CONVERGENCE PROBLEMS OF THE EIGENFUNCTION EXPANSIONS FOR POLYHARMONIC OPERATORS

SITI NOR AINI BINTI MOHD ASLAM

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By

SITI NOR AINI BINTI MOHD ASLAM

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfillment of the Requirements for the Degree of Doctor of Philosophy

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DEDICATIONS

I would like to dedicate this thesis to my loving family.
To my parents, Mum and Dad, Faridah and Mohd Aslam.

To my beloved husband,
Muhammad Sohail

To my dearest sisters, Siti Sariah, Siti Zaleha, Siti Nabisah and Siti Noor Saira
and my only brother, Mohd Imran

for their Patience, Caring, Love and Support.
I wish all of us will achieve our dreams and always be together forever.

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfillment of the requirement for the degree of Doctor of Philosophy

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Chairman: Assoc. Prof. Gafurjan Ibragimov, PhD
Institute: Institute for Mathematical Research

This research focuses on convergence and summability problems of the eigenfunctions expansions of differential operators related to polyharmonic operator in closed domain. The polyharmonic operator $(-\Delta)^m, m \in \mathbb{Z}^+$ is the elliptic operator of order $2m$ with domain consists of classes of infinitely differentiable functions with compact support, which is a symmetric and nonnegative linear operator and has a self-adjoint extension. For domains with smooth boundary, the solution to these differential operator problems involves eigenfunction expansions associated with polyharmonic operator with Navier boundary conditions. Suitable estimations for spectral function of the polyharmonic operator by using the mean value formula for the eigenfunctions of the polyharmonic operator is established. These estimations enable us to show the uniformly convergence of the Riesz means of the spectral expansions related to polyharmonic operator in closed domain. The classes of differentiable functions used are Sobolev and Nikolskii classes. Subsequently, the results are applied to study the sufficient conditions for localization properties of the spectral expansions related to distributions. The conditions and principles for the localization of the Riesz means spectral expansions of distributions associated with the polyharmonic operator in closed domain are considered.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

MASALAH-MASALAH PENUMPUAN BAGI PENGEMBANGAN FUNGSI EIGEN UNTUK PENGOPERASI POLIHARMONIK

Oleh

SITI NOR AINI BINTI MOHD ASLAM

November 2018

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Kajian ini bertumpu kepada mengkaji masalah penumpuan dan penjumlahan oleh pengembangan fungsi eigen bagi pengoperasi boleh beza terkait dengan pengoperasi poliharmonik di dalam domain tertutup. Pengoperasi poliharmonik $(-\Delta)^m, m \in \mathbb{Z}^+$ ialah pengoperasi elips berperingkat $2m$ dengan domain yang mengandungi kelas fungsi boleh beza tak terhingga dengan sokongan padat, yang simetri dan pengoperasi linear bukan negatif dan juga mempunyai sambungan adjoin tersendiri. Untuk domain dengan sempadan yang lancar, penyelesaian kepada masalah pengoperasi boleh beza ini adalah melibatkan pengembangan fungsi eigen berkaitan dengan pengoperasi poliharmonik dengan syarat-syarat sempadan Navier. Anggaran yang sesuai untuk fungsi spektrum bagi pengoperasi poliharmonik dengan menggunakan formula nilai purata fungsi eigen bagi pengoperasi poliharmonik diterbitkan. Anggaran ini membolehkan kita menunjukkan bahawa penumpuan seragam oleh pengembangan spektrum purata Riesz adalah terkait dengan pengoperasi poliharmonik di dalam domain tertutup. Kelas-kelas fungsi boleh beza yang digunakan adalah kelas-kelas Sobolev dan Nikolskii. Kemudiannya, keputusan yang didapati digunakan untuk mengkaji syarat-syarat cukup untuk ciri-ciri penyempatan bagi pengembangan spektrum yang berkaitan dengan pengagihan. Syarat-syarat dan prinsip penyempatan oleh pengembangan spektrum purata Riesz bagi pengagihan terkait dengan pengoperasi poliharmonik di dalam domain tertutup juga dipertimbangkan.

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This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Doctor of Philosophy. The members of the Supervisory Committee were as follows:

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LIST OF ABBREVIATIONS

$\frac{\partial}{\partial x}$	Partial derivative
$\partial\Omega$	Boundary on domain Ω
Δ	Laplace Operator
Δ^2	Biharmonic Operator
$(-\Delta)^m$	Polyharmonic Operator
$u_n(x)$	Eigenfunctions
λ_n	Eigenvalues
$\mathbb{R}$	Euclidean Space
$\mathbb{R}^{\mathbb{N}}$	N-dimensional Euclidean Space
C^∞	Class of infinitely differentiable function
C_0^∞	Class of infinitely differentiable function with compact support
PDEs	Partial Differential Equations
$\theta^\alpha(x,y,\lambda))$	Kernel of Riesz means
W_p^α	Sobolev Classes
H_p^α	Nikolskii Classes

CHAPTER 1

INTRODUCTION

In the present thesis we deal with summability and localization problems connected to eigenfunction expansions of the polyharmonic operator on the closed domain. There are various vibrating systems in physics, which can be described by the polyharmonic equations. The problems, which appear in the study of such vibrating systems, are the reasons to develop the theory of eigenfunction expansions of the polyharmonic operators. The mathematical description of the physical processes taking place in real space is based on the spectral theory of differential operators. The most difficult engineering problems concerning heat and mass transfer processes can be modelled by partial differential equations (PDEs), particularly by polyharmonic equations.

The natural generalization of the well-known Laplace operator Δ is polyharmonic operator $(-\Delta)^m, m \in \mathbb{Z}^+$ which is symmetric, nonnegative linear operator in the classes of infinite times differentiable functions with compact support. The spectral expansions coincides with the partial sums of the eigenfunction expansions of the polyharmonic operator. In the case of bounded domains in N-dimensional Euclidean space, the problems on the convergence and localization of the eigenfunction expansions of the polyharmonic operator become very complicated. This research is devoted to investigate the problems of uniform convergence of eigenfunction expansions of the functions from classes of Nikolskii and Sobolev in closed domain. In order to establish summability of eigenfunction expansions of the polyharmonic operator in closed domain, the suitable estimations for spectral function of the polyharmonic operator near the boundary of the domain are obtained. As an application to the modern theory of mathematical physics the localization problems of the spectral expansions related to the distributions on closed domain are researched.

This chapter will provide some fundamental background regarding the ideas and concepts from spectral theory of differential operators connected with convergence problems of eigenfunction expansions. For further details to be made, we present the research objectives, motivation and a brief outline of the thesis to complete this chapter.

1.1 Motivation

Linear elliptic equations become apparent in several models describing various phenomena in the applied sciences. Problems involving biharmonic and polyharmonic operator occurs in numerous applications in physics and engineering such as in gas dynamics, hydrodynamics, elasticity theory and in other area of mechanics. The study of some of these problems leads to the boundary value problems for polyharmonic operators with the Navier boundary conditions corresponding to biharmonic and polyharmonic equation. These partial differential equations are complemented with the Navier boundary conditons correspond to the hinged plate model such as for thin elastic plates and stationary surface diffusion flow.

The solution to these differential operator problems involves eigenfunction expansions associated with polyharmonic operator with Navier boundary conditions in order to describe physically relevant situations. Our first motivation arises from the case when eigenfunction expansions are not convergence. Then the problem of summability will occur. Secondly, the boundary or initial conditions of the differential operators can be expressed by nonsmooth function. This lead to the study of eigenfunction expansions in the spaces of distributions.

The theory for elliptic equations of order greater than two is much less well developed. However, the developments of past several years promise to lay a foundation for the general theory of spectral expansions related to differential operators. Lately, many applications and increasing interest to discover new tools suitable for polyharmonic operators contributing to the development of polyharmonic equations in modern mathematical physics and engineering.

1.2 Eigenvalues and Eigenfunctions for Polyharmonic Operator

We begin with introducing the mean value formula for eigenfunction of Laplace operator.

Let G be an N- dimensional space and $\{u_n(x)\}$ be an arbitrary eigenfunction of Laplace operator in N- dimensional domain $\Omega \subset G$ corresponding to the eigenvalue, $\lambda_n > 0$, with $x \in \Omega$ and radius $r > 0$. Then the following formula holds:

$$\int_{\omega} u_n(x + r\omega)\, d\omega = (2\pi)^{N/2}\, u_n(x)\, (r\sqrt{\lambda_n})^{(2-N)/2}\, J_{(N-2)/2}(r\sqrt{\lambda_n}).$$

The eigenfunction $\{u_n(x)\}$ is the solution of the equation

$$\Delta u_n + \lambda u_n = 0, \qquad x \in G.$$

For $N = 2$, the mean value formula for the eigenfunctions $\{u_n(x,y)\}$ of the Laplace operator

$$\int_{0}^{2\pi} u_n(x + r\cos\theta, y + r\sin\theta)\, d\theta = 2\pi\, J_0(r\sqrt{\lambda_n})\, u_n(x,y)$$

Moiseev (1977) considered the equation of Laplace operator and established theorem for eigenfunction of Laplace operator up to the boundary. He proved that the estimations for the solution in closed domain can be obtained by the following theorem.

Theorem 1.2.1 *(Moiseev (1977)) Let $u(x,y) \in C(\overline{\Omega}) \cap W_{\ell}^{2}(\Omega)$ be a solution of the equation:*

$$\Delta u(x,y) + \mu^2 u(x,y) = f(x,y), \qquad f \in L_2(\Omega)$$

where $u|_{\partial\Omega} = 0, \mu = \mu_0 + ia, \mu_0 > 0, a \neq 0$. Then for all $f \in L_2(\Omega)$ one has:

$$||u(x,y)||_{C(\overline{\Omega})} \leq C \sqrt{\frac{\ln^2 \mu_0}{\mu_0}} \, ||f||_{L_2(\Omega)}.$$

Note that, let $\Omega \subset \mathbb{R}^N$ be a domain with smooth boundary $\partial\Omega$. We define the space $C(\overline{\Omega})$ as a set of all continuous functions in bounded domain $\overline{\Omega}$ and the space $L_2(\Omega)$ is defined as a set of all square integrable functions in Ω.

The basic object of our study will be the N-dimensional Euclidean space, $\mathbb{R}^N$, consisting of N-tuples of real numbers, namely,

$$\mathbb{R}^N = \{(x_1, x_2, ..., x_N) : x_1, x_2, ..., x_N \in \mathbb{R}\},$$

where N is a positive integer, i.e $N \in \mathbb{N}$. Elements of $\mathbb{R}^N$ are referred to as vectors in N-space for $N > 1$, while the elements of $\mathbb{R}$ are referred to as scalars. Given a vector $\mathbf{x} = (x_1, x_2, ..., x_N)$ in $\mathbb{R}^N$ and $1 \leq i \leq N$, the scalar x_i is called the i-th coordinate of x.

The algebraic operations on $\mathbb{R}^N$ can be done easily in a componentwise form. Thus, the sum of $(x_1, x_2, ..., x_N)$ and $(y_1, y_2, ..., y_N)$ is given by $(x_1 + y_1, x_2 + y_2, ..., x_N + y_N)$ and $\alpha(x_1, x_2, ..., x_N) = (\alpha x_1, \alpha x_2, ..., \alpha x_N)$.

We define the scalar product of $\mathbf{x} = (x_1, x_2, ..., x_N)$ and $\mathbf{y} = (y_1, y_2, ..., y_N)$ denoted as $\mathbf{x} \cdot \mathbf{y}$ by

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + ... + x_N y_N.$$

.

The distance between the elements $\mathbf{x} = (x_1, x_2, ..., x_N)$ and $\mathbf{y} = (y_1, y_2, ..., y_N)$ in $\mathbb{R}^N$ is defined by

$$|\mathbf{x} - \mathbf{y}| = \sqrt{\sum_{i=1}^{N} (x_i - y_i)^2}.$$

We denote the N-dimensional integral of a function over the domain Ω by

$$\int_{\Omega} f(\mathbf{x})\, d\mathbf{x} = \int_{\Omega} \ldots \int f(x_1, x_2, \ldots, x_N)\, dx_1 dx_2 \ldots dx_N,$$

where $\mathbf{x}$ represents the N - tuple $(x_1, x_2, \ldots, x_N)$ and $d\mathbf{x}$ is the differential of N-dimensional volume.

The representation of integral transformation: Let $r = |\mathbf{x} - \mathbf{y}|$ and $\omega = \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} = \frac{\mathbf{x}-\mathbf{y}}{r}$ then $d\omega$ is the element of solid angle at $\mathbf{x}$ one has

$$\begin{aligned} \int_{|\mathbf{x}-\mathbf{y}| \le R} f(\mathbf{y})\, \mathbf{dy} &= \int_{r \le R} f(\mathbf{y})\, \mathbf{dy} \\ &= \int_0^R \left(r^{N-1} \int_{\omega} f(\mathbf{x} + r\omega)\, d\omega \right) dr. \end{aligned} \tag{1.1}$$

Here, we also consider L_p spaces of functions whose p-th powers are integrable. For $1 \le p < \infty$, the $L_p(\mathbb{R}^N)$ spaces are the usual complex spaces of p-integrable functions in the N-dimensional real Euclidean space $\mathbb{R}^N$ which is defined by

$$L_p(\mathbb{R}^N) = \left\{ f \,\middle|\, ||f||_{L_p} = \left(\int_{\mathbb{R}^N} |f(\mathbf{x})|^p\, d\mathbf{x} \right)^{1/p} < \infty \right\}.$$

The following theorems and definition are fundamental inequalities involved in proving triangle inequality which can be found in Reed and Simon (1978).

Theorem 1.2.2 *(Minkowski's inequality) If $f, g \in L_p(\mathbb{R}^N)$ where $1 \le p < \infty$, then $f + g \in L_p(\mathbb{R}^N)$ and*

$$||f+g||_{L_p} \le ||f||_{L_p} + ||g||_{L_p}.$$

For the second inequality, we first define the Holder conjugate.

Definition 1.2.1 *Let $1 \le p < \infty$. The Holder conjugate, p' of p is defined by*

$$\frac{1}{p} + \frac{1}{p'} = 1 \qquad if \;\; 1 < p < \infty.$$

Remark that, for $1 \le p < \infty$ the Holder conjugate of p' is p.

Theorem 1.2.3 *(Holder's inequality) For $1 \le p < \infty$. If $f \in L_p(\mathbb{R}^N)$ and $g \in$*

$L_{p'}(\mathbb{R}^N)$, *then* $fg \in L_p(\mathbb{R}^N)$ *and*

$$\int |fg| \, d\mu \leq ||f||_{L_p} ||g||_{L_{p'}}.$$

The Holder's inequality is valid for any p. But for the case when $p = p' = 2$, this inequality becomes the Cauchy-Schwartz inequality.

The L_2 space is a special case of L_p space which is also known as the Lebesgue space. It is a Hilbert space with its norm given in terms of its inner product. The L_2 space denotes the set of square integrable functions. It is a function space where the set of functions whose squares are summable which is not only a normed linear space but also an Euclidean space.

We define an inner product on L_2 by

$$< f, g >= \int f(x) \, g(x) \, d\mu(x).$$

This space is important in the study of complete orthonormal system of eigenfunctons, orthogonality of eigenfunctions and self-adjoint operator.

In the study of mathematical physics, a classical example of the problem of eigenfunctions for an elliptic operator is the problem of the eigenfunctions for the biharmonic operator. We consider the following problem

$$\begin{cases} \Delta^2 u + \lambda u = 0 & \text{in } \Omega, \\ u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0. \end{cases} \tag{1.2}$$

For this problem, the other types of the boundary conditions also can be considered to replace boundary condition with $u|_{\partial\Omega} = \frac{\partial u}{\partial \nu}|_{\partial\Omega} = 0$, where $\frac{\partial}{\partial \nu}$ denotes derivative by normal vector ν.

The biharmonic operator in (1.2) has a complete orthonormal system of eigenfunctions $\{u_k(x,y)\}$ in $L_2(\Omega)$ that corresponds to the sequence of eigenvalues $\{\lambda_k\}$. The Fourier series in eigenfunction of polyharmonic operator is represented as follows

$$\sum_{h=1}^{\infty} f_k u_k(\mathbf{x}), \tag{1.3}$$

where we denote $f_k = \int\limits_{\Omega} f(\mathbf{x}) \, u_k(\mathbf{x}) d\mathbf{x}$ as Fourier coefficients of $f(\mathbf{x})$ in the system $\{u_k(\mathbf{x})\}$.

The partial sums $E_\lambda f(\mathbf{x}) = \sum_{\lambda_k<\lambda} f_k u_k(\mathbf{x})$ of (1.3), can also be considered using the Riesz means of order $\alpha \geq 0$

$$(E_\lambda^\alpha f)(\mathbf{x}) = \sum_{\lambda_k<\lambda} f_k\, u_k(\mathbf{x}) \left(1-\frac{\lambda_k}{\lambda}\right)^\alpha .$$

The Fourier series (1.3) is said to be summable at $\mathbf{x}$ by Riesz means of order α to $f(\mathbf{x})$ such that

$$\lim_{\lambda\to\infty} \sum_{\lambda_k<\lambda} f_k\, u_k(\mathbf{x}) \left(1-\frac{\lambda_k}{\lambda}\right)^\alpha = f(\mathbf{x}).$$

For $\alpha = 0$, the summability of Riesz means of order α is simply an ordinary convergence.

Denote $C^\infty(\Omega)$ as a space of all smooth (infinitely differentiable) functions on domain Ω. Let $C_0^\infty(\Omega)$ be the space of infinitely differentiable functions in Ω with compact support in Ω. We say that a function is finite in Ω if it has compact support Ω_0. In general, a function which is finite in Ω and is defined in the space $\mathbb{R}^N$ is equal to zero outside a bounded set contained in Ω and a positive distance from the boundary $\partial\Omega$ of Ω.

We denote by A an operator in the Hilbert space $L_2(\Omega)$ with the domain of definiton $D(A) = C_0^\infty(\Omega)$ satisfying the following properties

$$Au = (-\Delta)^m\, u(\mathbf{x}), \qquad u \in C_0^\infty(\Omega).$$

This differential operator is self-adjoint if $(Au,v) = (u,Av)$ for any $u,v \in C_0^\infty(\Omega)$ (symmetric). Then, the operator A is semibounded if there exists a constant μ such that $(Au,u) \geq \mu(u,u)$ for all $u \in C_0^\infty(\Omega)$. From Friedrichs's theorem, every symmetric semibounded operator A has at least one self-adjoint extension $\hat{A}$ with the same lower bound μ. There exists an operator $\hat{A}$ with the following properties

1. $\hat{A}$ is self-adjoint:
2. $(\hat{A}u,u) \geq \mu(u,u), u \in D(\hat{A})$;
3. $D(A) \subset D(\hat{A})$.
4. $\hat{A}u = Au, u \in D(A)$.

(Von Neumann's spectral theorem) Similar to every self-adjoint operator, the operator $\hat{A}$ has a partition $\{E_\lambda\}$ of unity which is represented in the form of

$$\hat{A} = \int_\mu^\infty \lambda\, dE_\lambda .$$

The projections E_λ increase monotonically, are continous on the left and tend strongly to the unit operator

$$\lim_{\lambda\to\infty} ||E_\lambda u - u||_{L_2(\Omega)} = 0, \quad u \in L_2(\Omega).$$

We consider the polyharmonic operator $(-\Delta)^m$ for $m \in \mathbb{Z}^+$ with domain $D_B = \{u \in C^\infty(\overline{\Omega}) : u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = ... = \Delta^{m-1}u_n|_{\partial\Omega} = 0\}$, where Δ denotes the well-known Laplace operator:

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + ... + \frac{\partial^2}{\partial x_N^2}.$$

The iterations of Laplace operator can be represented as follows

$$\Delta^m u = \sum_{\ell_1+\ell_2+...+\ell_N=m} \frac{m!}{\ell_1!\ell_2!...\ell_N!} \frac{\partial^{2m}u}{\partial x_1^{2\ell_1}\partial x_2^{2\ell_2}...\partial x_N^{2\ell_N}},$$

which can be established by using mathematical induction and the following relation between powers of Laplace operator

$$\begin{aligned}
\Delta^{k+1} = \Delta(\Delta^k) &= \Delta\Big(\sum_{\ell_1+\ell_2+...+\ell_N=k} \frac{\partial^{2k}}{\partial x_1^{2\ell_1}\partial x_2^{2\ell_2}...\partial x_N^{2\ell_N}}\Big),\\
&= \sum_{\ell_1+\ell_2+...+\ell_N=k} \Big(\frac{\partial^{2k+2}}{\partial x_1^{2\ell_1+2}\partial x_2^{2\ell_2}...\partial x_N^{2\ell_N}} + \frac{\partial^{2k+2}}{\partial x_1^{2\ell_1}\partial x_2^{2\ell_2+2}...\partial x_N^{2\ell_N}} +\\
&\quad + ... + \frac{\partial^{2k+2}}{\partial x_1^{2\ell_1}\partial x_2^{2\ell_2}...\partial x_N^{2\ell_N+2}}\Big),\\
&= \sum_{\ell'_1+\ell'_2+...+\ell'_N=k+1} \frac{\partial^{2(k+1)}}{\partial x_1^{2\ell'_1}\partial x_2^{2\ell'_2}...\partial x_N^{2\ell'_N}}.
\end{aligned}$$

In an abstract way, the polyharmonic operator $(-\Delta)^m$ may also be seen through the polynomial $L_m : \mathbb{R}^N \to \mathbb{R}$ defined by

$$L_m(\xi) = \sum_{\ell_1+...+\ell_N=m} \frac{m!}{\ell_1!...\ell_N!}\Big(\prod_{i=1}^{n} \xi_i^{2\ell_i}\Big) = |\xi|^{2m}, \qquad \xi \in \mathbb{R}^N.$$

Note that, $(-\Delta)^m = L_m(\nabla)$. In particular, this shows that $L_m(\xi) > 0$ for all $\xi \neq 0$ so that $(-\Delta)^m$ is an elliptic operator [Agmon et al. (1959)].

The equation for eigenfunctions and eigenvalues for the polyharmonic operator has a form

$$(-\Delta)^m u(\mathbf{x}) - \lambda\, u(\mathbf{x}) = 0, \qquad \mathbf{x} \in \Omega,$$

with Navier boundary condition

$$u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = ... = \Delta^{m-1}u|_{\partial\Omega} = 0.$$

We denote by $\Theta(\mathbf{x},\mathbf{y},\lambda)$ the kernel of the integral operator E_λ and referred as the spectral function of $\hat{A}$ and the expression

$$(E_\lambda f)(\mathbf{x}) = \int_\Omega \Theta(\mathbf{x},\mathbf{y},\lambda)\, f(\mathbf{y})d\mathbf{y}, \quad f \in L_2(\Omega), \tag{1.4}$$

is called as spectral decomposition of the element f corresponding to the self-adjoint operator $\hat{A}$. The properties of the spectral function established in Gårding's theorem enable us to define the spectral decomposition of any function in $L_1(\Omega)$ that is finite in Ω.

The Riesz means $E_\lambda^\alpha f$ of a spectral decomposition of order $\alpha \geq 0$ has the following integral form

$$E_\lambda^\alpha f = \int_\mu^\lambda \left(1-\frac{t}{\lambda}\right)^\alpha dE_t f. \tag{1.5}$$

It is easy to see that E_λ^α is an integral operator

$$E_\lambda^\alpha f(\mathbf{x}) = \int_\Omega \Theta^\alpha(\mathbf{x},\mathbf{y},\lambda)\, f(\mathbf{y})d\mathbf{y}, \tag{1.6}$$

with the kernel $\Theta^\alpha(\mathbf{x},\mathbf{y},\lambda)$ which are also the Riesz means of order α of the spectral function

$$\Theta^\alpha(\mathbf{x},\mathbf{y},\lambda) = \int_\mu^\lambda \left(1-\frac{t}{\lambda}\right)^\alpha d_t\Theta(\mathbf{x},\mathbf{y},t). \tag{1.7}$$

The operator $\hat{A}$ is an extension of $(-\Delta)^m$, that is $C_0^\infty(\Omega) \subset D(\hat{A})$ and $(-\Delta)^m f(\mathbf{x}) = \hat{A}f(\mathbf{x})$ for any function $f \in C_0^\infty(\Omega)$.

Let $f(\mathbf{x}) \in C_0^\infty(\Omega)$. Then

$$((-\Delta)^m f, v_n) = \int_\Omega [(-\Delta)^m f(\mathbf{x})]\, \overline{v_n(\mathbf{x})}\, d\mathbf{x},$$

and since $(-\Delta)^m$ is formally self-adjoint

$$((-\Delta)^m f, v_n) = \int_\Omega f(\mathbf{x})\, \overline{(-\Delta)^m\, v_n(\mathbf{x})}\, d\mathbf{x} = \int_\Omega f(\mathbf{x})\, \lambda_n\, \overline{v_n(\mathbf{x})}\, d\mathbf{x} = \lambda_n\, f_n.$$

Since $(-\Delta)^m f(\mathbf{x}) \in L_2(\Omega)$, by Bessel's inequality [Gradshtein and Ryzik (1943)] we have $f \in D(\hat{A})$ and since the functions $(-\Delta)^m f(\mathbf{x})$ and $\hat{A}f$ have the same Fourier coefficients $\lambda_n f_n$ and the system $\{v_n(\mathbf{x})\}$ is complete, the required equality $(-\Delta)^m f = \hat{A}f$, understood as an equality of ele ments of $L_2(\Omega)$ is hold.

Shishmarev (1969) studied the mean value formula of polyharmonic operator. He proposed the following:

Let $u(\mathbf{x})$ be a solution of the polyharmonic equation

$$\Delta^m\, u(\mathbf{x}) - (-1)^m\, \mu^{2m}\, u(\mathbf{x}) = f(\mathbf{x}), \qquad f \in L_2(\Omega), \quad u \in C(\overline{\Omega}) \cap W_2^{2m}(\Omega),$$

in a bounded domain $\Omega \subset \mathbb{R}^N$ with sufficiently smooth boundary $\partial\Omega$, subject to the homogeneous boundary conditions

$$u_n\Big|_{\partial\Omega} = \Delta u_n\Big|_{\partial\Omega} = \ldots = \Delta^{m-1} u_n\Big|_{\partial\Omega} = 0,$$

where $\mu = \mu_0 + ia, \mu_0 \to \infty, a \neq 0$.

For eigenfunctions $\{u_k(\mathbf{x})\}$of the polyharmonic operator, the mean value formula was given by

$$\int \ldots \iint_{\theta} u_k(\mathbf{x}+r\theta)\, d\theta = 2^{\frac{N-2}{2}}\, \Gamma(\frac{N}{2})\, \frac{J_{\frac{N-2}{2}}(r\sqrt{\lambda_n})}{(r\, \sqrt[2m]{\lambda_k})^{\frac{N-2}{2}}}\, u_k(\mathbf{x}) + O(e^{-\beta_0 (r_0 - R)\, \sqrt[2m]{\lambda_k}}),$$

where $\beta_0 = max_{j\geq 2} |\zeta(\sqrt{-\alpha_j})|, r_0 < R$.

Example 1.2.1 *Let $(-\Delta)^m$ be a polyharmonic operator, which is self-adjoint elliptic operator of order $2m$ in a domain $\Omega \subset \mathbb{R}^N$. Let Ω be a bounded domain with a smooth boundary $\partial\Omega$ and $\Omega + \partial\Omega$. We denote by A_0 an operator in $L_2(\Omega)$ whose domain of definition consists of all the functions $u(x)$ that are infinitely differentiable in $\Omega + \partial\Omega$ and satisfy the conditions*

$$u(\mathbf{x}) = \frac{\partial u}{\partial v}(\mathbf{x}) = \ldots = \frac{\partial^{m'-1} u}{\partial^{m'-1} v}(\mathbf{x}) = 0, \qquad \mathbf{x} \in \partial\Omega, \tag{1.8}$$

and which acts according to the rule $A_0 u = (-\Delta)^m\, u(\mathbf{x})$. ■

Garding (1953) proved that the closure $\hat{A}$ of A_0 is a self-adjoint operator whose spectrum consists of the eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \lambda_n \to \infty$ with the corresponding orthonormal system of eigenfunctions $\{v_n(\mathbf{x})\}$ being complete in $L_2(G)$. Every eigenfunction $u_n(\mathbf{x})$ which is infinitely differentiable in $\Omega \cup \partial\Omega$ satisfies the equality $A(x,D)\, v_n(\mathbf{x}) = \lambda_n\, v_n(\mathbf{x})$ in Ω and the boundary condition (1.8).

The functions $v_n(x)$ are called eigenfunctions of the first boundary problem (or Dirichlet problem) for the operator $(-\Delta)^m$. The spectral function of $\hat{A}$ is of the form

$$\theta(\mathbf{x},\mathbf{y},\lambda) = \sum_{\lambda_n<\lambda} v_n(\mathbf{x})\, \overline{v_n(\mathbf{y})}, \tag{1.9}$$

and the spectral decomposition of the element f is of the form

$$E_\lambda f = \sum_{\lambda_n<\lambda} f_n\, v_n(\mathbf{x}), \quad \text{where} \quad f_n = (f, v_n). \tag{1.10}$$

Example 1.2.2 *Let $T^N = \{\mathbf{x} \in \mathbb{R}^N : -\pi < x_k < \pi, k = 1,2,...,N\}$ and let $(-\Delta)^m$ be an elliptic formally self-adjoint differential operator with constant coefficients. One of the self adjoint extensions $\hat{A}$ of this operator is specified by the condition of periodicity, that is, $\hat{A}$ is the closure in $L_2(T^N)$ of the operator $(-\Delta)^m$ defined on the functions in $C^\infty(\mathbb{R}^N)$ that are periodic in each argument with the period 2π.* ■

It can be verified by a direct computation that the eigenfunctions of $\hat{A}$ are $(2\pi)^{-N/2}e^{i(n,\mathbf{x})}$ corresponding to the eigenvalues $|n|^{2m}$; its spectral function is

$$\theta(\mathbf{x},\mathbf{y},\lambda) = (2\pi)^{-N} \sum_{|n|^{2m}<\lambda} e^{i(n,\mathbf{x})}. \tag{1.11}$$

The spectral decomposition of any element $f \in L(T^N)$ has the form

$$E_\lambda f(\mathbf{x}) = \sum_{|n|^{2m}<\lambda} f_n e^{i(n,\mathbf{x})}, \tag{1.12}$$

where

$$f_n = (2\pi)^{-N} \int_{T^N} f(y)\, e^{i(n,\mathbf{y})}\, d\mathbf{y}, \tag{1.13}$$

and the Riesz means of order α of the spectral decomposition are

$$E_\lambda^\alpha f = \sum_{|n|^m<\lambda} \left(1 - \frac{|n|^m}{\lambda}\right)^\alpha f_n e^{i(n,\mathbf{x})}. \tag{1.14}$$

The spectral decomposition (1.12) purposes a summation method for the multiple Fourier series of a function $f \in L(T^N)$. In particular, the method corresponding to the Laplace operator $A(D) = \Delta$ is the summation over spheres and for an arbitrary second order elliptic operator the summation is over expanding ellipsoids.

Example 1.2.3 *Let $\Omega = \mathbb{R}^N$ and $(-\Delta)^{2m}$ an elliptic polynomial with real coefficients. In this case, by using the Fourier transform one can show that there exists a unique self adjoint extension $\hat{A}_0$ whose spectral function is*

$$\theta_0(\mathbf{x},\mathbf{y},\lambda) = (2\pi)^{-N} \int\limits_{|\xi|^{2m}<\lambda} e^{i(\mathbf{x}-\mathbf{y},\xi)}\, d\xi. \tag{1.15}$$

The spectral decomposition of an arbitrary element $f \in L_2(\mathbb{R}^N)$ is determined by the formula

$$E_\lambda f(\mathbf{x}) = (2\pi)^{-N} \int\limits_{|\xi|^{2m}<\lambda} \hat{f}(\xi)\, e^{i(\mathbf{x},\xi)}\, d\xi, \tag{1.16}$$

where

$$\hat{f}(\xi) = \int\limits_{\mathbb{R}^N} f(\mathbf{x})\, e^{-i(\mathbf{x},\xi)}\, d\mathbf{x} \tag{1.17}$$

and the integral converges in $L_2(\mathbb{R}^N)$ by Plancherel's theorem. Accordingly, the Riesz means of the spectral function (1.15) *are of the form*

$$\theta_0^\alpha(\mathbf{x},\mathbf{y},\lambda) = (2\pi)^{-N} \int\limits_{|\xi|^{2m}<\lambda} \left(1-\frac{|\xi|^{2m}}{\lambda}\right)^\alpha e^{i(\mathbf{x}-\mathbf{y},\xi)}\, d\xi, \tag{1.18}$$

while the Riesz means of the spectral decomposition (1.16) *are*

$$E_\lambda^\alpha f(\mathbf{x}) = (2\pi)^{-N} \int\limits_{|\xi|^{2m}<\lambda} \left(1-\frac{|\xi|^{2m}}{\lambda}\right)^\alpha \hat{f}(\xi) e^{i(\mathbf{x},\xi)}\, d\xi, \tag{1.19}$$

■

As in Example 1.2.2, every elliptic operator purposes a summation method of multiple Fourier integrals, while the Laplace operator corresponds to the summation method over spheres.

1.3 Eigenfunction Expansions Associated with Polyharmonic Operator

The linear differential elliptic operators

$$u \mapsto Au = (-\Delta)^m u, \tag{1.20}$$

is a linear operator containing all the lower order partial derivatives of the function u. The coefficients of the derivatives are measurable functions of $\mathbf{x} \in \Omega$. For elliptic differential operators A of the form (1.20) and under assumptions on f, we consider solutions $u = u(\mathbf{x})$ of the equation

$$(-\Delta)^m u = f \qquad \text{in } \Omega, \tag{1.21}$$

satisfies m boundary conditions on $\partial\Omega$. These conditions can be represented by linear differential operators

$$B_j(\mathbf{x};D)u = h_j, \qquad j = 1,...,m \text{ on } \partial\Omega, \tag{1.22}$$

where the functions h_j belong to the corresponding functional spaces. Each B_j has a maximal order of derivatives $m_j \in \mathbb{N}$ and the coefficients of the derivatives are sufficiently smooth functions on $\partial\Omega$. For the problems considered in this thesis, we take

$$m_j \leq 2m - 1 \qquad \text{for all } j = 1,...,m. \tag{1.23}$$

Therefore, we assume that (1.23) holds. The meaning of (1.22) will remain unclear until the exact definition of solution to (1.21) is given and it is satisfied the operators

$$\gamma_j u = \frac{\partial^j u}{\partial \nu^j}\Big|_{\partial\Omega}, \qquad u \in C^m(\overline{\Omega}) \text{ and } j = 0,...,m,$$

where ν denotes the unit outer normal to $\partial\Omega$.

Since the choice of the B_j's is not completely free, we need to impose the so-called complementing condition. For any j, let B'_j denote the B_j of order m_j, then for equation (1.20) of the polyharmonic operator has a crucial condition in order to obtain estimates for its solutions. The solvability of (1.20) depends on the assumptions made on f, B_j and h_j. Here, we will provide the homogeneous problem as follows.

Let assume $f = 0$ in Ω and h_j on $\partial\Omega$ for all $j = 1,...,m$ then for any B_j's there exists a set of solutions in $\mathbb{R}$ such that

$$\begin{aligned} (-\Delta)^m u = 0 \qquad & \text{in } \Omega, \\ B_j(\mathbf{x};D)u = 0 \qquad & \text{with } j = 1,...,m \text{ on } \partial\Omega, \end{aligned} \tag{1.24}$$

which only has a trivial solution.

We shall now give the boundary conditions as follows.

- *Dirichlet boundary conditions*. For the case $B_j(\mathbf{x},D)u = B'_j(\mathbf{x},D)u = \frac{\partial^{j-1}u}{\partial \nu^{j-1}}$ for $j = 1,...,m; m_j = j-1$ and (1.22) becomes

$$u = h_1, ..., \frac{\partial^{m-1}u}{\partial \nu^{m-1}} = h_m \qquad \text{on } \partial\Omega. \tag{1.25}$$

- *Navier boundary conditions*. In the case of $B_j(\mathbf{x},D)u = B'_j(\mathbf{x},D)u = \Delta^{j-1}u$ for $j = 1,...,m; m_j = 2(j-1)$ and (1.22) becomes

$$u = h_1, ..., \Delta^{m-1}u = h_m \qquad \text{on } \partial\Omega. \tag{1.26}$$

- *Mixed Dirichlet-Navier boundary conditions*. They are suitable combination of (1.25) and (1.26). For instance, if m is odd, we have $B_j(\mathbf{x},D)u = \frac{\partial^{j-1}u}{\partial \nu^{j-1}}$ for $j = 1,...,m-1$ and $B_m(\mathbf{x},D)u = \Delta^{(m-1)/2}u$.

- *Steklov boundary conditions*. These conditions only considered for the biharmonic operator. Let $a \in C^0(\partial\Omega)$ and equation $\Delta^2 u = f$ in Ω corresponds with the boundary operators $B_1(\mathbf{x},D)u = u$ and $B_2(\mathbf{x},D)u = \Delta u - a\frac{\partial u}{\partial \nu}$. Then (1.22) becomes

$$u = h_1 \qquad \text{and} \qquad \Delta u - a\frac{\partial u}{\partial \nu} = h_2 \qquad \text{on } \partial\Omega.$$

Note that the Navier boundary conditions is the main boundary conditions considered in this thesis.

As an example, for $\alpha \in (\frac{1}{2}\pi, \pi)$ we fix the domain

$$\Omega_\alpha = \{(r\cos\theta, r\sin\theta) \in \mathbb{R}^2; 0 < r < 1 \text{ and } |\theta| < \alpha\}.$$

Let $f \in L^2(\Omega_\alpha)$ and we consider the homogeneous Navier problem

$$\begin{aligned} \Delta^2 u &= f \text{ in } \Omega_\alpha, \\ u &= 0 \text{ on } \partial\Omega_\alpha, \\ \Delta u &= 0 \text{ on } \partial\Omega_\alpha \setminus \{0\}. \end{aligned} \tag{1.27}$$

We say that u is a system of solution to (1.27) if

$$\begin{aligned} \Delta u = w \text{ and } -\Delta w = f \text{ in } \Omega_\alpha, \\ u = 0 \text{ and } w = 0 \text{ on } \partial\Omega_\alpha. \end{aligned} \tag{1.28}$$

This system of solution is a solution to an iterated Dirichilet Laplace problem on a bounded domain for any $f \in L^2(\Omega_\alpha)$. We call this the energy solution, since its second derivatives are square summable.

1.4 Resolvent Operator

The spectral theory of operators on infinite-dimensional spaces is more complicated and it is important for an understanding of the operators themselves.

If T is a linear transformation on $\mathbb{C}^n$, then the eigenvalues of T are the complex numbers λ such that the determinant of $\lambda I - T$ is equal to zero. The set of such λ is called the spectrum of T. It can consist of at most n points since $det(\lambda I - T)$ is a polynomial of degree n. If λ is not an eigenvalue, then $\lambda I - T$ has an inverse since $det(\lambda I - T) \neq 0$.

The following definitions and theorem was given in Kolmogorov and Fomin (1970).

Definition 1.4.1 *Let $T \in \mathscr{L}(X)$. A complex number λ is said to be in the resolvent set $\rho(T)$ of T if $\lambda I - T$ is a bijection with a bounded inverse. $R_\lambda(T) = (\lambda I - T)^{-1}$ is called the resolvent of T at λ. If $\lambda \notin \rho(T)$, then λ is said to be in the spectrum $\sigma(T)$ of T.*

By the inverse mapping theorem, $\lambda I - T$ has a bounded inverse if it is bijective. We provide two subsets of the spectrum.

Definition 1.4.2 *Let $T \in L(X)$*

- *An $x \neq 0$ which satisfies $Tx = \lambda x$ for some $\lambda \in \mathbb{C}$ is called an eigenvector of T; λ is called the corresponding eigenvalue. If λ is an eigenvalue, then $\lambda I - T$ is not injective so λ is in the spectrum of T. The set of all eigenvalues is called the point spectrum of T.*
- *If λ is not an eigenvalue and if $Rank(\lambda I - T)$ is not dense, then λ is said to be in the residual spectrum.*

The residual spectrum does not occur for a large class of operators, for instance, self-adjoint operators. Note that, the resolvent set $\rho(T)$ is open and $R_\lambda(T)$ is an analytic operator-valued function on $\rho(T)$.

Theorem 1.4.1 *Let X be a Banach space and suppose $T \in L(X)$. Then $\rho(T)$ is an open subset of $\mathbb{C}$ and $R_\lambda(T)$ is an analytic $L(X)$-valued function on each component (maximal connected subset) of D. For any two points λ, $\mu \in \rho(T), R_\lambda(T)$ and $R_\mu(T)$ commute and*

$$R_\lambda(T) - R_\mu(T) = (\mu - \lambda)\, R_\mu(T)\, R_\lambda(T). \tag{1.29}$$

Example 1.4.1 *Let D be a region in the complex plane, i.e. a connected open subset of $\mathbb{C}$ and let T be a closed operator on a Hilbert space H. A complex number λ is in the resolvent set, $\rho(T)$, if $\lambda I - T$ is a bijection of $D(T)$ onto H with a bounded inverse. If $\lambda \in \rho(T), R_\lambda(T) = (\lambda I - T)^{-1}$ is called the resolvent of T at λ.* ■

For a point to be in the resolvent set of T, several conditions must be satisfied. These conditions are not all independent. For example, if $\lambda I - T$ is a bijection of $D(T)$ onto H, its inverse is said to be bounded.

Example 1.4.2 *Let consider the equation:*

$$Tf - \lambda f = g.$$

Then T has an eigenfunction $\{u_n(x)\}$ and an eigenvalues $\{\lambda_n\}$ defined by

$$Tu_n(x) = \lambda_n u_n(x). \tag{1.30}$$

If $f \in L_2(\Omega)$: $f = \sum c_n u_n(x)$, we have

$$Tf = \sum c_n T u_n(x) = \sum c_n \lambda_n u_n(x),$$

then by subtracting λf:

$$Tf - \lambda f = \sum c_n \left(\lambda_n - \lambda\right) u_n(x).$$

Assume that $g(x) = \sum d_n u_n(x)$, then

$$\sum d_n u_n(x) = g = (T - \lambda I)f = \sum c_n \left(\lambda_n - \lambda\right) u_n(x).$$

Hence

$$d_n = c_n \left(\lambda_n - \lambda\right) \quad \textit{and} \quad c_n = \frac{d_n}{\lambda_n - \lambda},$$

thus

$$(T_n - \lambda I)^{-1} g = R_\lambda \, g = \sum \frac{d_n}{\lambda_n - \lambda} \, u_n(x).$$

■

The expression

$$R_\lambda(T) - R_\mu(T) = R_\lambda(T) \left(\mu I - T\right) R_\mu(T) - R_\lambda(T) \left(\lambda I - T\right) R_\mu(T),$$

proves (1.30). Interchanging μ and λ shows that $R_\lambda(T)$ and $R_\mu(T)$ commute. Equation (1.30) is called the first resolvent formula.

Example 1.4.3 *Let polyharmonic operator $\Delta^m = T$, then*

$$\begin{aligned}
[\Delta^m - (-1)^m \lambda^{2m}]\, u(x) &= f(x), \\
(T - \lambda I)\, u(x) &= f(x), \\
u(x) &= (T - \lambda I)^{-1}\, f(x), \\
&= \int_\Omega R(x,y,\lambda)\, f(y)\, dy.
\end{aligned}$$

Here,

$$\begin{aligned}
R_\lambda\, f &= \int_\Omega R(x,y,\lambda)\, f(y)\, dy, \\
&= \int_\Omega \sum \frac{u_n(x)\, u_n(y)}{\lambda_n - \lambda}\, f(y)\, dy, \\
&= \sum \frac{c_n\, u_n(x)}{\lambda_n - \lambda}, \\
&= (T - \lambda I)^{-1}\, f(x),
\end{aligned}$$

where

$$R(x,y,\lambda) = \sum \frac{u_n(x)\, u_n(y)}{\lambda_n - \lambda}.$$

Applying into (1.30)*, the resolvent operator for the polyharmonic operator is represented as follows*

$$\begin{aligned}
Tu_n(x) &= \lambda_n\, u_n(x), \\
(T - \lambda I)\, u_n(x) &= (\lambda_n - \lambda)\, u_n(x), \\
u_n(x) &= (T - \lambda I)^{-1}\, (\lambda_n - \lambda)\, u_n(x), \\
&= (\lambda_n - \lambda)\, R_\lambda\, u_n(x), \\
&= (\lambda_n - \lambda) \int_\Omega R(x,y,\lambda)\, u_n(y)\, dy.
\end{aligned}$$

■

1.5 Green's Identities

Green's functions provide a powerful tool for solving linear problems that consists of an ordinary differential equation and partial differential equation with boundary conditions which resulting a unique solution. The Green's function can be defined by a similar linear problem where all boundary conditions are homogeneous and the inhomogeneous term in the differential equation is a delta function. In electrodynamics and quantum field theory, Green's functions are widely used in which relevant differential operators are often difficult or impossible to solve exactly. They can be solved perturbatively using Green's functions.

Let consider a differential equation

$$\mathscr{L}u(\mathbf{x}) = f(\mathbf{x}), \quad B_1(u) = \alpha_1,\ B_2(u) = \alpha_2, ..., B_N(u) = \alpha_N,$$

for some function $u = u(\mathbf{x}), \mathbf{x}$ in some subset Ω of $\mathbb{R}^N$, $\mathscr{L} = \mathscr{L}_x$ some linear differential operator acting on the variable x and the B_j defining linear boundary conditions. Here, the notation $\mathscr{L}$ is known as ordinary differential operator if $\mathscr{L} = d^2/dx^2$ and $\mathscr{L}$ is a partial differential if $\mathscr{L} = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + ... + \partial^2/\partial x_N^2$, the Laplacian.

Then the corresponding Green's function $G = G(\mathbf{x}, \mathbf{x}')$ is defined as a solution of the following problem,

$$\mathscr{L}_x G(\mathbf{x}, \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}'), \quad B_1(G) = B_2(G) = ... = B_N(G) = 0,$$

where $\delta(\mathbf{x} - \mathbf{x}')$ is the delta function localized at $\mathbf{x}' \in \Omega$ (if B_j involve differentiations they are to act on the variables x'). If all boundary conditions are homogeneous: $\alpha_j = 0\ \forall j$, then the solution of the problem is

$$u(\mathbf{x}) = \int_{\Omega} G(\mathbf{x}, \mathbf{x}')\ f(\mathbf{x}')\ d\mathbf{x}.$$

Otherwise one also has to add integrals involving G and α_j, over the boundary regions where α_j is defined.

The Green's first identity for the pair of functions (u, v) in N- dimensional can be written as follows

$$\int_{\Omega} v\Delta u\ d\mathbf{x} = \int_{\partial\Omega} v\frac{\partial u}{\partial n}\ dS - \int_{\Omega} \nabla u \cdot \nabla v\ d\mathbf{x}.$$

Interchanging u and v, we can also write the Green's first identity for the pair (v, u),

$$\int_{\Omega} u\Delta v\ d\mathbf{x} = \int_{\partial\Omega} u\frac{\partial v}{\partial n}\ dS - \int_{\Omega} \nabla v \cdot \nabla u\ d\mathbf{x}.$$

Notice that the last term of the Green's first identity for the pair (u, v) and (v, u) are identical, thus subtracting the first identity from the second, one yields

$$\int_{\Omega} \left(u\Delta v - v\Delta u\right)\ d\mathbf{x} = \int_{\partial\Omega} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)\ dS.$$

This is the Green's second identity for the pair of function (u,v).

Similar to the notion of symmetric boundary conditions for the heat and wave equation, the symmetric boundary conditions can be defined for Laplace's equation, by requiring that the right hand side of the Green's second identity vanishes for any functions u,v satisfying the boundary conditions. Furthermore, the homogeneous Dirichlet, Neumann and Robin boundary conditions are all symmetric. In this thesis, it is useful to provide the following relations.

Let $u,v \in C^4(\Omega)$. We consider bilinear forms:

$$L_0(u,v) = \sum_{|\beta|=2} \frac{|\beta|!}{\beta!}(D^\beta u)D^\beta v + \sum_{|\gamma|=1} \frac{|\gamma|!}{\gamma!}(D^\gamma \Delta u)D^\gamma v. \tag{1.31}$$

For $\beta = (\beta_1, \beta_2)$,

$$\begin{aligned}
&\sum_{|\beta|=2} \frac{(\beta_1+\beta_2)!}{\beta_1!\beta_2!} D^{(\beta_1,\beta_2)}u.D^{(\beta_1+\beta_2)}v \\
&= \frac{(2+0)!}{2!0!} D^{(2,0)}u.D^{(2,0)}v + \frac{(1+1)!}{1!1!} D^{(1,1)}u.D^{(1,1)}v, \\
&= \frac{\partial^2 u}{\partial x_1^2}.\frac{\partial^2 v}{\partial x_1^2} + 2\frac{\partial^2 u}{\partial x_1 \partial x_2}.\frac{\partial^2 v}{\partial x_1 \partial x_2} + \frac{\partial^2 u}{\partial x_2^2}.\frac{\partial^2 v}{\partial x_2^2}.
\end{aligned}$$

Second term of (1.31), we have

$$\sum_{|\gamma|=1} \frac{|\gamma|!}{\gamma!}(D^\gamma \Delta u)D^\gamma v = \Big(\frac{\partial}{\partial x_1}\Delta u\Big)\frac{\partial v}{\partial x_1} + \Big(\frac{\partial}{\partial x_2}\Delta u\Big)\frac{\partial v}{\partial x_2}.$$

Then (1.31) can be written as

$$L_0(u,v) = \sum_{|\beta|=2} \frac{2!}{\beta!}(D^\beta u)D^\beta v + \sum_{|\gamma|=1} (D^\gamma \Delta u)D^\gamma v,$$

and

$$L_1(u,v) = \sum_{|\beta|=1} (D^\beta \Delta u)D^\beta v + (\Delta^2 u)v,$$

where these will take the form

$$L_0 = \sum_{j=1}^{N} \frac{\partial^2 u}{\partial x_j^2}.\frac{\partial^2 v}{\partial x_j^2} + 2\sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\partial^2 u}{\partial x_i \partial x_j}.\frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{i=1}^{N} \frac{\partial(\Delta u)}{\partial x_i}.\frac{\partial v}{\partial x_i},$$

and

$$L_1 = \sum_{i=1}^{N} \frac{\partial(\Delta u)}{\partial x_i} \cdot \frac{\partial v}{\partial x_i} + (\Delta^2 u) \cdot v.$$

1.6 Spaces of Distributions

The notion of generalized derivation was developed in the early of twentieth century. During 1950's, Laurent Schwartz found a precise formulation of the theory of distributions with some significant ideas by Sergei Lvovich Sobolev. The theory of distribution is a concept that generalizes the notion of function, also called generalized functions. Distributions are the objects that generalize the classical notion of functions in mathematical analysis which make possible to differentiate function for which the derivatives do not exist in the classical sense. It is a class of objects which is more larger than the class of differentiable functions. Theory of function spaces and spaces of distribution was studied as part of functional analysis. The main field of application is the theory of ordinary and partial differential equation. The fundamental idea of the theory of distributions is that it is generally easier to work with linear functionals acting on spaces of good functions which lead to the study of distributions spaces i.e. the space of continuous linear functionals on the set of test function.

Let T be a topological space of functions defined on N-dimensional Euclidean space, $\mathbb{R}^N$. Then we consider T a space of test functions and define the space of distributions on T.

We begin with some preliminaries. Let γ denote a multi-index, i.e. N-dimensional vector with nonnegative integer components $\gamma = (\gamma_1, \gamma_2, ..., \gamma_N)$ and the length of multi-index is set by $|\gamma| = \gamma_1 + \gamma_2 + ... + \gamma_N$. We use the notation $x^\gamma = (x_1^{\gamma_1}, x_2^{\gamma_3}, ..., x_N^{\gamma_N})$ where $x \in \mathbb{R}^N$. Then we can write a partial mixed derivative of order γ as follows

$$D^\gamma f(x) = \frac{\partial^{|\gamma|} f(x)}{\partial x_1^{\gamma_1}\, \partial x_2^{\gamma_2} \ldots \partial x_N^{\gamma_N}}.$$

Let U be an open set in $\mathbb{R}^N$. If $f : U \to \mathbb{R}$ is a continuous function, we define its support as

$$supp\ f = \overline{\{x \in U; f(x) \neq 0\}}.$$

The function f is said to have a compact support if $supp\ f$ is a bounded set.

First, we state an important space of test functions $C^\infty(\mathbb{R}^N)$ as a space of all smooth (infinitely differentiable) functions in the N-dimensional Euclidean space $\mathbb{R}^N$ and use the notation $\varepsilon(\mathbb{R}^N)$ to denote the set $C^\infty(\mathbb{R}^N)$. Let K be a compact subset of $\mathbb{R}^{\mathbb{N}}$.

The topology of $\varepsilon(\mathbb{R}^N)$ is defined by the family of seminorms

$$P_{K,\gamma}(\varphi) = \max_{x \in K} |D^\gamma \varphi(x)|,$$

as topological space with the following definition of topology. We say that a sequence $\varphi_n \in \varepsilon(\mathbb{R}^N)$ converges to a function $\varphi \in \varepsilon(\mathbb{R}^N)$ in topology of ε if uniformly on each compact subset of $\mathbb{R}^{\mathbb{N}}$, $|D^\gamma \varphi_n(x) - D^\gamma \varphi(x)| \to 0$, $\forall \gamma \in \mathbb{N}$ as $n \to \infty$. The $\varepsilon(\mathbb{R}^N)$ space equipped with the topology of uniform convergence of every derivative on each compact subset of $\mathbb{R}^{\mathbb{N}}$.

We consider the space of test functions $J(\mathbb{R}^N)$ denote as the Schwartz space of the smooth functions and define the space

$$J(\mathbb{R}^N) = \Big\{ f : f \in C^\infty(\mathbb{R}^N),\ \sup_{x \in \mathbb{R}^N} (1+|x|)^\alpha\ |D^\gamma f(x)| < \infty;\ \forall \gamma, \alpha \in \mathbb{N} \Big\}.$$

Let $f \in J(\mathbb{R}^N)$, then all derivatives $D^\gamma f(x)$ are rapidly decreasing as $|x| \to \infty$. The space $J(\mathbb{R}^N)$ is a topological space and let introduce topology in $J(\mathbb{R}^N)$ as follows. We say that a sequence $\varphi_n \in J(\mathbb{R}^N)$ converges to a function $\varphi \in J(\mathbb{R}^N)$, if $\sup_{x \in \mathbb{R}^N} (1+|x|)^\alpha\ |D^\gamma \varphi_n(x) - D^\gamma \varphi(x)| \to 0$, $\forall \gamma, \forall \alpha \in \mathbb{N}$ as $n \to \infty$. For example, the function $e^{-|x|^2}$ belongs to $J(\mathbb{R}^N)$.

Another important space of test functions is $C_0^\infty(\mathbb{R}^N)$. This space is the subsets of $C^\infty(\mathbb{R}^N)$ which contains the functions from $C^\infty(\mathbb{R}^N)$ with compact support defined in the N-dimensional Euclidean space $\mathbb{R}^N$. The notation $\mathscr{D}(\mathbb{R}^N)$ is used to denote the set $C_0^\infty(\mathbb{R}^N)$ equipped with the topology defined as follows. We consider $\mathscr{D}(\mathbb{R}^N)$ with a topology that corresponds to the the following convergence of test functions. We say that a sequence $\varphi_n \in \mathscr{D}(\mathbb{R}^N)$ converges to a function $\varphi \in \mathscr{D}(\mathbb{R}^N)$, if there is a number $R > 0$ such that $supp\ \varphi_n \subset \mathbf{U}_R$ and $x \in \mathbb{R}^N$ then $|D^\gamma \varphi_n(x) - D^\gamma \varphi(x)| \to 0$, $\forall \gamma \in \mathbb{N}$ as $n \to \infty$ where $\mathbf{U}_R = \mathbf{U}(0;R)$ denotes the sphere of radius R centered at the origin.

As an example of a function from $\mathscr{D}(\mathbb{R}^N)$, we consider the so called cap-shaped function

$$\omega_\varepsilon(x) = \begin{cases} C_\varepsilon e^{-\frac{\varepsilon^2}{\varepsilon^2 - |x|^2}}, & |x| \le \varepsilon, \\ 0, & |x| > \varepsilon. \end{cases}$$

We choose a constant C_ε to give $\int \omega_\varepsilon(x)\, dx = 1$, resulting

$$C_\varepsilon \varepsilon^n \int_{U_1} e^{-\frac{1}{1-|\varepsilon|^2}}\ d\varepsilon = 1.$$

We conclude with presenting the embedding relationship between all the three spaces of test functions by

$$\varepsilon(\mathbb{R}^N) \supset J(\mathbb{R}^N) \supset \mathscr{D}(\mathbb{R}^N).$$

The generalized function in the sense of Sobolev-Schwartz refers to each linear continuous functional over the space of test functions T. In order to define the generalized function, it is important to give the following definition.

Definition 1.6.1 *We call f as a linear continuous functional on T if f is mapping from T to $\mathbb{R}$ (set of real numbers) such that*

(i) linearity:

$$f(\alpha\varphi + \beta\psi) = \alpha f(\varphi) + \beta f(\psi),$$

for any $\varphi, \psi \in T$ and $\alpha, \beta \in \mathbb{R}$.

(ii) continuity: If φ_n converges uniformly to φ in topology of T then $f(\varphi_n) \to f(\varphi)$.

The linear continuous function is called a distributions on T. We use the notation $< f, \varphi >$ to denote the action of the functional (generalized function) f over the test function φ. Further, we represent $f(x)$ as generalized function f corresponding to the argument x of the test functions on which the functional f acts.

In line with the previous subsection, we first let $\varepsilon(\mathbb{R}^N)$ be the space of test functions. Then the space of distributions on $\varepsilon(\mathbb{R}^N)$ denote by $\varepsilon'(\mathbb{R}^N)$.

From the definition, it is possible to say that the distributions becomes zero in a domain. We say that a distribution f is equal to zero in a domain $\Omega \subset \mathbb{R}^N$ if

$$< f, \varphi >= 0,$$

for any function $\varphi \in C^\infty(\mathbb{R}^N)$ with $supp\ \varphi \subset \Omega$.

Let $\mathbf{N}_f \subset \mathbb{R}^N$ is a set where a distribution is zero. This set called nill set of f. Then support of distribution is the following set

$$supp\ f = \overline{\mathbb{R}^N \setminus \mathbf{N}_f}.$$

If $supp\ f$ is finite in $\mathbb{R}^N$ (means can be included in the ball with finite radius) then we say that a distribution of f has compact support. Note that the space $\varepsilon'(\Omega)$ is the space of distributions with compact support.

Subsequently, let consider $J(\mathbb{R}^N)$ be a space of test functions. Then the corresponding space of distributions $J'(\mathbb{R}^N)$ is called the space of tempered distributions. Sometimes, $J'(\mathbb{R}^N)$ is called the space of slowly increasing distributions. If $f \in J'(\mathbb{R}^N)$ and $\varphi \in J(\mathbb{R}^N)$, then the action of functional f on test function φ is denoted by $< f, \varphi >$. We now define convergence in $J'(\mathbb{R}^N)$ as a weak convergence of the sequence of functionals. We say that a sequence $f_n \in J'(\mathbb{R}^N)$ converges to the generalized function $f \in J'(\mathbb{R}^N); f_n \to f$ as $n \to \infty$ in $J'(\mathbb{R}^N)$, if $< f_n, \varphi > \to < f, \varphi >$ for any $\varphi \in J(\mathbb{R}^N)$ as $n \to \infty$.

Finally, let $\mathscr{D}(\mathbb{R}^N)$ be the space of test functions. The space of continuous linear functional on $\mathscr{D}(\mathbb{R}^N)$ is called the space of distributions. The spaces of distributions on $\mathscr{D}(\mathbb{R}^N)$ are denoted by $\mathscr{D}'(\mathbb{R}^N)$. The convergence in $\mathscr{D}'(\mathbb{R}^N)$ is defined as follows. We say that a sequence of generalized functions $f_n \in \mathscr{D}'(\mathbb{R}^N)$ converges to the generalized function $f \in \mathscr{D}'(\mathbb{R}^N); f_n \to f$ as $n \to \infty$ in $\mathscr{D}'(\mathbb{R}^N)$, if for any $\varphi \in \mathscr{D}(\mathbb{R}^N)$ we have $< f_n, \varphi > \to < f, \varphi >$ as $n \to \infty$.

Hence, it follows from all these definitions that the embedding is valid.

$$\varepsilon'(\mathbb{R}^N) \subset J'(\mathbb{R}^N) \subset \mathscr{D}'(\mathbb{R}^N).$$

Note that, we briefly introduce $C^\infty(\Omega)$ and $C_0^\infty(\Omega)$ spaces in Section 1.2. We will discuss further for $C^\infty(\Omega)$ and $C_0^\infty(\Omega)$ topological spaces in the domain of $\Omega \in \mathbb{R}^N$ in Chapter 5 of this thesis.

1.7 Classes of Differentiable Functions

1.7.1 Sobolev Classes

We consider Ω as bounded domain in the N-dimensional Euclidean space $\mathbb{R}^N$ and its boundary is denoted by $\partial\Omega$. Let $1 < p < \infty$. We define Sobolev space as

Definition 1.7.1 *Let α be a positive integer and $1 < p < \infty$. The space $W_p^\alpha(\Omega)$ is the collection of functions in $L_p(\Omega)$ such that*

$$W_p^\alpha(\Omega) = \Big\{u : u \in L_p(\Omega), ||u||_{W_p^\alpha} = \Big(\sum_{|\gamma|\le\alpha} \int_\Omega |D^\gamma u|^p \, dx\Big)^{1/p}\Big\}.$$

Also, for $p = \infty$

$$||u||_{W_\infty^\alpha} = \sum_{|\gamma|\le\alpha} ess \sup_{x\in\Omega} |D^\gamma u|.$$

Note that, $W_p^0(\Omega) = L_p(\Omega)$.

We define subspace $\mathring{W}_p^{\alpha}(\Omega) \subseteq W_p^{\alpha}(\Omega)$ as a closure of $C_0^{\infty}(\Omega)$ in the norm of $W_p^{\alpha}(\Omega)$. In other words, we say that $u \in \mathring{W}_p^{\alpha}(\Omega)$ if and only if there exists a sequence of functions $u_n \in C_0^{\infty}(\Omega)$ such that

$$||u - u_n||_{W_p^{\alpha}} \to 0.$$

More generally, $\mathring{W}_p^{\alpha}(\Omega)$ is a space of functions whose derivatives $D^{\gamma}u$ vanish along $\partial\Omega$, for $|\gamma| < \alpha - 1$.

In the case $p = 2$, we define the Hilbert-Sobolev space $H^{\alpha}(\Omega) = W_2^{\alpha}(\Omega)$. The space $H^{\alpha}(\Omega)$ is equipped with the inner product

$$< u, v >_{H^{\alpha}} = \sum_{|\gamma| \leq \alpha} \int_{\Omega} D^{\gamma}u \, D^{\gamma}v \, dx.$$

Similarly, we define $\mathring{H}^{\alpha}(\Omega) = \mathring{W}_2^{\alpha}(\Omega)$.

Remark 1.7.1

- Each Sobolev space $W_p^{\alpha}(\Omega)$ is a Banach space.
- The space $\mathring{W}_p^{\alpha}(\Omega)$ is a closed subspace of $W_p^{\alpha}(\Omega)$ which is also a Banach space, with same norm.
- The spaces $H^{\alpha}(\Omega)$ and $\mathring{H}^{\alpha}(\Omega)$ are Hilbert spaces.

1.7.2 Nikolskii Classes

Let $\Omega \subset \mathbb{R}^N$ be a domain with smooth boundary $\partial\Omega$. We denote by $L_p(\Omega)$ a classs of the measurable functions which are p-integrable over Ω.

Definition 1.7.2 *Let $\alpha = \ell + \varkappa, \ell$- positive integer and $0 < \varkappa < 1, p \geq 1$. We say that a function $f(x) \in L_p(\Omega)$ belongs to the Nikolskii class $H_p^{\alpha}(\Omega)$, if for any $h \in \mathbb{R}^N$ and for all $\alpha = (\alpha_1, \alpha_2, ..., \alpha_N)$ satisfying $|\alpha| < \ell$:*

$$\left|\left|\partial^{\alpha} f(x+h) - 2\partial^{\alpha} f(x) + \partial^{\alpha} f(x-h)\right|\right|_{L_p(\Omega)} \leq C\left|\left|h\right|\right|^{\varkappa},$$

where $\partial^{\alpha} = i^{|\alpha|D^{\alpha}}$.

Using the notation $\Delta_h^2 f(x) = f(x+h) - 2f(x) + f(x-h)$ we define a norm in $H_p^{\alpha}(\Omega)$ by the following

$$\left|\left|f\right|\right|_{p,\alpha} = \left|\left|f\right|\right|_{L_p(\Omega)} + \sum_{|\alpha|=\ell} \sup_h |h|^{\varkappa} \left|\left|\Delta_h^2 \partial^{|\alpha|} f(x)\right|\right|.$$

The closure of the space $C_0^{\infty}(\Omega)$ in the norm of $H_p^{\alpha}(\Omega)$ denoted by $\mathring{H}_p^{\alpha}(\Omega)$.

1.8 Research Objectives

The thesis is devoted

1. To estimate the spectral function of the polyharmonic operator in closed domain.
2. To establish the uniformly convergence of eigenfunction expansions of continuous functions in closed domain.
3. To obtain the sufficient conditions for uniformly convergence of the eigenfunction expansions from Nikolskii classes in closed domain.
4. To obtain sufficient conditions for localization of the distributions from the classes of Sobolev with negative smoothness.

1.9 Thesis Outline

This thesis consists of six chapters as follows.

- Chapter 1 - Introduction: In this chapter we provide the basic tools of the research background related to the area of our research subject to the objectives and methods applied in solving problems in this thesis.
- Chapter 2 - Literature review: This chapter is basically deals with the previous research works done by well-known mathematicians related to our research topic. All these works are the factors and motivation leads to the idea of our research.
- Chapter 3 - In this chapter, we investigate the uniform convergence of eigenfunction expansions of the biharmonic operator of continuous functions and also from Nikolskii classes in closed domain by using the estimated Riesz means.
- Chapter 4 - This chapter provides a more precise respresentation of eigenfunction expansions of the polyharmonic operator. We estimate the Riesz means of the spectral function of polyharmonic operator which we will apply in the investigation of the uniform convergence of eigenfunction expansions of the polyharmonic operator of continuous functions and Nikolskii classes in closed domain.
- Chapter 5 - In this chapter, we use the results of the estimation of eigenfunctions of biharmonic operator from Chapter 3 and also the estimation of eigenfunctions of polyharmonic operator from Chapter 4 to study the localization

principle of spectral expansions of distributions in the Sobolev space with negative smoothness in closed domain.

- Chapter 6 - Discussions and future work: In the last chapter of this thesis, we give the conclusions of the results obtained in the end of this research study and its remarkable contributions to the development of the partial differential equation theory particularly problems involving polyharmonic operator. We also discuss and give suggestion on the open problems related to polyharmonic operator for future researchers to explore.

REFERENCES

Agmon, S., Douglis, A., and Nirenberg, L. (1959). Estimates Near the Boundary for Solutions of Elliptic Partial Differential Equations Satisfying General Boundary Conditions. *Comm. Pure Appl. Math.*, 12:623–727.

Ahmedov, A. (2009). The Principle of General Localization on Unit Sphere . *Journal of Mathematical Analysis and Applications*, 356(1):310–321.

Ahmedov, A. (2010). The Generalized Principle of Localization for the N- fold Fourier Integrals. *Malaysian Journal of Mathematical Sciences*, 4(2):209–227.

Ahmedov, A. (2011). The Estimations for Maximal Operators. *Applied Mathematics Letters*, 24(5):615–619.

Ahmedov, A. and Ashurov, R. R. (2011). On the Divergence of Spectral Expansions of Elliptic Differential Operators. *Malaysian Journal of Mathematical Sciences*, 5(2):185–196.

Ahmedov, A., Aziz, N. A., and Mohtar, M. N. (2011). On Convergence Almost Everywhere of Multiple Fourier Integrals. *Journal of Quality Measurement and Analysis*, 7(1):109–115.

Ahmedov, A. and Cholpon, A. (2008a). Even Similarity of Spectral Decompositions of Biharmonic Operator. *International Journal Science Education Engineering*, 2:94–99.

Ahmedov, A. and Cholpon, A. (2008b). The Generalized Principle of Localization for the N-fold Fourier Integrals. *International Journal Science Education Engineering*, 4:56–65.

Ahmedov, A. and Rakhimov, A. A. (2012). On the Fundamental Solution of the Cauchy Problem for time Fractional Diffusion Equation of the Sphere. *Malaysia Journal of Mathematical Sciences*, 6(1):105–112.

Ahmedov, A., Rasedee, A. F. N., and Rakhimov, A. (2013a). Localization of Fourier-Laplace Series of Distributions. *Malaysian Journal of Mathematical Sciences*, 7(2):315–326.

Ahmedov, A., Rasedee, A. F. N., and Rakhimov, A. (2013b). On the Sufficient Conditions of the Localization of the Fourier-Laplace Series of Distributions from Liouville Classes. *Journal of Physics: Conference Series*, 435(1):12–16.

Ahmedov, A. and Zainuddin, H. (2009a). The Convergence Problems of Spectral Expansions of Polyharmonic Operator on Closed Domain. In *Proceedings of the 5^{th} Asean Mathematical Conference*, PWTC, Kuala Lumpur Malaysia. Universiti Sains Malaysia.

Ahmedov, A. and Zainuddin, H. (2009b). The Uniform Convergence of Spectral Expansions of the Laplace Operator on Closed Domain $\overline{\Omega} \subset R^2$. *MathDigest Bulletin of Universiti Putra Malaysia*, 2(1):6–11.

Alimov, S. A. (1970). The Summability of Fourier Series of Functions in L_p in Terms of Eigenfunctions. *Journal of Differential Equations*, 6:416–423.

Alimov, S. A. (1972). Fractional Orders of the Elliptic Operators and Isomorphism of the Class of Differentiable Functions. *Journal of Differential Equations*, 8(9):1609–1626.

Alimov, S. A. (1973). Uniform Convergence and Summability of the Spectral Decompositions of Functions in L_p^a. *Journal of Differential Equations*, 9(4):669–681.

Alimov, S. A. (1976). On Spectral Decompositions of Functions in H_p^α. *Mathematical of the USSR-Sbornik*, 30(1):1–16.

Alimov, S. A. (1979). On Expanding Continuous Functions from Sobolev Classes in Eigenfunctions of Laplace Operator. *Sib Math. J.*, 19(1):721–734.

Alimov, S. A. (1993). On the Spectral Expansions of the Distributions. *Doclady Acad. Nauk*, 331(6):661–662.

Alimov, S. A. (2004). Sets of Uniform Convergence of Fourier Expansions of Piecewise Smooth Functions. *Journal of Fourier Analysis and Applications*, 10(6):635–644.

Alimov, S. A., Ashurov, R. R., and Pulatov, A. K. (1991). Multiple Fourier Series and Fourier Integral. *Encyclopaedia of Mathematical Sciences*, 42:3–95.

Alimov, S. A. and Rakhimov, A. A. (1986). On the Uniformly Convergence of Spectral Expansions in a Closed Domain. *Dokl. Acad Nauk UzSSR*, 10:5–7.

Alimov, S. A. and Rakhimov, A. A. (1996). On the Localization Spectral Expansions of Distributions. *Journal of Differential Equations*, 32(6):798–802.

Alimov, S. A. and Rakhimov, A. A. (1997). On the Localization of Spectral Expansions of Distributions in a Closed Domain. *Journal of Differential Equations*, 33(1):80–82.

Almansi, E. (1899a). Sull'integrazione Dell'equazione Differenziale $\Delta^2\Delta^2$. *Rom. Acc. L. Rend.*, 81(5):104–107.

Almansi, E. (1899b). Sull'integrazione Dell'equazione Differenziale Δ^{2n} . *Annali di Mat.*, 2(3):1–51.

Ashurov, R. R. (1981a). Conditions for Riesz Means of Spectral Resolutions to be a Basis in $L_p(R^n)$. *Journal of Mathematical Notes*, 29(5):342–348.

Ashurov, R. R. (1981b). Divergence of Spectral Expansions Connected with Elliptic Operators. *Journal of Mathematical Notes*, 30(2):599–605.

Ashurov, R. R. (1990). Decomposability of Continuous Functions from Nikol'skii Classes into Multiple Fourier Integrals. *Mathematical Notes of the Academy of Sciences of USSR*, 47(2):107–110.

Ashurov, R. R. (2008). On the Summability of Spectral Expansions of Piecewise Smooth Functions. *Journal of Differential Equations*, 44(9):1197–1202.

Ashurov, R. R., Ahmedov, A., and Mahmud, A. R. (2010). The Generalized Localization for Mulltiple Fourier Integrals. *Journal of Mathematical Analysis and Applications*, 371(2):832–841.

Ashurov, R. R. and Anvarjon, A. (2010). On Summability of N-fold Fourier Integrals Corresponding to Pseudodifferential Operators . *Journal of Pseudo-Differential Operators and Applications*, 1(4):417–432.

Bochner, S. (1936). *Trans. Amer. Math. Soc*, 40:175–270.

Boggio, T. (1905). Sulle Funzioni di Green D'ordine. *Rend. Circ. Mat. Palermo*, 20:97–135.

Brandolini, L. and Colzani, L. (1999). Localization and Convergence of Eigenfunction Expansions. *Journal of Fourier Analysis and Applications*, 5(5):431–447.

Egorov, Y. V. (1985). *Lectures on Linear Partial Differential Equations*. Nauka Fizmatlit, Moscow.

Egorov, Y. V. (1986). *Linear Differential Equations of Principal Type*. Consultants Bureau, New York.

Eskin, G. (1975). Asymptotics near the Boundary of Spectral Functions of Elliptic Self-Adjoint Boundary Problems. *Israel Journal of Mathematics*, 22(3-4):214–246.

Garding, L. (1953). Dirichlet's Problem for Linear Elliptic Partial Differential Equations. *Math Scand*, 1(MR 16 - 366):55–72.

Garding, L. (1957). Eigenfunction Expansions Connected with Elliptic Differential Operators. *Matematika*, 1(3):107–116.

Gazzola, F., Grunau, H. C., and Sweers, G. (1991). *Polyharmonic Boundary Value Problems.* Springer-Verlag Berlin Heidelberg.

Gradshtein, I. S. and Ryzik, I. M. (1943). *Table of Integrals, Series and Products.* Academic Press, New York.

Hörmander, L. (1966). On the Riesz Means of Spectral Functions and Eigenfunction Expansions for Elliptic Differential Operators. *Belfer Graduate School of Science Annual Science Conference Proceedings: Some Recent Advances in the Basic Sciences,*, 2:1965–1966.

Il'in, V. A. (1957a). On the Uniform Convergence of Expansions by Eigenfunctions under Summation in the Order of Increasing Eigenvalues. *Dokl. Akad. Nauk SSSR*, 114:698–701.

Il'in, V. A. (1957b). Uniform Convergence of Expansions by Eigenfunctions in Odd-dimensional Domains. *Dokl. Akad. Nauk SSSR*, 115:650–652.

Il'in, V. A. (1958a). On Convergence of Expansions in Eigenfunctions of the Laplace Operator. *Uspekhi Matematicheskikh Nauk,*, 13(1):87–180.

Il'in, V. A. (1958b). On the Uniformly Convergence Eigenfunction Expansions Associated with Laplace Operator in a Closed Domain. *Matematicheskii Sbornik*, 45(2):195–232.

Il'in, V. A. (1967). Solutions Exhaustive in the Classes W_2 and $C^{(n,\alpha)}$ of the Localization Problem for Fourier Series by Fundamental Function Systems of the Laplace Operator. *Soviet Math. Dokl.*, 8:1389–1392.

Il'in, V. A. (1968). Localization and Convergence Problems for Fourier Series by Fundamental Function Systems of the Laplace Operator. *Russian Math. Surveys*, 23(2):59–116.

Il'in, V. A. (1991). *Spectral Theory of Differential Operators*. Nauka Fizmatlit, Moscow.

Il'in, V. A. and Alimov, S. A. (1971a). Conditions for Convergence of the Spectral Expansions that Correspond to Self-adjoint Extensions of the Elliptic Operators. I. Self-adjoint Extension of the Laplace Operator with a Point Spectrum. *Journal of Differential Equations*, 7(4):670–710.

Il'in, V. A. and Alimov, S. A. (1971b). Conditions for the Convergence of Spectral Decompositions that Correspond to Self-adjoint Extensions of Elliptic Operators. II. Self-adjoint Extension of the Laplace Operator with an Arbitrary Spectrum. *Journal of Differential Equations*, 7(5):851–882.

Il'in, V. A. and Alimov, S. A. (1974). Conditions for the Convergence of Spectral Decompositions that Correspond to Self-adjoint Extensions of Elliptic Operators. *Journal of Differential Equations V*, 10:360–377.

Il'in, V. A. and Moiseev, E. I. (1971). On Spectral Expansions Connected with nonnegative Selfadjoint extension of the Second Order Elliptic Operator. *Doklady Akademii Nauk SSSR*, 191(4):770–772.

Jamaludin, N. A. and Ahmedov, A. (2013). Uniform Convergence of the Spectral Expansions of the Schrodinger Operator on Closed Domain. *Journal of Physics: Conference Series*, 435(1):12–14.

Jamaludin, N. A. and Ahmedov, A. (2017). The Uniform Convergence of Eigenfunction Expansions of Schrodinger Operator in the Nikolskii Classes. *Journal of Physics: Conference Series*, 890(1):12–51.

Kolmogorov, A. N. and Fomin, S. V. (1970). *Introductory Real Analysis*. DOVER PUBLICATIONS, INC, New York.

Kozlova, N. N. (1984). Riesz Summability of Continuous Functions from Nikol'skii Classes. *Journal of Differential Equations*, 20(1):46–56.

Levitan, B. M. (1954). On Expansion in Eigenfunctions of the Laplace Operator. *Matematicheskii Sbornik*, 77(2):267–316.

Meleshko, V. V. (2003). Selected topics in the History of the Two Dimensional Biharmonic Problem. *Applied Mechanics Reviews*, 56:33–85.

Moiseev, E. I. (1977). On Uniform Convergence of Spectral Decompositions on Closed Domain. *Dokl. Akad. Nauk SSSR*, 223(6):1042–1045.

Nikol'skii, S. M. (1971). Approximation of Functions of Several Variables and Embedding Theorems. *Army Foreign Science and Technology Center Charlottesville*.

Pinsky, M. A. (1993). Fourier Inversion for Piecewise Smooth Functions in Several Variables. *Proceedings of the American Mathematical Society*, 118(3):903–910.

Pinsky, M. A. and Taylor, M. E. (1997). Pointwise Fourier Inversion: A Wave Equation Approach. *Journal of Fourier Analysis and Applications*, 3(6):647–703.

Rakhimov, A. A. (1987a). About Uniform Convergence of the Spectral Decomposition for the Functions from Nikolskii classes in Closed Domain. *VINITI 1564-B87*, pages 1–18.

Rakhimov, A. A. (1987b). On Uniform Convergence of Spectral Resolutions of a Continuous Function in a Closed Domain. *Izv. Akad. Nauk UzSSR, Ser. Fiz.-Mat. Nauk*, 6:17–22.

Rakhimov, A. A. (1988). *Thesis of Doctor of Philosophy*. PhD thesis, Tashkent State University.

Rakhimov, A. A. (1996a). On the Localization Conditions of Spectral Expansions Connected with Elliptic Operators from the Class A_r. *Journal of Mathematics Notes*, 59(3):421–427.

Rakhimov, A. A. (1996b). Spectral Expansions of Distributions from Negative Sobolev Spaces. *Journal of Differential Equations*, 32(7):1010–1013.

Rakhimov, A. A. (2000). On the Localization of Multiple Trigonometric Series of Distributions. *Doclades of Russian. Acad. Science*, 374(1):20–22.

Rakhimov, A. A. (2003). Localization of the Spectral Decompositions of Distributions Connected with the Schrodinger Operator. *Modern Problems of Mathematical Physics and Information Technologies*, 1:167–172.

Rakhimov, A. A. (2004a). On the Sharp Conditions of Localization Multiple Trigonometric Series of Distributions. *Doclades of Uzbek Acad. Science*, 3.

Rakhimov, A. A. (2004b). On the Sharp Conditions of Localization of Spectral Expansions Connected with One Elliptic Operator. *Doclades of Uzbek Acad. Science*, 3:14–17.

Rakhimov, A. A. (2006). Localization Conditions of Spectral Expansions of Distributions Connected with Laplace Operator on Sphere. *Acta NUUz*, pages 47–49.

Rakhimov, A. A. (2013). On an Uniform Convergence of the Eigenfunction Expansions. *ournal of Physics: Conference Series*, 435:012010.

Rakhimov, A. A. (2016). On the Equiconvergence of the Fourier Series and the Fourier Integral of Distributions. *AIP Journal Conf. Proc. 1739 020060 http://dx.doi.org/10.1063/1.4952540*, pages 1–6.

Rakhimov, A. A. (2017). On the Uniform Convergence of Fourier Series on a Closed Domain. *Eurasian Mathematical Journal ISSN 2077-9879*, 8(3):70–79.

Rakhimov, A. A., Ahmedov, A., and Zainuddin, H. (2011). Localization Principle of the Spectral Expansions of Distributions Connected with Schrodinger Operator. *Australian Journal of Basic and Applied Sciences*, 5(5):1–4.

Rakhimov, A. A., Ahmedov, A., and Zainuddin, H. (2012). On the Spectral Expansions of Distributions Connected with Schrodinger Operator. *Applied Mathematics Letters*, 25(6):921–924.

Rakhimov, A. A. and Kamran, Z. (2010). On an Estimation of Eigenfunctions of Schrdingers Operator on a Closed Domain. *MathDigest. Bulletin of Universiti Putra Malaysia*, June Edition:1–4.

Rakhimov, A. A., Kamran, Z., and Nazir, A. (2010). Uniform Convergence Spectral Expansions Connected with Schrodinger's Operator of Continuous Function in a Closed Domain. *J. The Nucleus*, 47(4):261–265.

Rakhimov, A. A., Torla, H., and Rasedee, A. F. N. (2017). On Equiconvergence of Fourier Series and Fourier Integral. *IOP Conf. Series: Journal of Physics: Conf. Series*, 819:012025.

Reed, M. and Simon, B. (1978). *Methods of Modern Mathematical Physics*. Academics Press, Inc, New York.

Shishmarev, V. A. (1969). The Mean Value Theorem for Polyharmonic Equations and its Applications. *Journal of Functional Analysis*, 4:69–76.

Shishmarev, V. A. and Fourier, I. A. (1969). Series by Fundamental Systems of Functions of a Polyharmonic Operator. *Doklady Akademii Nauk SSSR*, 10:1473–1485.

Stein, E. and Weiss, G. (1971). *Introduction to Fourier Analysis on Euclidean Spaces*. Princeton University Press, New Jersey.

Titschmarch, E. C. (1958). *Eigenfunction Expansions Associated with Second-order Differential Equations. Part II*. Clarendon Press, Oxford.

Titschmarch, E. C. (1961). Eigenfunction Expansions Connected with Differential Equations of Second Order.

Triebel, H. (1978). *Interpolation Theory, Function Spaces, Differential Operators.* North-Holland Publishing Company, New York.