \in N} \in C([0;1];\|.\|_\infty)$, where

$$f_n = \frac{\sin(2\pi nx)}{n^2}.$$

*The Figure 1.3 shows the first few terms of the sequence. We notice that the terms* seem to converge to the zero function.

$$\|f_n - 0\|_\infty = \frac{1}{n^2}\|\sin(2\pi nx)\|_\infty = \frac{1}{n^2} < \varepsilon.$$

Since, $\|\sin(2\pi nx)\|_\infty = \sup_{n \in N} |\sin(2\pi nx)| = 1$, for all $n \geq N > \sqrt{\frac{1}{\varepsilon}}$.

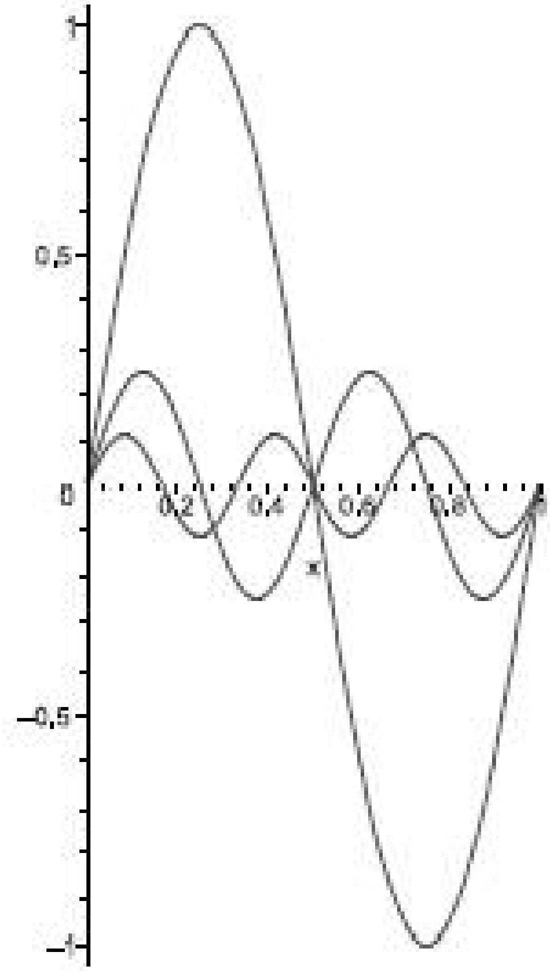


**Figure 1.3: The First Few Terms of the Sequence $(f_n)_{n\in\mathbb{N}}$:**

**Example 1.6** **Let the space $l^p$, for $1 \leq p$ . Then $l^p$** *can defined as:*

$$l^p \quad \left\{ x \quad (x_n)_{n\in\mathbb{N}} \,\middle|\, {}_{n\ 1}\ |x_n|^p \qquad \right\};$$

**with the norm**

$$\|x\|_p \quad \left( {}_{n\ 1}\ |x_n|^p \right)^{1\ p}:$$

**For p , the space l** *is defined by*

$$l \quad \left\{ x \quad (x_n)_{n\in\mathbb{N}} \,\middle|\, \sup_{n\in\mathbb{N}} |x_n| \qquad \right\}$$

with the norm

$$\|x\|_\infty = \sup_{n \in \mathbb{N}} |x_n| .$$

**Theorem 1.2 (Debnath and Mikusinski (2005))** The space $l^p$ is Banach spaces for $1 \leqslant p \leqslant \infty$.

**Proof:**
We prove this, for instance, in the case of the space $l^2$. From the inequality

$$|f_n + g_n|^2 \leq 2|f_n|^2 + 2|g_n|^2;$$

note that $l^2$ is equipped with the operations

$$(f_n)_{n \in \mathbb{N}} + (g_n)_{n \in \mathbb{N}} = (f_n + g_n)_{n \in \mathbb{N}};$$

where $(f_n)_{n \in \mathbb{N}}, (g_n)_{n \in \mathbb{N}} \in l^2$; since $a(f_n)_{n \in \mathbb{N}} = (a f_n)_{n \in \mathbb{N}}$, for $a \in K$ and $(f_n)_{n \in \mathbb{N}} \in l^2$ is a vector space. To show that $l^2$ is complete. Let $(f_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $l^2$. The proof of the completeness will be obtained by three steps:
Step 1. We seek a candidate limit $f$ for the sequence $(f_n)_{n \in \mathbb{N}}$. We have

$$\|f_k - f_n\|_2 < e;$$

for all $n, k \geq N$, that is

$$\sum_{i=1}^{\infty} |f_{ki} - f_{ni}|^2 < e^2 : \tag{1.18}$$

Thus for each $i \in \mathbb{N}$ and for every $n, k \geq N$

$$|f_{ki} - f_{ni}| < e :$$

That is, the sequence $(f_n)_{n \in \mathbb{N}}$ is convergent Cauchy sequence in $\mathbb{R}$

$$\lim_{n \to \infty} f_{ni} = f_i :$$

Step 2. We show that $f$ belong to the desired space (here $l^2$).
The sequence $f = (f_k)_{k \in \mathbb{N}}$ belongs to $l^2$. Let $m \in \mathbb{N}$ from the Equation (1.18).

$$\sum_{i=1}^{m} |f_{ki} - f_{ni}|^2 < e^2;$$

for all $n, k \geq N$. Let $k \to \infty$, then we have

$$\sum_{i=1}^{m} |f_i - f_{ni}|^2 \leq e^2;$$

for all n  N. Since for all m 2 N this is true , we obtain

$$j f_i - f_{ni} j^2 \quad e^2; \qquad \text{(1.19)}$$
$$i \;\; 1$$

for all n  N. This means that the sequence $f_n - f$, and the sequence f  $f - f_n$  $f_n$ belongs to $l^2$, for n  N.
Step 3. We show that

$$k f_n - f k \quad 0;$$

thus, $f_n$ converges to f in the normed space $l^2$.
The Equation (1.19) is equivalent with

$$k f_n - f k_2 \leq e$$

for all n  N, and so it follows that $\lim_{n} f_n$  f in the normed space $l^2$.
■

Theorem 1.3 (Holder s Inequality): For p  1, q  1 and $\frac{1}{p}$  $\frac{1}{q}$  1. If ( $f_n$) 2 $l^p$ and $g_n$ 2 $l^q$, then

$$\underset{n \;\; 1}{} j f_n g_n j \quad \left( \underset{n \;\; 1}{} j f_n j^p \right)^{1 \;\; p} \left( \underset{n \;\; 1}{} j g_n j^q \right)^{1 \;\; q}$$

Proof:
See Debnath and Mikusinski (2005) ■

Definition 1.9 (Orthogonality and Biorthogonality): Recall $d_{j;j^0}$ is the Kronecker delta function given by,

$$d_{j;j^0} \quad \begin{cases} 1 & \text{if} \quad j \quad j^0 \\ 0 & \text{otherwise:} \end{cases}$$

The elements of sequence $f x_j g_j^m \;_1$ in a Banach space X are orthogonal if

$$\left\langle x_j ; x_{j^0} \right\rangle \quad d_{j;j^0}:$$

The sequences $f x_j g_j^m \;_1$ 2 X and $f x_{j^0} g_{j^0}^m \;_1$ 2 X (X is a dual of Banach space X in which it is a Banach space of continuous linear maps X  R) are biorthogonal if

$$\left\langle x_j ; x_{j^0} \right\rangle \quad d_{j;j^0};$$

for $j; j^0$ 2 Z:

### 1.4.4 Hilbert Spaces

**Definition 1.10** *Hilbert space H refers to a vector space ( possibly infinite dimensional) that has an inner product* $\langle f; g \rangle$ *so that the norm is defined by* $\|f\|_2 \; \langle f; f \rangle^{1\;2}$. *For example, an infinite-dimensional Hilbert space is represented by* $L^2$, having a set of all functions $f : R \quad R$ *in such a way that there is a finite integral of* $f$ $(\int_R |f(x)|^2 dx \quad)$ over the whole real line. The inner product in this case is then

$$\langle f; g \rangle \quad \int_{-} f(x)g(x)dx$$

A Hilbert space will always be a Banach space, but the opposite is not always true. For more details refer the reader to Sasane (2017).

### 1.4.5 $C_c$ Class

**Definition 1.11** Let $C(\;)$ *to be a function space defined on a set of numbers* , the set of functions $\{f(x)\} \in C(\;)$ represent the class $C_c$ if the following conditions hold:

1. $f(x)$ *are infinitely smooth functions, i.e.* $\;^a f$ is uniformly continuous in the closure set $\;$, $\forall a \in N$.

2. $f(x)$ are compactly supported, i.e. $f \in C_c$ , $\; f \in C_c(\;)$ and $supp\{f\}$ is a compact subset of .

Note: The closure set is the set together with all of its limit points.

### 1.4.6 Schwartz and Dual Spaces

Schwartz theory is described in this section with several facts:

**Definition 1.12** A set of functions $\{y\} \in C \;(R)$ is called the Schwartz class $\;(R)$

if for every indices $s\ ;a\ 2\ Z$ , there exists a positive constant $C_{a\ ;s}$ such that

$$\sup_{x 2 R} \left| x^{a} \frac{{}^{s} y\ (x)}{x^{s}} \right| \quad C_{a\ ;s} \qquad : \tag{1.20}$$

**Definition 1.13** Let $f\ 2\ \ (R)$. Then Fourier transform operator $F :\ \ (R)\ \ \ \ (R)$ *defined as*

$$f(x) \quad (2p)^{\frac{-1}{2}} \int_{R} f(x) e^{-i(xx)} dx: \tag{1.21}$$

It is known that $f\ 2\ \ (R)$ when $f\ 2\ \ (R)$.
The inverse transformation $F^{-1} :\ \ (R)\ \ \ \ (R)$ is given by

$$f(x) \quad (2p)^{\frac{-1}{2}} \int_{R} f(x) e^{i(xx)} dx\ ; \tag{1.22}$$

**Definition 1.14 (Dual Space):** The space of continuous linear functional on Schwartz class $(R)$ s testing functions can be represented by $^{0}(R)$ and is then referred to as a dual space or a class of tempered distribution. Thus, when $v\,2\ \ ^{0}(R)$ and $f\ 2\ \ (R)$, the functional $v$ on function $f$ is denoted by $hv; fi$.

**Example 1.7** Let $y\ \ 2\ \ ^{0}(R)$, a functional $y\ \ 2\ \ ^{0}(R)$ is called the Fourier image of $y$ , if

$$hy\ ; fi \quad \langle f;y \rangle;$$

$8\,f\ 2\ \ (R)$:

We now recall the explanation of distribution theory. The main motivation behind distribution theory is extending the common linear operators on functions like the convolution, derivative and the Fourier transform. This is performed so that they can also be applicable to the non-smooth, singular, or non-integrable functions that regularly seen in both applied and theoretical analysis.

The distribution theory also aims to define appropriate structures on the spaces of the involved functions to make sure that suitable approximating functions converge, as well as the continuity of certain operators. For instance, the limit of derivatives has to be the same as the derivative of the limit, as well as some definitions for the limiting operation.

A distribution on R represents a linear mapping that takes a smooth function (having compact support) on R and turns it into a real number. For instance, the delta

**distribution refers to the map,**

$$f \quad f(0);$$

**while on R, any smooth function g brings about a distribution**

$$f \quad \int_{R} fg:$$

Thus, putting a formal definition for distribution is important.

**Definition 1.15 A distribution refers to a linear continuous functional on the smooth functions vector space with compact support on an open set U $\subseteq$ R. Thus, f C$^{0}$(U )g is denoted to the set of all distributions on U which is a linear continuous mapping C(U ) C.**

### 1.4.7 Sobolev Space

**Within the one-dimensional case (functions on R)**, one can define the Sobolev space **$H^{s;p}$ as the subset of functions f in $L^{p}(R)$ in such a way that the function f as well as its weak derivatives going up to some order s** have a finite **$L^{p}$ norm, for any given p, for (1 6 p 6 ).**

**Definition 1.16 For – s and p 2, the Sobolev spaces $H^{s;2}$** *is defined as* **follows:**

$$H^{s;2}(R) \quad \left\{ f\,2 \quad {}^{0}(R) : kfk_{H^{s;2}} \quad \right\};$$

**where**

$$kfk_{H^{s;2}} \quad \int_{R} (1 \quad jxj^{2})^{s} \left| f(x) \right|^{2} dx; \qquad \textbf{(1.23)}$$

**for all f;g 2 $H^{s;2}(R)$ and dx is the Lebesgue measure on R. The inner product in $H^{s}(R)$** *is defined by*

$$hf;gi \quad \int_{R} \left( j(x)j^{2} \right)^{s} f(x)\overline{g(x)}dx:$$

**Thus, $H^{s;2}$ space contains all tempered distributions f such that its Fourier transform belongs to $L^{2}_{1oc}(R)$.**

**For more information about functional spaces, we refer the reader to Debnath and Mikusinski (2005) and Dorina (2013).**

## 1.5 Hardy-Littlewood Maximal Operator

In this section the definition of Hardy-Littlewood maximal operator **M is re-described as:**

**Definition 1.17** **Let $|B|$ is a Lebesgue measure of a set B. For any $x \in \mathbf{R}^n$ and $r > 0$ let $B(x;r) = \{y \in \mathbf{R}^n : \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{\frac{1}{p}} < r\}$** *denote the open ball defined by* $L^p-$**norm in $\mathbf{R}^n$ with radius r and centred at point x. For any locally integrable function f** *defined on* $\mathbf{R}^n$*, we define the Hardy-Littlewood maximal operator M as:*

$$Mf(x) = \sup_{r>0} |A_r(f)(x)|;$$

**$A_r$ is a maximal function with**

$$A_r(f)(x) = \frac{1}{|B(x;r)|} \int_{B(x;r)} f(y)dy$$

$$|B(x;r)| = W_n r^n;$$

**where $W_n$ is the volume of unit ball B in $\mathbf{R}^n$ (i.e. Lebesgue measure of unit ball).** *More generally, we start with a fixed ball* $B(0;1)$ *containing the origin and define a* **maximal function using all the family of balls obtained using dilations and translations of $B(0;1)$.**
**For more details one can refer to see Grafakos (2014)(pp 85-104).**

**Example 1.8** **On R, let f be a characteristic function of interval $[a;b]$ of a set R as**

$$f(x) = \begin{cases} 1; & x \in [a;b]; \\ 0; & x \notin [a;b]: \end{cases}$$

**For $x \in (a;b)$, $x + d \geq b$ and $x - d \leq a$,**
**the Hardy-Littlewood maximal operator of f over all intervals $(x-d;x+d)$ is obtained as**

$$M(f)(x) = \sup_{d} \frac{1}{2d} \int_{x-d}^{x+d} |f(y)|dy$$

$$= \sup_{d} \frac{1}{2d} \int_{x-d}^{x+d} dy = 1:$$

For $x \geq b$, the maximum of $d \quad x - a$,

$$M(f)(x) \quad \sup_{d} \frac{1}{2d} \int_{x-d}^{x \quad d} |f(y)|dy$$

$$\frac{1}{2|x-a|} \int_{x-(x-a)}^{b} dy \quad \frac{b-a}{2|x-a|}:$$

For $x \leq a$, the maximum of $d \quad b - x$,

$$M(f)(x) \quad \sup_{d} \frac{1}{2d} \int_{x-d}^{x \quad d} |f(y)|dy$$

$$\frac{1}{2|b-x|} \int_{a}^{x \quad (b-x)} dy \quad \frac{b-a}{2|x-b|}:$$

## 1.6 Lebesgue Integrable Functions

**Definition 1.18** A function $f(x)$ *defined on* $R^n$ be Lebesgue integrable if there exists a sequence of step functions $f \, f_n g$ such that $\int f$ *can be defined by*

$$\int f(x)dx \quad \sum_{n \quad 1} \int f_n(x)dx;$$

and the following two conditions are hold:

1. $\sum_{n \quad 1} \int |f_n(x)|dx \quad ;$

2. $f(x) \quad \sum_{n \quad 1} f_n(x)$ for every $x \, 2 \; R^n$ such that $\sum_{n \quad 1} |f_n(x)| \quad :$

The space of all Lebesgue integrable functions on $R^n$ is denoted by $L^1(R^n)$.

## 1.7 Locally Integrable Function

**Definition 1.19 (Measurable Set): A set S is called measurable if the characteristic function of S is a locally integrable function.**

**Definition 1.20 (Locally Integrable Function): Let A is an open set in $\mathbf{R}^n$ and f : A C** *is a Lebesgue integrable function. When the Lebesgue integral of f is finite* **on all compact subsets K of A**

$$\int_K jfjdx \qquad ;$$

**then f is called locally integrable and the set f fg is denoted by $L^1_{1oc}(A)$.**

**Example 1.9 Let [a;b] and [c;d] be any compact intervals of R and f(x;y) 4xy is** *a function defined on* $\mathbf{R}^2$**. The function f 2 $L^1_{1oc}(K)$ is locally integrable function for all K [a;b] × [c;d] 2 $\mathbf{R}^2$ but f 2 $L^1(\mathbf{R}^2)$.**

$$\iint_K jf(x;y)jdxdy \qquad \iint_K j4xyjdxdy$$

$$\int_a^b 2jxjdx \int_c^d 2jyjdy \quad c \qquad ;$$

**where c is a constant. For explanation, since**

$$\int_a^b 2jxjdx \quad x^2\Big|_a^b \quad b^2 - a^2;$$

**for 0 a ≤ b:**

$$\int_a^b 2jxjdx \quad -x^2\Big|_a^b \quad -\left(b^2 - a^2\right);$$

**for a ≤ b 0:**

$$\int_a^b 2jxjdx \quad -\int_a^0 2xdx \quad \int_0^b 2xdx \quad a^2 \quad b^2;$$

**for a 0 b. Hence,**

$$\int_a^b 2jxjdx \quad \begin{cases} \left|b^2 - a^2\right|; & ab \quad 0 \\ a^2 \quad b^2; & ab \, 6 \; 0 \end{cases}$$

similar steps can follow with $\int_c^d 2|y|dy$.

## 1.8 Lebesgue Points Set

**Definition 1.21** (Lebesgue Points Set): Let $f(x) \in L^1_{loc}(\mathbb{R}^n)$, a point $x \in \mathbb{R}^n$ is a Lebesgue point of $f$ if the derivative $\left(\lim_{r \to 0} \frac{1}{|B(x;r)|} \int_{B(x;r)} f(y)dy\right)$ of Lebesgue integral of function $f$ exists and is equal to $f(x)$, i.e.

$$\lim_{r \to 0} \frac{1}{|B(x;r)|} \int_{B(x;r)} (f(y) - f(x))\, dy = 0; \tag{1.24}$$

where $B(x;r)$ is a ball centered at $x$ with radius $r > 0$, and $|B(x;r)|$ represents its Lebesgue measure.

**Definition 1.22** Let $f \in L^1_{loc}(\mathbb{R}^n)$, the set Lebesgue points of the function $f$ is $x \in \mathbb{R}^n$ such that

$$\lim_{r \to 0} \frac{1}{\omega_n r^n} \int_{|x| \le r} (f(x-t) - f(x))\, dt = 0.$$

The Lebesgue set points of a locally integrable function are closely related to the set of points $x \in \mathbb{R}^n$ such that the integral of $f$ is differentiable and the condition below holds:

$$\lim_{r \to 0} \frac{1}{\omega_n r^n} \int_{|x| \le r} f(x-t)dt = f(x);$$

where $\omega_n$ denotes the volume of the unit ball $B(0;1)$ in $\mathbb{R}^n$.

**Corollary 1.1** For $f \in L^1_{loc}(\mathbb{R}^n)$, the almost every $x \in \mathbb{R}^n$ is a Lebesgue point of $f$.

**Proof:**
One obtain that for any rational number $q$, the function $(f(x) - q)$ is a locally integrable function. Equation (1.24) implies that

$$\lim_{r \to 0} \frac{1}{\omega_n r^n} \int_{|x| \le r} (f(x-t) - q + q - f(x))\, dt = 0;$$

$$\lim_{r \to 0} \frac{1}{\omega_n r^n} \int_{|x| \le r} ((f(x-t) - q) - (f(x) - q))\, dt = 0; \tag{1.25}$$

for almost every $x\,2\ R^n$.

Let $G_q$ be a set, such that the statement in Equation (1.25) is not true, it has a zero measure, and does the set $G \quad \bigcup_{q2} \quad G_q$

Consider $x\,2\ R^n\, n\, G$ and let $e \quad 0$ and $q\,2 \quad$ such that

$$(f(x)-q) \quad {}^{e}\,2:$$

Thus, one get

$$\lim_{r \quad 0} \frac{1}{W_n r^n} \int_{jxj \quad r} (f(x-t)-f(x))dt$$

$$6 \lim_{r \quad 0} \frac{1}{W_n r^n} \int_{jxj \quad r} (f(x-t)-q)dt - \lim_{r \quad 0} \frac{1}{W_n r^n} \int_{jxj \quad r} (f(x)-q)dt:$$

By applying Definition 1.22 the first part of inequality becomes

$$\lim_{r \quad 0} \frac{1}{W_n r^n} \int_{jxj \quad r} (f(x-t)-q)dt \quad \lim_{r \quad 0} \frac{1}{W_n r^n} \int_{jxj \quad r} f(x-t)dt - \lim_{r \quad 0} \frac{1}{W_n r^n} \int_{jxj \quad r} qdt$$

$$\lim_{r \quad 0} \frac{1}{W_n r^n} \int_{jxj \quad r} f(x-t)dt - \frac{q(W_n r^n)}{W_n r^n}$$

$$f(x)-q:$$

Also, the second part of inequality

$$\lim_{r \quad 0} \frac{1}{W_n r^n} \int_{jxj \quad r} (f(x)-q)dt \quad \lim_{r \quad 0} \frac{1}{W_n r^n} \int_{jxj \quad r} \frac{e}{2}dt \quad \frac{e(W_n r^n)}{2(W_n r^n)} \quad \frac{e}{2}:$$

Since $x\,2\ G$, the first part of previous inequality converges to $(f(x)-q) \quad \frac{e}{2}$ as $r \quad 0$, while the second part is smaller than $\frac{e}{2}$. Thus,

$$\lim_{r \quad 0} \frac{1}{W_n r^n} \int_{jxj \quad r} (f(x-t)-f(x))dt\, 6\, (f(x)-q) - \frac{e}{2} \quad \frac{e}{2} - \frac{e}{2} \quad 0:$$

This shows that the Lebesgue set of $f$ is contained in $R^n$, and hence almost every point in $R^n$ is a Lebesgue point of $f$.

■

For more in information we refer the reader to Debnath and Mikusinski (2005).

## 1.9 Convergence

### 1.9.1 Almost Everywhere Convergence

**(Almost Everywhere Convergence of Wavelet Expansions): A wavelet expansion of $L^p(R^n)$ functions $f$, for $1 \leqslant p \leq \infty$ under a family of partial sum operators $\{P_{j;k} : j;k \in Z^n\}$, is almost everywhere convergent if**

$$\lim_{j;k \to \infty} P_{j;k} f(x) = f(x);$$

**for almost every $x$ in the Lebesgue set of $f$.**

### 1.9.2 Uniform Convergence

**(Uniformly Convergence of Wavelet Expansions): Suppose $E$ is a set and $P_n f(x) = \sum_{j=1}^{n} f_j(x)$ is a partial sums operator, for $x \in E$. We say that the sequence of partial sums $(P_n)_{n \in N}$ is uniformly convergent to function $f(x)$,** if we define

$$a_n = \sup_{x \in E} |P_n f(x) - f(x)|;$$

**then $P_n f(x)$ converges to $f$ uniformly if and only if $a_n \to 0$ as $n \to \infty$.**

**Example 1.10 Let a function $f(x) = x^2$, the Fourier series expansion of $f(x)$ converges uniformly on the interval $[-p;p]$. By doing some calculations, one obtains that on the interval $[-p;p]$ the Fourier series of $f(x) = x^2$ is given by**

$$f_N(x) = f(x) = \int_{-p}^{p} x^2 e^{-2pixx}\,dx = \frac{p^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos(nx):$$

**Hence,**

$$|f_N(x) - f(x)| = \left|4\sum_{n=1}^{N} \frac{(-1)^n}{n^2}\cos(nx) - 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos(nx)\right|$$

$$= \left|4\sum_{n=N+1}^{\infty} \frac{(-1)^n}{n^2}\cos(nx)\right| \leqslant 4\sum_{n=N+1}^{\infty}\left|\frac{(-1)^n}{n^2}\cos(nx)\right| \leqslant 4\sum_{n=N+1}^{\infty}\frac{1}{n^2}:$$

$$\lim_{N\to\infty}\sum_{n=N+1}^{\infty}\frac{1}{n^2} \le \lim_{N\to\infty}\int_{N+1}^{\infty}\frac{1}{x^2}dx = \lim_{N\to\infty}\left(\frac{-1}{x}\Big|_{N+1}^{\infty}\right) = \lim_{N\to\infty}\frac{1}{N+1} = 0.$$

Thus,

$$\lim_{N\to\infty}\left(\sup_{x\in[-\pi,\pi]}|f_N(x)-f(x)|\right) = 0.$$

This implies that the Fourier series expansion of $f(x) = x^2$ converges uniformly.

### 1.9.3 Absolutely Convergence

(Convergent and absolutely convergent series): A series $\sum_{n=1}^{\infty}|f_n|$ in a normed space $X$ is characterized as convergent if the sequence of partial sums converges in $X$, i.e. there exists $f \in X$ such that

$$\|f_1 + f_2 + \cdots + f_n - f\| \to 0$$

as $n\to\infty$. We then write $\sum_{n=1}^{\infty} f_n = f$. If $\sum_{n=1}^{\infty}|f_n| < \infty$, then the series is characterized absolutely convergent.

Example 1.11 To check the convergence of the series $\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{n}\right|$.

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n},$$

since

$$\int_1^{\infty}\frac{1}{x}dx = \ln(|x|)|_1^{\infty} = \infty.$$

from the integral test the alternating harmonic series $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ is absolutely divergent.

Example 1.12 To proof the absolutely convergent of the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+2}}{n^2}$, we need to check the convergence of

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^{n+2}}{n^2}\right| = \sum_{n=1}^{\infty}\frac{1}{n^2};$$

to estimate the partial sums,

$$\lim_{t} \int_{1}^{t} \frac{1}{x^2} dx \quad \lim_{t} \left(-\frac{1}{x}\right)\Big|_{1}^{t} \quad \lim_{t} \left(1 - \frac{1}{t}\right) \quad 1 \qquad :$$

This series is convergent by the integral test and so the series $\sum_{n \; 1} \frac{(-1)^{n \; 2}}{n^2}$ is absolute convergent.

## 1.10 Motivation

In numerous applications of mathematics, computer science, engineering, and physics, problems are encountered when a function is significantly more complex than the standard functions. Given these issues, this thesis will aim to provide a solution that will minimize the difficulties with the use of wavelet expansions through one of the partial sum operators. We will construct complicated functions using the powerful wavelet bases functions of wavelet expansions. They are advantageous because they only need a smaller amount of coefficients to accurately represent large data sets and general functions.

The most difficult problems in mathematics deal with complicated high-dimensional functions, which one can model them using partial sums operator. The main ingredient to solve these problems is re-finding a new formulas of Wavelet Projections Operator. Furthermore, the simple classical wavelets constructions are only restricted to simple domains like real line and intervals, that will not be enough for modelling scientific applications in high-dimensional spaces. This is especially true in physics, mathematics, and engineering. This is paves an idea to improve the performance of these operators by employing classical wavelet basis functions that are defined on the $R^2$ and $R^4$. Some problems in wavelet theory are noticeable when attempting to reconstruct the spherical surface functions $L^p(S^2)$ by wavelet basis functions in Wavelet Projection Operators. We employ the solutions by generate Spherical Wavelet Projection Operator with the use of spherical wavelet basis functions defined on $S^2$. They are then considered as keys that can be used to solve mathematical problems that take place within any dimension of spaces $\;\; 2$.

Furthermore, the motivation comes from convergence theory. This is due to the fact that mathematical meaning of wavelet expansion processes of functions are based on multi-resolutions analysis. This study will present the expansions of functions go to infinite-level of analysis. Therefore, determining if these processes do not result in any information loss during the analysis is important matter; for more explicit when the number of terms of partial sums increases, the partial sums should be nearer and nearer to a certain function (i.e. convergent). Motivated by this matter, we need to establish some conditions on wavelet that are suitable with the above new formulations of partial sums operator. In this thesis, the investigations are conducted mainly on the developments in convergence issues for partial sums operators of wavelet expansions of spaces function.

## 1.11 Problem Statement and Research Objective(s)

The methods that used to converge the wavelet expansion under some kind of partial sums operators should developed to cover the current gaps of the previous studies, as well as this, the performance of these operators should optimized such that the new forms of operators can model some open problems in the sciences. For that the following objectives are considered:

1. To prove some new properties of wavelet functions defined on $(\mathbf{R}^2)$ and $(\mathbf{S}^2)$.

2. To derive new form of partial sums operators of wavelet expansion in classical and spherical forms, that can study the behavior of $L^p(\mathbf{R}^2)$ and $L^p(\mathbf{S}^2)$ functions almost everywhere.

3. To derive new form of Wavelet Projection operators of Meyer wavelet expansion, that can study the behavior of distributions from Sobolev space $H^s(\mathbf{R}^2)$ at the singularity points.

4. To find the sufficient conditions on wavelet function and its expansion in the classical and spherical forms, that are important for convergence of the partial sums operators.

5. To establish new techniques to achieve the convergence of wavelet expansion of $L^p(\mathbf{R}^2)$ and $L^p(\mathbf{S}^2)$ functions almost everywhere and Sobolev space functions $H^s(\mathbf{R}^2)$ uniformly, under some kinds of partial sums operators (Wavelet Projection Operator, Soft Sampling Operator, Hard Sampling Operator, Spherical Wavelet Projection Operator).

6. To prove the almost everywhere convergence of Spherical Wavelet Projection Operator of SOHO wavelet expansion of $L^p(\mathbf{S}^2)$ functions.

## 1.12 Thesis Organization

**Chapter 1** is a review of some background tools that were utilized for our subject matter. It also introduces the basic definitions that are related to wavelet expansions. The motivation behind the idea, the problem statement and objectives, and the methods utilized for problem solving are also presented.

**Chapter 2** displays the literary works that have previously studied the related issues to this current study.

**Chapter 3** provides improvements in the work of wavelet expansions after modifying structures of main operators (Wavelet Projection Operator, Soft Sampling Operator, Hard Sampling Operator, Spherical Wavelet Projection Operator), These operators have a relationship with wavelet expansions.

**Chapter 4** addressed the problems related to the convergence of the some operators ; Wavelet Projection Operator, Soft Sampling Operator, Hard Sampling Operator of functions in $L^p(R^2)$ and Sobolev space $H^s(R^2)$ with the utilization of two-dimensional classical wavelet expansions.

**Chapter 5** interprets the issue of having the almost everywhere convergence of Spherical Wavelet Projection Operators of $L^p(S^2)$ functions. In this chapter spherical wavelet functions with their features are employed.

**Chapter 6** presents the conclusion based on the main findings of this thesis and how it contributes to the field of wavelet analysis. This chapter also gives the possibility of future works by suggesting a few problems in this area that future researchers can work on.

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