RUNGE-KUTTA TYPE METHODS FOR SOLVING HIGH-ORDER ORDINARY DIFFERENTIAL EQUATIONS

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By

NIZAM GHANNAM FAYEZ GHAWADRI

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Doctor of Philosophy

December 2018

DEDICATIONS

To my beloved family and friends

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of **the requirement for the degree of Doctor of Philosophy**

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NIZAM GHANNAM FAYEZ GHAWADRI

December 2018

Chair: Associate Professor Norazak Senu, PhD
Faculty: Institute for Mathematical Research

This study is focused on developing Runge-Kutta type methods to solve two types of ordinary differential equations (ODEs). The first type is the special third-order ODEs in the forms of $y''' = f(x\ y)$ and $y''' = f(x\ y\ y')$: The second type is the special fourth-order in the forms of $y^{(4)} = f(x\ y\ y')$ and $y^{(4)} = f(x\ y\ y'\ y'')$ and the general fourth-order ODEs. These types of ODEs often used to describe the mathematical models for problems arises in several fields of applied sciences and engineering.

The first part of this thesis is focused on construction an exponentially-fitted explicit modified Runge-Kutta type (MRKT) method for solving special third-order ordinary differential equations (ODEs) in the form of $y''' = f(x\ y)$: The new three-stage fourth-order explicit MRKT method is called EFMRKT4 for solving initial value problems is derived. Meanwhile, exponentially- and trigonometrically-fitted explicit modified Runge-Kutta type methods denoted as EFMRKT and TFMRKT respectively for solving special third-order ODEs in the form of $y''' = f(x\ y\ y')$ are derived. The new four-stage fifth-order explicit MRKT methods are called EFMRKT5 and TFMRKT5 respectively for solving initial value problems whose solutions involving exponential or trigonometric form.

The second part of this thesis is focused on derivation of new explicit Runge-Kutta type RKDF, RKTF and RKTGF methods for directly solving $y^{(4)} = f(x\ y\ y')$ $y^{(4)} = f(x\ y\ y'\ y'')$ and $y^{(4)} = f(x\ y\ y'\ y''\ y''')$ respectively . The order conditions of the RKDF, RKTF and RKTGF approaches are constructed by using two methods the first method is using the Taylor series expansion and the second method is B-series and quad-colored trees. Based on algebraic order conditions, fourth- and fifth-order explicit RKDF and RKTF methods using constant step length and an embedded explicit RKDF

and RKTF methods of 5(4) pair for variable step size have been derived respectively. The new three-stage fourth- and fifth-order explicit RKDF methods are called RKDF4 and RKDF5 for solving $y^{(4)} = f(x\ y\ y^{0})$ is constructed respectively and three-stage fourth-order and four-stage fifth-order explicit RKTF methods are called RKTF4 and RKTF5 for $y^{(4)} = f(x\ y\ y^{0}\ y^{00})$ is developed respectively. The new fourth-order four-stage explicit RKTGF method that denoted as RKTGF4 using constant step size have been constructed.

The third part of this thesis is focused on derivation of diagonally implicit Runge-Kutta type (DIRKT) approach to solve special fourth-order ODEs. To see the accurocy and effectiveness of the method, the constant step size code is developed and numerical results are compared with current methods given in literature.

In conclusion, the proposed methods constructed in this study are suitable to solve special third-order, special fourth-order and general fourth-order ODEs. The proposed methods are also more efficient than the existing RK type methods in the terms of accuracy, maximum global error and number of function evaluations.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

KAEDAH RUNGE-KUTTA UNTUK MENYELESAIKAN PERSAMAAN PEMBEZAAN BIASA PERINGKAT TINGGI

Oleh

NIZAM GHANNAM FAYEZ GHAWADRI

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Kajian ini memberi tumpuan kepada pembangunan kaedah-kaedah jenis Runge-Kutta untuk menyelesaikan dua jenis Persamaan Pembezaan Biasa (PPB). Jenis pertama PPB ialah peringkat-ketiga khas dalam bentuk $y^{000} = f(x\ y)$ dan $y^{000} = f(x\ y\ y^{0})$: Bentuk kedua ialah peringkat-keempat khas dalam bentuk $y^{(4)} = f(x\ y\ y^{0})$ dan $y^{(4)} = f(x\ y\ y^{0}\ y^{00})$ dan PPB peringkat-keempat umum. PPB jenis ini sering digunakan untuk menerangkan model matematik untuk masalah yang timbul dalam beberapa bidang aplikasi sains dan kejuruteraan.

Bahagian pertama tesis ini memberi tumpuan kepada pembangunan kaedah suai-secara-eksponen tak tersirat Jenis Runge-Kutta Terubahsuai (MRKT) untuk menyelesaikan PPB khas peringkat-ketiga dalam bentuk $y^{000} = f(x\ y)$: Kaedah MRKT tak tersirat tahap-tiga peringkat-keempat yang baharu dipanggil EFMRKT4 untuk menyelesaikan masalah nilai awal yang diperolehi.

Sementara itu, Kaedah Jenis Runge-Kutta Terubahsuai Suai-secara-eksponen dan Suai-secara-Trigonometrik masing-masing dinyatakan sebagai EFMRKT dan TFMRKT untuk menyelesaikan PPB peringkat-ketiga khas dalam bentuk $y^{000} = f(x\ y\ y^{0})$ telah diterbitkan. Kaedah MRKT tak tersirat tahap-empat peringkat-kelima yang baharu masing-masing dipanggil EFMRKT5 dan TFMRKT5 untuk menyelesaikan masalah nilai awal yang penyelesaiannya melibatkan bentuk eksponen atau trigonometrik.

Bahagian kedua daripada tesis ini difokuskan kepada pembentukan kaedah baharu tak tersirat Runge-Kutta bentuk RKDF, RKTF dan RKTGF masing-masing untuk menyelesaikan secara langsung $y^{(4)} = f(x\ y\ y^{0})$ $y^{(4)} = f(x\ y\ y^{0}\ y^{00})$ dan

$y^{(4)} = f(x\ y\ y^{0}\ y^{00}\ y^{000})$. Syarat peringkat bagi pendekatan RKDF, RKTF dan RKTGF dibina dengan menggunakan dua kaedah kaedah pertama menggunakan pengembangan siri Taylor dan kaedah kedua ialah siri-B dan pokok empat-warna. Berdasarkan kepada syarat-syarat peringkat algebra, kaedah RKDF dan RKTF yang tak tersirat peringkat-keempat dan kelima yang menggunakan panjang langkah tetap dan kaedah benaman tak tersirat RKDF dan RKTF bagi pasangan 5(4) untuk saiz langkah berubah masing-masing telah diterbitkan. Kaedah RKDF tak tersirat tahap-empat peringkat-kelima yang baharu disebut sebagai RKDF4 dan RKDF5 untuk menyelesaikan $y^{(4)} = f(x\ y\ y^{0})$ masing-masing dibina dan kaedah RKTF tak tersirat tahap-tiga peringkat-keempat dan tahap-empat peringkat-kelima yang masing-masing dipanggil RKTF4 dan RKTF5 untuk $y^{(4)} = f(x\ y\ y^{0}\ y^{00})$ dibangunkan. Kaedah baharu tak tersirat RKTGF tahap-empat peringkat-keempat yang ditandakan sebagai RKTGF4 menggunakan saiz langkah tetap telah dibina.

Bahagian ketiga daripada tesis ini difokuskan kepada pembentukan kaedah Jenis Runge-Kutta Pepenjuru Tersirat (DIRKT) untuk menyelesaikan PPB peringkat-keempat khas. Untuk melihat ketepatan dan keberkesanan kaedah ini, kod saiz langkah tetap dibangunkan dan keputusan berangkanya dibandingkan dengan kaedah terkini yang diberikan dalam kajian lepas.

Kesimpulannya, kaedah yang dicadangkan dalam kajian ini sesuai untuk menyelesaikan PPB peringkat-ketiga khas, peringkat-keempat khas dan peringkat-keempat umum. Kaedah yang dicadangkan juga lebih cekap daripada kaedah jenis RK sedia ada dari segi kejituan, ralat global maksimum dan bilangan penilaian fungsi.

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This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Doctor of Philosophy. The **members of the Supervisory Committee were as follows:**

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ODEs	Ordinary Differential Equations
IVPs	Initial Value Problems
LTE	Local Truncation Error
MAXE	Maximum Error
RKM	Runge-Kutta method
MRK	Modified Runge-Kutta Method
RKT	Runge-Kutta type method
RK4	Fourth-order Runge-Kutta Method
EFMRKT4	The new three-stage fourth-order exponentially-fitted modified explicit Runge-Kutta method
RKDM4	The three-stage fourth-order explicit Runge-Kutta method given by Mechee et al. (2014a)
EFRKS4	The four-stage fourth-order exponentially-fitted Runge-Kutta method as given in Simos (2000)
TFMRKT5	The new four-stage fifth-order trigonometrically-fitted modifid RK type method
EFMRKT5	The new four-stage fifth-order exponentially-fitted RK type method
RKT5	The four-stage fifth-order RK type method given by Fawzi et al. (2017)
RK5B	The six-stage fifth-order RK method given in Butcher (2008)
RKF5	The six-stage fifth-order RK method given in Lambert (1991)
TFRK	The six-stage fifth-order trigonometrically-fitted RK method given in Anastassi and Simos (2005)
RKDF5	The new three-stage fifth-order RKT method
RKDF4	The new three-stage fourth-order RKT method
RKDF	The new explicit Runge-Kutta type methods for solving $y^{(4)} = f(x\ y\ y^{0})$
RKF5	The six-stage fifth-order RK method given by Lambert (1991)
DOPRI5	The seven-stage fifth-order RK method derived by Dormand (1996)
RKM4	The five-stage fourth-order RK method given by Hairer et al. (2010)
RKTF	The new explicit Runge-Kutta type methods for solving $y^{(4)} = f(x\ y\ y^{0}\ y^{00})$
RKTF5	The new four-stage fifth-order RKTF method
RKTF4	The new four-stage fourth-order RKTF method
RKDF5(4)	The new Runge-Kutta type 5(4) pair
RKF5(4)	Runge-Kutta 5(4) pair as given in Fehlberg (1969)
DOPRI(5)4	Runge-Kutta 5(4) pair introduced in Butcher (2008)
RKTGF5(4)	The new Runge-Kutta type 5(4) pair
RKTGF4	The new three-stage fourth-order RKT method

RKTGF	**The new explicit Runge-Kutta type methods for solving $y^{(4)} = f(x\ y\ y^{0}\ y^{00}\ y^{000})$**
DIRKT	**The new diagonally implicit Runge-Kutta type methods for solving special fourth-order**
DIRKT5	The three-stage fifth-order diagonally implicit Runge-Kutta **type method which was derived in this chapter.**
DIRKT6	**The four-stage sixth-order diagonally implicit Runge-Kutta type method which was derived in this chapter.**
RKRI5	The fifth-order three-stage implicit Runge-Kutta Radau I **method given by Lambert (1991).**
RKRIIA5	The fifth-order three-stage implicit Runge-Kutta Radau IIA **method as given by Butcher (2008).**
DIRK5	The fifth-order five-stage diagonally implicit Runge-Kutta **method given by Ababneh et al. (2009).**
RKLIIIC6	**The sixth-order four-stage implicit Runge-Kutta Lobatto IIIC method as given by Lambert (1991).**

CHAPTER 1

INTRODUCTION

1.1 Ordinary Differential Equations (ODEs)

Ordinary differential equations (ODEs) are equations that involve an unknown function with independent variable and one or more of its derivatives. ODEs arise in many contexts of engineering and science such as fluid dynamics, radioactive decay and population growth. Many theoretical and numerical studies for such equations have appeared in the literature. The analytical way to solve ODEs is via application of integration technique. However, the anti-derivatives for most realistic systems of ODEs are difficult or impossible to find. Thus, numerical methods for ODEs have attracted considerable attention.

1.2 Numerical Methods for ODEs

Many differential equations can not be solved analytically. For practical purposes however, such as in engineering, a numerical approximation to the solution is often sufficient. Numerical methods for ODEs are methods to find numerical approximations to the solutions of ODEs.

Here, we consider the n-th order ODEs can be written as:

$$y^{(n)} = f\left(x\ y\ :::\ y^{(n-1)}\right)\quad n = 2\ 3\ 4\ :::: \tag{1.1}$$

with initial conditions:
$y(a) = y_0$ and $y^{(i)}(a) = h_i\ \ 0\quad i \leq n-1\ \ x\,2\ [a\ \ b]$
while the first order ODEs can be written as:

$$\frac{dy}{dx} = f\left(x\ y(x)\right)\ y(a) = y_0 \tag{1.2}$$

is a continuous vector-valued function.For $x\,2\ [a\ \ b]$. In (1.2), the quantity being differentiated, y is named as the dependent variable, while the quantity with respect to which y is differentiated, x is named as independent variable.

1.3 Initial Value Problem of ODEs

The initial value problems (IVPs) of a system first order differential equation is defined as:

$$y^{0}(x) = f(x\ y) \tag{1.3}$$

with initial conditions

$$y(x_0) = y_0, \quad x \in [a, b]$$

where $f : R \times R^n \rightarrow R^n$ is a continuous vector-valued function and $y_0 \in R^n$.

$$y(x) = [y_1(x), y_2(x), \ldots, y_s(x)]^T$$
$$f(x, y) = [f_1(x, y), f_2(x, y), \ldots, f_s(x, y)]^T$$

and y_0 is a given vector of initial conditions and their solution may oscillatory.

The initial value problems (IVPs) of general second-order is defined as:

$$y''(x) = f(x, y, y') \tag{1.4}$$

with initial conditions

$$y(x_0) = y_0, \quad y'(x_0) = y'_0, \quad x \in [a, b]$$

where $f : R \times R^n \times R^n \rightarrow R^n$ is a continuous vector-valued function and $y_0, y'_0 \in R^n$.

$$y(x) = [y_1(x), y_2(x), \ldots, y_s(x)]^T$$
$$y'(x) = \left[y'_1(x), y'_2(x), \ldots, y'_s(x)\right]^T$$
$$f(x, y, y') = \left[f_1(x, y, y'), f_2(x, y, y'), \ldots, f_s(x, y, y')\right]^T$$

and y_0, y'_0 are the vector of initial conditions

The initial value problems (IVPs) of general third-order is defined as:

$$y'''(x) = f(x, y, y', y'') \tag{1.5}$$

with initial conditions

$$y(x_0) = y_0, \quad y'(x_0) = y'_0, \quad y''(x_0) = y''_0, \quad x \in [a, b]$$

where $f : R \times R^n \times R^n \times R^n \rightarrow R^n$ is a continuous vector-valued function and $y_0, y'_0, y''_0 \in R^n$.

$$y(x) = [y_1(x), y_2(x), \ldots, y_s(x)]^T$$
$$y'(x) = \left[y'_1(x), y'_2(x), \ldots, y'_s(x)\right]^T$$
$$y''(x) = \left[y''_1(x), y''_2(x), \ldots, y''_s(x)\right]^T$$
$$f(x, y, y', y'') = \left[f_1(x, y, y', y''), f_2(x, y, y', y''), \ldots, f_s(x, y, y', y'')\right]^T$$

and y_0, y'_0 and y''_0 are the vector of initial conditions a special type of fourth-order IVPs can be stated as

$$y^{(4)}(x) = f(x, y, y') \tag{1.6}$$

with initial conditions
$y(x_0) = y_0,\ y'(x_0) = y'_0,\ y''(x_0) = y''_0,\ y'''(x_0) = y'''_0,\ x \in [a, b]$ **where** $f : R \times R^n \times R^n \to R^n$ **is a continuous vector-valued function and** $y_0, y'_0, y''_0, y'''_0 \in R^n$.

$$y(x) = [y_1(x), y_2(x), \ldots, y_s(x)]^T$$
$$y'(x) = [y'_1(x), y'_2(x), \ldots, y'_s(x)]^T$$
$$y''(x) = [y''_1(x), y''_2(x), \ldots, y''_s(x)]^T$$
$$y'''(x) = [y'''_1(x), y'''_2(x), \ldots, y'''_s(x)]^T$$
$$f(x, y, y') = [f_1(x, y, y'), f_2(x, y, y'), \ldots, f_s(x, y, y')]^T$$

and y_0, y'_0, y''_0 **and** y'''_0 **are the vector of initial conditions.**

Some other forms of fourth-order IVPs that are under consideration can be defined as

$$y^{(4)}(x) = f(x, y, y', y'') \tag{1.7}$$

with initial conditions
$y(x_0) = y_0,\ y'(x_0) = y'_0,\ y''(x_0) = y''_0,\ y'''(x_0) = y'''_0,\ x \in [a, b]$ **where** $f : R \times R^n \times R^n \times R^n \to R^n$ **is a continuous vector-valued function and** $y_0, y'_0, y''_0, y'''_0 \in R^n$.

$$y(x) = [y_1(x), y_2(x), \ldots, y_s(x)]^T$$
$$y'(x) = [y'_1(x), y'_2(x), \ldots, y'_s(x)]^T$$
$$y''(x) = [y''_1(x), y''_2(x), \ldots, y''_s(x)]^T$$
$$y'''(x) = [y'''_1(x), y'''_2(x), \ldots, y'''_s(x)]^T$$
$$f(x, y, y', y'') = [f_1(x, y, y', y''), f_2(x, y, y', y''), \ldots, f_s(x, y, y', y'')]^T$$

and y_0, y'_0, y''_0 **and** y'''_0 **are the vector of initial conditions.**

The initial value problems (IVPs) of general fourth-order is defined as:

$$y^{(4)}(x) = f(x, y, y', y'', y''') \tag{1.8}$$

with initial conditions

$y(x_0) = y_0, \quad y'(x_0) = y'_0, \quad y''(x_0) = y''_0, \quad y'''(x_0) = y'''_0, \quad x \in [a, b]$

where $f : R \times R^n \times R^n \times R^n \times R^n \to R^n$ is a continuous vector-valued function and $y_0, y'_0, y''_0, y'''_0 \in R^n$.

$$y(x) = [y_1(x), y_2(x), \ldots, y_s(x)]^T$$

$$y'(x) = \left[y'_1(x), y'_2(x), \ldots, y'_s(x)\right]^T$$

$$y''(x) = \left[y''_1(x), y''_2(x), \ldots, y''_s(x)\right]^T$$

$$y'''(x) = \left[y'''_1(x), y'''_2(x), \ldots, y'''_s(x)\right]^T$$

$$f(x, y, y', y'', y''') = \left[f_1(x, y, y', y'', y'''), f_2(x, y, y', y'', y'''), \ldots, f_s(x, y, y', y'', y''')\right]^T$$

and y_0, y'_0, y''_0 and y'''_0 are the vector of initial conditions. One way to solve (1.3)–(1.8) is by Runge-Kutta method (RKM). RKM can be divided into two groups which are explicit and implicit methods. An easy and quick way to distinguish the type of these methods is that the implicit methods need an iteration scheme, usually Newton type iteration during the integration, whereas the explicit methods do not. Therefore, the computation for implicit methods is more expensive than explicit methods. In addition to the implementation of the methods, accuracy and stability are further factors for judging the efficiency of a method. In this study, we are focusing on solving problem **(1.6)–(1.8) by using explicit Runge-Kutta type methods for directly solving fourth-order differential equations.**

1.4 Existence and Uniqueness of Solution

Initial value problems describe a problem together with the behavior of it's path taken at some initial points of the independent variable x. Some of the characteristic of initial value problems that answer this question, as given by Butcher (2008), are existence of solution, uniqueness of the solution if it exists and the sensitivity of the solution to a small perturbation to the initial information. One of the well known conditions that guarantees these characteristics is the Lipschitz condition.

Definition 1.1 **A function $f : R \times R^d \to R^d$ is said to satisfy Lipschitz condition in its second variable if there exist a constant L such that for any $x \in [a, b]$ and $y_1, y_2 \in R^d$,**

$$\| f(x, y_1) - f(x, y_2) \| \leq L \| y_1 - y_2 \| \tag{1.9}$$

where L is called Lipschitz constant.

Theorem 1.1 : (Existence and Uniqueness)
Let $f(x, y(x))$ *be defined and continuous* $\forall$ **points** $(x, y(x))$ *in a domain D defined by* $x \in [a, b]$, $y \in (-\infty, \infty)$ *a and b are finite, and that* $f(x, y(x))$ *satisfies Lipschitz condition.* **Then for any given number z, $\exists$ a unique solution y(x) of the IVP (1.3), where $\forall$ $(x, y(x)) \in D$, y(x) is continuous and differentiable.**

The proof is given by Henrici (1962).

1.5 Runge-Kutta Type Method for Directly Solving Third-Order ODEs in the form of $y''' = f(x, y, y')$

A Runge-Kutta type (RKT) method can be devided into two kind which is explicit RKT methods and implicit RKT methods. If $a_{ij} = \bar{a}_{ij} = 0$ for $i \leq j$, a RKT method is an explicit and it is an implicit if $a_{ij} \neq 0$ and $\bar{a}_{ij} \neq 0$ for $i \leq j$. In this research context, our focus is mainly on explicit RKT method.

The general form of RKT method with m-stage for solving the IVP

$$y'''(x) = f\left(x, y(x), y'(x)\right) \tag{1.10}$$

with initial conditions

$$y(x_0) = y_0, \quad y'(x_0) = y'_0, \quad y''(x_0) = y''_0.$$

can be written as follows:

$$y_{n+1} = y_n + h y'_n + \frac{h^2}{2} y''_n + h^3 \sum_{i=1}^{m} b_i k_i \tag{1.11}$$

$$y'_{n+1} = y'_n + h y''_n + h^2 \sum_{i=1}^{m} b'_i k_i \tag{1.12}$$

$$y''_{n+1} = y''_n + h \sum_{i=1}^{m} b''_i k_i \tag{1.13}$$

where

$$k_1 = f(x_n, y_n, y'_n)$$

$$k_i = f\left(x_n + c_i h,\ y_n + c_i h y'_n + \frac{h^2}{2} c_i^2 y''_n + h^3 \sum_{j=1}^{i-1} a_{ij} k_j,\ y'_n + c_i h y''_n + h^2 \sum_{j=1}^{i-1} \bar{a}_{ij} k_j\right) \tag{1.14}$$

for $i = 2, 3, \ldots, m$. The parameters b_i, b'_i, b''_i, a_{ij}, $\bar{a}_{ij}$ and c_i of the RKT method are assumed to be real for $i, j = 1, 2, \ldots, m$.

The method can be represented by Butcher tableau as follows:

c	A	A	
	b^T	b^{0T}	b^{00T}

(1.15)

1.5.1 Algebraic Order Conditions for RKT Method

The order conditions for RKT method may be obtained from direct expansion of Taylor series using the Local Truncation Error (LTE). The m-stage up to order six, the method RKT can be expressed as follows:

$$
\begin{aligned}
y_{n\ 1} &= y_n \quad hF(x_n\ y_n\ y_n^0) \\
y^0_{n\ 1} &= y^0_n \quad hF^0(x_n\ y_n\ y_n^0) \\
y^{00}_{n\ 1} &= y^{00}_n \quad hF^{00}(x_n\ y_n\ y_n^0): \qquad (1.16)
\end{aligned}
$$

where the increment functions are

$$
\begin{aligned}
F(x_n\ y_n\ y_n^0) &= y_n^0 \quad \frac{h}{2}y_n^{00} \quad h^2 \sum_{i=1}^{m} b_i k_i \\
F^0(x_n\ y_n\ y_n^0) &= y_n^{00} \quad h \sum_{i=1}^{m} b_i^0 k_i \\
F^{00}(x_n\ y_n\ y_n^0) &= \sum_{i=1}^{m} b_i^{00} k_i \qquad (1.17)
\end{aligned}
$$

with k_i is given in (1.14). If we assume that D D^0 and D^{00} are the Taylor series increment function. Then, the local truncation errors of y(x), $y^0(x)$ and $y^{00}(x)$ can be obtained by substituting the accurate solution of (1.10) into (1.17) as follows:

$$
\begin{aligned}
t_{n\ 1} &= h[F - D] \\
t^0_{n\ 1} &= h[F^0 - D^0] \\
t^{00}_{n\ 1} &= h[F^{00} - D^{00}]: \qquad (1.18)
\end{aligned}
$$

In the terms of elementary differentials, these expressions are best given and the Taylor series can be expressed as follows:

$$
\begin{aligned}
D &= y^0 \quad \frac{1}{2}hy^{00} \quad \frac{1}{6}h^2F_1^{(3)} \quad \frac{1}{24}h^3F_1^{(4)} \quad O(h^4) \\
D^0 &= y^{00} \quad \frac{1}{2}hF_1^{(3)} \quad \frac{1}{6}h^2F_1^{(4)} \quad \frac{1}{24}h^3F_1^{(5)} \quad O(h^4) \\
D^{00} &= F_1^{(3)} \quad \frac{1}{2}hF_1^{(4)} \quad \frac{1}{6}h^2F_1^{(5)} \quad O(h^3): \qquad (1.19)
\end{aligned}
$$

The first few elementary differentials for the scalar case are

$$F_1^{(3)} = f$$
$$F_1^{(4)} = f_x \quad f_y y_x \quad f_{y0} y_{xx}$$
$$F_1^{(5)} = f_{xx} \quad y_x f_{xy} \quad f_{xy0} y_{xx} \quad y_x^2 f_{yy} \quad f_{yy0} y_x y_{xx} \quad f_y y_{xx} \quad f_{y0y0} y_{xx}^2 \quad f_{y0} f : \qquad (1.20)$$

Substituting (1.20) into (1.17), the increment functions F F 0 and F 00 for the method will become

$$\sum_{i=1}^{m} b_i k_i = \sum_{i=1}^{m} b_i f \quad \sum_{i=1}^{m} b_i c_i (f_x \quad f_y y_x \quad f_{y0} y_{xx}) h \quad \frac{1}{2} \sum_{i=1}^{m} b_i c_i^2 (f_{xx} \quad y_x f_{xy} \quad f_{xz} y_{xx}$$
$$y_x^2 f_{yy} \quad f_{yy0} y_x y_{xx} \quad f_y y_{xx} \quad f_{y0y0} y_{xx}^2 \quad f_{y0} f) h^2 \quad O(h^3)$$
$$\sum_{i=1}^{m} b_i^0 k_i = \sum_{i=1}^{m} b_i^0 f \quad \sum_{i=1}^{m} b_i^0 c_i (f_x \quad f_y y_x \quad f_{y0} y_{xx}) h \quad \frac{1}{2} \sum_{i=1}^{m} b_i^0 c_i^2 (f_{xx} \quad y_x f_{xy} \quad f_{xy0} y_{xx}$$
$$y_x^2 f_{yy} \quad f_{yy0} y_x y_{xx} \quad f_y y_{xx} \quad f_{y0y0} y_{xx}^2 \quad f_{y0} f) h^2 \quad O(h^3)$$
$$\sum_{i=1}^{m} b_i^{00} k_i = \sum_{i=1}^{m} b_i^{00} f \quad \sum_{i=1}^{m} b_i^{00} c_i (f_x \quad f_y y_x \quad f_{y0} y_{xx}) h \quad \frac{1}{2} \sum_{i=1}^{m} b_i^{00} c_i^2 f_{xx} \quad y_x f_{xy} \quad f_{xy0} y_{xx}$$
$$y_x^2 f_{yy} \quad F_{yy0} y_x y_{xx} \quad f_y y_{xx} \quad F_{y0y0} y_{xx}^2 \quad f_{y0} f) h^2 \quad O(h^3) : \qquad (1.21)$$

From (1.19) and (1.21), the local truncation error (1.18) can be expressed as follows:

$$t_{n \; 1} = h^3 \left[\sum_{i=1}^{m} b_i k_i - \left(\frac{1}{6} F_1^{(3)} \quad \frac{1}{24} h F_1^{(4)} \quad ::: \right) \right]$$
$$t_{n \; 1}^0 = h^2 \left[\sum_{i=1}^{m} b_i^0 k_i - \left(\frac{1}{2} F_1^{(3)} \quad \frac{1}{6} h F_1^{(4)} \quad ::: \right) \right]$$
$$t_{n \; 1}^{00} = h \left[\sum_{i=1}^{m} b_i^{00} k_i - \left(F_1^{(3)} \quad \frac{1}{2} h F_1^{(4)} \quad \frac{1}{6} h^2 F_1^{(5)} \quad ::: \right) \right] : \qquad (1.22)$$

Substituting (1.21) into (1.22) and expanding as a Taylor expansion using Maple package (see Gander and Gruntz (1999)), the local truncation errors or the order conditions for m-stage up to order six for method can be expressed as follows:
Order conditions for y:

Order 3:

$$b_i = \frac{1}{6} : \qquad (1.23)$$

Order 4:

$$b_i c_i = \frac{1}{24} : \qquad (1.24)$$

Order 5:

$$b_i c_i^2 = \frac{1}{60} \qquad b_i a_{ij} = \frac{1}{120} : \qquad (1.25)$$

Order 6:

$$b_i a_{ij} c_j = \frac{1}{720} \qquad b_i c_i^3 = \frac{1}{120} \tag{1.26}$$

$$b_i a_{ij} c_i = \frac{1}{240} \qquad b_i a_{ij} = \frac{1}{720}: \tag{1.27}$$

Order conditions for y^0:

Order 2:

$$b_i^0 = \frac{1}{2}: \tag{1.28}$$

Order 3:

$$b_i^0 c_i = \frac{1}{6}: \tag{1.29}$$

Order 4:

$$b_i^0 c_i^2 = \frac{1}{12} \qquad b_i^0 a_{ij} = \frac{1}{24}: \tag{1.30}$$

Order 5:

$$b_i^0 c_i^3 = \frac{1}{20} \qquad b_i^0 a_{ij} c_j = \frac{1}{120} \tag{1.31}$$

$$b_i^0 a_{ij} c_i = \frac{1}{40} \qquad b_i^0 a_{ij} = \frac{1}{120}: \tag{1.32}$$

Order 6:

$$b_i^0 c_i^2 a_{ij} = \frac{1}{60} \qquad b_i^0 c_i a_{ij} c_j = \frac{1}{180} \qquad b_i^0 c_i^4 = \frac{1}{30}\;\frac{1}{2} \qquad b_i^0 c_j^2 a_{ij} = \frac{1}{720}$$
$$b_i^0 a_{ij} c_i = \frac{1}{180} \qquad b_i^0 a_{ij} c_j = \frac{1}{720} \qquad b_i^0 a_{ij} a_{jk} = \frac{1}{720}\;\frac{1}{2} \qquad b_i^0 a_{ik} a_{ij} = \frac{1}{240}: \tag{1.33}$$

Order conditions for y^{00}:

Order 1:

$$b_i^{00} = 1: \tag{1.34}$$

Order 2:

$$b_i^{00} c_i = \frac{1}{2}: \tag{1.35}$$

Order 3:

$$b_i^{00} c_i^2 = \frac{1}{3} \qquad b_i^{00} a_{ij} = \frac{1}{6}: \tag{1.36}$$

Order 4:

$$b_i^{00} c_i^3 = \frac{1}{4} \qquad b_i^{00} a_{ij} c_j = \frac{1}{24} \qquad b_i^{00} a_{ij} = \frac{1}{24} \qquad b_i^{00} c_i a_{ij} = \frac{1}{8}: \tag{1.37}$$

Order 5:

$$b_i''c_i^4 = \frac{1}{5} \quad b_i''a_{ij}c_j^2 = \frac{1}{60} \quad b_i''a_{ij}c_i = \frac{1}{30} \quad b_i''a_{ij}a_{jk} = \frac{1}{120}$$

$$\frac{1}{2} \quad b_i''a_{ik}a_{ij} = \frac{1}{40} \quad b_i''a_{ij}c_j = \frac{1}{120} \quad b_i''a_{ij}c_ic_j = \frac{1}{30}: \qquad (1.38)$$

Order 6:

$$b_i''c_i^5 = \frac{1}{6} \; \frac{1}{2} \quad b_i''a_{ij}c_i^2c_j = \frac{1}{72} \; \frac{1}{6} \quad b_i''a_{ij}c_i^3 = \frac{1}{72} \quad b_i''a_{ij}c_ic_j = \frac{1}{144}$$

$$\frac{1}{2} \quad b_i''a_{ij}c_j^3 = \frac{1}{240} \quad b_i''a_{ij}c_ja_{jk} \quad b_i''c_ia_{ij}a_{jk} = \frac{1}{90} \quad b_i''c_ia_{ij}c_j^2 = \frac{1}{72}$$

$$\frac{1}{2} \quad b_i''c_i^2a_{ij} = \frac{1}{72} \quad b_i''a_{ij}a_{jk}c_k = \frac{1}{720} \quad b_i''a_{ij}a_{ik} = \frac{1}{72} \; \frac{1}{2} \quad b_i''a_{ij}c_j^2 = \frac{1}{720}$$

$$b_i''a_{ij}a_{jk} \quad b_i''a_{ij}a_{jk} = \frac{1}{360} \; \frac{1}{2} \quad b_i''c_ia_{ij}a_{ik} = \frac{1}{48} \quad b_i''a_{ij}a_{ik}c_j = \frac{1}{72}$$

$$b_i''a_{ij}a_{ik}c_k = \frac{1}{72} \quad b_i''a_{ij}a_{jk}c_i \quad b_i''a_{ij}a_{jk}c_j \quad b_i''a_{ij}a_{ik}c_j = \frac{1}{40}: \qquad (1.39)$$

All indexes are run from one to m, (i j and k = 1 2 3 ::: m). The following simplifying assumption is used in order to reduce the number of equations to be solved:

$$a_{ij} = \frac{c_i^2}{2}: \qquad (1.40)$$

$$b_i' = b_i''(1 - c_i) \qquad (1.41)$$

$$b_i = b_i''\frac{(1 - c_i)^2}{2}: \qquad (1.42)$$

1.5.2 Local Truncation Error (LTE)

The global local truncation error for the p order RKT method is defined as follows:

$$k\, t_g^{(p\ 1)}\, k_2 = \left(\sum_{i=1}^{n_p\ 1} \left(t_i^{(p\ 1)}\right)^2 \; \sum_{i=1}^{n_p'\ 1} \left(t_i'^{(p\ 1)}\right)^2 \; \sum_{i=1}^{n_p''\ 1} \left(t_i''^{(p\ 1)}\right)^2 \right)^{\frac{1}{2}} \qquad (1.43)$$

where

$t^{(p\ 1)}$ $t'^{(p\ 1)}$ and $t''^{(p\ 1)}$ are the local truncation error terms for y y' and y'' respectively, $t_g^{(p\ 1)}$ is the global local truncation error.

1.5.3 Third-Order Linear Differential Equation with Oscillating and Nonoscillating Solutions

This subsection discusses the oscillatory and nonoscillatory properties of the third-order linear differential equation

$$y^{000}(x) \quad p(x)y^{0} \quad q(x)y = 0 \tag{1.44}$$

A solution of (1.44) is said to be oscillatory if it changes signs for arbitrarily large values of x. Then other solutions are said to be nonoscillatory. If $p(x) \quad 0$ and $q(x) \quad 0$ are independant of y, then it is easy to show that if (1.44) has an oscillatory solution then there are two linearly independent oscillatory solutions of (1.44) whose zeros separate and such that any oscillatory solution of (1.44) is a linear combination of them. Assuming that $p(x) \quad p^{0}(x)$ and $q(x)$ are continuous on $[0 \quad)$ the following definition can established (see Hanan (1961), Rovder (1975), Lazer (1966), Jones (1974)).

Definition 1.2 *A solution of (1.44) iscalled oscillatory iff it has an infinity zeros in* $(0 \quad)$ *and nonoscillatory iff it has a finite number of zeros in this interval. Equation* (1.44) is said to be oscillatory iff it has at least one (nontrivial) oscillatory solution, and nonoscillatory iff all of its (nontrivial) solutions are nonoscillatory.

1.6 Modified Runge-Kutta (MRK) Method

An explicit m-stage MRK formula is given by

$$y_{n \quad 1} = y_n \quad h \sum_{i=1}^{m} b_i f(x_n \quad c_i h \; Y_i) \tag{1.45}$$

where

$$Y_i = g_i y_n \quad h \sum_{j=1}^{i-1} a_{ij} f(x_n \quad c_j h \; Y_j): \tag{1.46}$$

The method is said to be explicit when $a_{ij} = 0$ for $i \leq j$ otherwise it is implicit. The method in (1.45) and (1.46) can be reduced into Butcher tableau form (see Table 1.1)

1.7 Exponantially-Fitted Explicit RK Method

The m-stage explicit Runge-Kutta method in the matrix form for first-order equations or systems of equations can be written in the form given in Table 1.2.
Based on this table, we have that for an explicit m-stage Runge-Kutta method, the

Table 1.1: m-stage modified explicit Runge-Kutta method

0						
c_2	g_2	a_{21}				
c_3	g_3	a_{31}	a_{32}			
.	.	.	.			
.	.	.	.			
.	.	.	.			
c_m	g_m	a_{m1}	a_{m2}	...	$a_{m\,m-1}$	
		b_1	b_2	...	b_{m-1}	b_m

Table 1.2: m-stage explicit Runge-Kutta method

0					
c_2	a_{21}				
c_3	a_{31}	a_{32}			
.	.	.			
.	.	.			
.	.	.			
c_m	a_{m1}	a_{m2}	...	$a_{m\,m-1}$	
	b_1	b_2	...	b_{m-1}	b_m

approximation $y_{n\ 1}$ is given by $y_0 = y_n$,

$$y_{n\ 1} = y_n \quad h \mathop{\ }\limits_{i=1}^{m} b_i f(x_n \quad c_i h\ Y_i) \tag{1.47}$$

where

$$Y_i = y_n \quad h \mathop{\ }\limits_{j=1}^{i-1} a_{ij} f(x_n \quad c_j h\ Y_j) \qquad i = 1(1)m \tag{1.48}$$

The method (1.47) is associated with the operator

$$L(x) = z(x \quad h) \quad z(x) - h \mathop{\ }\limits_{i=1}^{m} b_i z^0(x_n \quad c_i h\ Z_i) \tag{1.49}$$

$$Z_i = z(x) \quad h \mathop{\ }\limits_{j=1}^{m} a_{ij} z^0(x_n \quad c_j h\ Z_j) \qquad i = 1(1)m\ \ Z_0 = z(x) \tag{1.50}$$

where z is a continuously differentiable function.

Definition 1.3 The method (1.49) is called exponential of order p if the associated linear operator L vanishes for any linear combination of the linearly independent functions $f \exp(v_0 x) \ \exp(v_1 x) \ ::: \ \exp(v_p x) g$ where v_i $i = 1 \ 2 \ :: \ p$ are real or complex numbers.

1.8 Order Conditions of Runge-Kutta Type Method for Directly Solving Special Fourth-Order ODEs.

The algebraic order conditions for the Runge-Kutta type (RKFD) method up to order seven given in Hussain et al. (2016) which can be presented as follows:

The order terms for y:

4th-order:

$$b_i = \frac{1}{24} \tag{1.51}$$

5th-order:

$$b_i c_i = \frac{1}{120} \tag{1.52}$$

6th-order:

$$b_i c_i^2 = \frac{1}{360} \tag{1.53}$$

7th-order:

$$b_i c_i^3 = \frac{1}{840} \tag{1.54}$$

The order terms for y^0:

3rd-order:

$$b_i^0 = \frac{1}{6} \tag{1.55}$$

4th-order:

$$b_i^0 c_i = \frac{1}{24} \tag{1.56}$$

5th-order:

$$b_i^0 c_i^2 = \frac{1}{60} \tag{1.57}$$

6th-order:

$$b_i^0 c_i^3 = \frac{1}{120} \tag{1.58}$$

7th-order:

$$b_i^0 c_i^4 = \frac{1}{210} \qquad b_i^0 a_{ij} = \frac{1}{5040} \tag{1.59}$$

The order terms for $y^{(I)}$:

2nd-order:

$$b_i^{(I)} = \frac{1}{2} \tag{1.60}$$

3rd-order:

$$b_i^{(I)} c_i = \frac{1}{6} \tag{1.61}$$

4th-order:

$$b_i^{(I)} c_i^2 = \frac{1}{12} \tag{1.62}$$

5th-order:

$$b_i^{(I)} c_i^3 = \frac{1}{20} \tag{1.63}$$

6th-order:

$$b_i^{(I)} c_i^4 = \frac{1}{30} \qquad b_i^{(I)} a_{ij} = \frac{1}{720} \tag{1.64}$$

7th-order:

$$b_i^{(I)} c_i^5 = \frac{1}{42} \qquad b_i^{(I)} a_{ij} c_j = \frac{1}{5040} \qquad b_i^{(I)} c_i a_{ij} = \frac{1}{1008}; \tag{1.65}$$

The order terms for $y^{(II)}$:

1st-order:

$$b_i^{(II)} = 1 \tag{1.66}$$

2nd-order:

$$b_i^{(II)} c_i = \frac{1}{2} \tag{1.67}$$

3th-order:

$$b_i^{(II)} c_i^2 = \frac{1}{3} \tag{1.68}$$

4th-order:

$$b_i^{(II)} c_i^3 = \frac{1}{4} \tag{1.69}$$

5th-order:

$$b_i^{(II)} c_i^4 = \frac{1}{5} \qquad b_i^{(II)} a_{ij} = \frac{1}{120} \tag{1.70}$$

6th-order:

$$b_i^{(II)} c_i^5 = \frac{1}{6} \qquad b_i^{(II)} a_{ij} c_j = \frac{1}{720} \qquad b_i^{(II)} c_i a_{ij} = \frac{1}{144} \tag{1.71}$$

7^{th}-order:

$$b_i^{(00)}c_i^6 = \frac{1}{7} \qquad b_i^{(00)}c_ia_{ij}c_j = \frac{1}{840} \qquad b_i^{(00)}a_{ij}c_j^2 = \frac{1}{2520} \qquad b_i^{(00)}c_i^2a_{ij} = \frac{1}{168}: \tag{1.72}$$

1.9 Exponentially- and Trigonometrically-Fitting Explicit Modified Runge-Kutta-Nyström Method

A generalized explicit modified Runge-Kutta-Nyström **MRKN method formula is given in**

$$y_{n\ 1} = y_n \quad hg_iy_n^0 \quad h^2 \ {}_{i=1}^{m} b_ik_i \tag{1.73}$$

$$y_{n\ 1}^0 = y_n^0 \quad h \ {}_{i=1}^{m} b_i^0k_i: \tag{1.74}$$

where

$$k_1 = f(x_n \ y_n) \tag{1.75}$$

$$k_i = f\left(x_n \quad c_ih \ y_n \quad g_ihc_iy_n^0 \quad h^2 \ {}_{j=1}^{m} a_{ij}k_j\right) \tag{1.76}$$

for $i = 2 \ 3 \ ::: \ m$: The method in (1.73) to (1.76) can be reduced into Butcher tableau form $($see Table 1.3$)$

Table 1.3: m-stage modified explicit Runge-Kutta-Nyström method

0						
c_2	g_2	a_{21}				
c_3	g_3	a_{31}	a_{32}			
.	.	.	.			
.	.	.	.			
.	.	.	.			
c_m	g_m	a_{m1}	a_{m2}	...	$a_{m\,m-1}$	
		b_1	b_2	...	b_{m-1}	b_m
		b_1^0	b_2^0	...	b_{m-1}^0	b_m^0

To construct the exponentially-fitted Runge-Kutta method we want to integrate e^{wx} **and** e^{-wx} **exactly at each stage as follows :**

$$e^{\pm c_i v} = 1 \pm g_i c_i v \quad v^2 \sum_{j=1}^{m} a_{ij} e^{\pm c_j v} \tag{1.77}$$

and the following that corresponds to y and y^0**:**

$$e^{\pm v} = 1 \pm v \quad v^2 \sum_{i=1}^{m} b_i e^{\pm c_i v} \tag{1.78}$$

$$e^{\pm v} = 1 \pm v \sum_{i=1}^{m} b_i^0 e^{\pm c_i v}: \tag{1.79}$$

where $v = wh, w\ 2\ R$. For the trigonometrically-fitted, we want to integrate exactly **sin(wx) and cos(wx) at each stage to have :**

$$\cos(vc_i) = 1 - v^2 \sum_{j=1}^{i-1} a_{ij} \cos(vc_j) \tag{1.80}$$

$$\sin(vc_i) = g_i c_i v - v^2 \sum_{j=1}^{i-1} a_{ij} \sin(vc_j) \tag{1.81}$$

and the following that corresponds to y and y^0**:**

$$\cos(v) = 1 - v^2 \sum_{i=1}^{m} b_i \cos(vc_i) \tag{1.82}$$

$$\sin(v) = v - v^2 \sum_{i=1}^{m} b_i \sin(vc_i) \tag{1.83}$$

$$\cos(v) = 1 - v \sum_{i=1}^{m} b_i^0 \sin(vc_i) \tag{1.84}$$

$$\sin(v) = v \sum_{i=1}^{m} b_i^0 \cos(vc_i): \tag{1.85}$$

for $i = 2\ 3\ :::\ m$**: Solving Equations (1.80) and (1.81) to obtain** $a_{i\,i-1}$ g_i **and solving** Eqations (1.82)-(1.85), to find $b_1\ b_2\ :::\ b_m, b_1^0\ b_2^0\ :::\ b_m^0$:

1.10 Embedded Runge-Kutta Methods

In RK method the embedded pair q(p) is based on the RKM, (c A b) of order q and another RKM, (c A b*) of order p q. An embedded pair is charaterized by Butcher tableau

c	A
	b
	b*

An embedded pair of explicit Runge-Kutta method is used in variable step-size algorithm because its provide a cheap error estimation. From embedded method we obtain an estimate

$$EST_{n\ 1} = k\, y^*_{n\ 1} - y_{n\ 1}\, k: \qquad (1.86)$$

For the numerical integration of the equation $y^0(x) = f(x\ y)$ $y(x_0) = y_0$ we used step-size control procedure by Raptis and Cash (1985):

if $EST_{n\ 1}$ $\frac{TOL}{100}$ $h_{n\ 1} = 2h_n$

if $\frac{TOL}{100} \leq EST_{n\ 1}$ TOL $h_{n\ 1} = h_n$

if $EST_{n\ 1} \geq TOL$ $h_{n\ 1} = \frac{h_n}{2}$ and repeat the step,

where TOL is the requested local error. It should be noted that the q th-order approximation y_n is used as the initial value for the (n 1) th step, that mean the embedded pair is applied in local extrapolation mode or higher order mode.

1.11 Problem Statement

In general, procedure for solving higher order ODEs is by reducing the problems into a system of first order ODEs and the system is solved by a appropriate numerical method **in the literature. The disadvantage of this approach is that more function evaluations are requared to be evaluated or computed, which lead to a longer procedure time and more computational effort. Therefore, the direct numerical method for solving higher** order ODEs becomes essential in the field of numerical analysis. Because these direct methods demonstrated the efficiency in terms of accuracy and computational time.

Here, we are going to construct exponentially- and trigonometrically-fitted explicit modified Runge-Kutta type (MRKT) methods for solving **$y^{000}(x) = f(x\ y\ y^0)$ ODEs and** derive exponentially-fitted explicit MRKT method to solve special fourth-order ODEs.

Iinitial value problem (IVP) of the special and general fourth-order ODEs often arise in many fields of applied sciences such as electromagnetic wave, ship dynamic and gravity driven flow. The aim of this research is to develop algebraic order conditions for explicit Runge-Kutta type methods to directly solve special and general fourth-order ODEs. Then derive explicit Runge-Kutta type method based on the order conditions developed. In additionally, the construction of diagonally implicit Runge-Kutta type (DIRKT) methods for solving spesial fourth-order ODEs.

1.12 Scope of Study

This study focuses on RKT method for solving high-order ODEs. First of all, the aim and scope of this research is to use Taylor's series and the quad-colored trees theory to derive one-step explicit Runge-Kutta type for directly solving $y^{(4)} = f(x, y, y')$, $y^{(4)} = f(x, y, y', y'')$ and $y^{(4)} = f(x, y, y', y'', y''')$. Then, the secondary aim for this research is to derive explicit exponentially-fitted and trigonometrically-fitted modified Runge-Kutta type methods for solving special third-order ODEs and variable step size mode for solving special fourth-order ODEs. Thirdly aim for this research is to derive diagonally implicit Runge-Kutta type (DIRKT) methods to solve special fourth-order ODEs in the form $y^{(4)} = f(x, y)$.

1.13 Objectives of the Study

The objectives of this thesis are:

- To develop exponentially- and trigonometrically-fitted explicit modified Runge-Kutta type methods for solving special third-order ODEs.
- To construct algebraic order conditions and to derive explicit Runge-Kutta type methods for directly solving special and general fourth-order ODEs.
- To derive embedded of explicit Runge-Kutta type methods for directly solving special fourth-order ODEs.
- To derive diagonally implicit Runge-Kutta type (DIRKT) methods for solving special fourth-order ODEs.

1.14 Outline of Thesis

The brief description for the organization of the thesis will be discussed here. In chapter 1, a brief explanation of introduction on ODEs and initial value problem from first-order to fourth-order as well as the existence and uniqueness theorem are given in this chapter. Runge-Kutta type method for directly solving third-order ODEs in the form of $y''' = f(x, y, y')$. A RKT method and local truncation error (LTE) for RKT are presented. In addition, exponentially-fitted and trigonometrically-fitted explicit

modified Runge-Kutta methods for solving first and second-order ODEs. Definitions **and theories that are related methods are discussed.**
Chapter 2, reviews some of a brief history about the numerical solutions ODEs.

In chapter 3, we construct three-stage fourth-order the exponentially-fitted explicit modified Runge-Kutta type method for solving directly third-order ODEs of the form $y''' = f(x\ y)$. We derive four-stage fifth-order exponentially- and trigonometrically-fitted MRKT for solving third-order ODEs of the form $y''' = f(x\ y\ y')$. **The numerical outcomes of the new methods for solving directly third-order ODEs of the form $y''' = f(x\ y)$ and $y''' = f(x\ y\ y')$ have been compared with methods that reduced the** third-order ODEs to the system of first-order ODEs.

Chapter 4 deals with three types, first type focuses the derivation of the order conditions **of explicit Runge-Kutta type (RKTDF) methods for directly solving $y^{(4)} = f(x\ y\ y')$** and three-stage fourth-order and fifth-order RKDF methods are derived. The results **have been compared with existing methods for directly solving $y^{(4)} = f(x\ y\ y')$.** Second type, we derive three-stage fourth-order and four-stage fifth-order explicit **Runge-Kutta type (RKTF) methods for directly solving $y^{(4)} = f(x\ y\ y'\ y'')$ and we discuss the strategies to obtain the new methods. The numerical results have been compared with existing RK methods for directly solving fourth-order ODEs in the form of $y^{(4)} = f(x\ y\ y'\ y'')$. third type focuses on derivation of the explicit four-stage fourth-order explicit Runge-Kutta type (RKTGF) method and the strategies for obtained the new method is discussed. Numerical results have been compared with the existing RK methods for directly solving general fourth-order ODEs.**

Chapter 5, we construct the embedded pairs for RKDF and RKTF methods for variable step-size where the higher order of the methods are based on the methods derived in chapter 4. The methods have been compared with embedded existing RK methods for solving special fourth-order ODEs.

Chapter 6, we present a new diagonally implicit Runge-Kutta type fourth-order two- stage DIRKT4, fifth-order three stage DIRKT5 and sixth-order four-stage DIRKT6 **methods.**
Lastly, the summary of the entire thesis, conclusions and future studies are given in chapter 7.

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