UNIVERSITI PUTRA MALAYSIA

PURSUIT AND EVASION DIFFERENTIAL GAMES DESCRIBED BY INFINITE TWO-SYSTEMS OF FIRST ORDER DIFFERENTIAL EQUATIONS

PUTERI NUR AIZZAT BINTI KAMAL MUSTAPHA

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By

PUTERI NUR AIZZAT BINTI KAMAL MUSTAPHA

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfillment of the Requirements for the Master of Science

April 2018

DEDICATIONS

This thesis dedicated to;

Both my parents
Kamal Mustapha Bin Muhaiyat & Jamilah Binti Abu Samah
for their endless love, patience and support.

My husband & son
Mohd Johari Bin Ibrahim & Muhammad Qhideer Bin Mohd Johari
for their full encouragement, love and care.

My friends in Differential Game Field
for their kindness and encouragement for me to finish this thesis

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the Master of Science

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Chairman:Gafurjan Ibragimov, PhD
Faculty: Institute for Mathematical Research

Differential games are a special kind of problems for dynamic systems particularly for moving objects. Many reseachers had drawn interests on control and differential game problems described by parabolic and hyperbolic partial differential equations which can be reduced to the ones described by infinite systems of ordinary differential equations by using decomposition method.

The main purpose of this thesis is to study pursuit and evasion differential game problems described by first order infinite two-systems of differential equations in Hilbert space, l_2. The control functions of the players are subjected to the geometric constraints. Pursuit is considered completed, if the state of the system coincides with the origin. In the game, the pursuer's goal is to complete the pursuit while oppositely, the evader tries to avoid this.

First, we solve for first order non-homogenous system of differential equations to obtain the general solution $z_k(t),\ k = 1, 2, \ldots.$ Then, to validate the existence and uniqueness of the general solution, we first prove that the general solution exists in Hilbert space. Next, we prove that the general solution is continous on time interval $[0, T]$.

Our main contribution is that we examine the game by solving an auxiliary control problem, validating a control function and find a time for which state of the system can be steered to the origin. Then, we solve pursuit problem by constructing

pursuit strategy and obtain guaranteed pursuit time, θ_1, under the speed of pursuer $\sum_{k=1}^{\infty} |u_k(t,v(t))|^2 \leq \rho^2$ for any $v(\cdot) \in S(\sigma)$. However, for evasion differential game problem, we prove that evasion is possible when the speed of evader, σ, is greater or equal than that of pursuer, ρ, on the interval $[0,T]$.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk Ijazah Sarjana Sains

PERMAINAN PEMBEZAAN PENGEJARAN DAN PENGELAKKAN YANG DI TERANGKAN OLEH DUA-SISTEM PERSAMAAN PEMBEZAAN TAK TERHINGGA TERTIB PERTAMA

Oleh

PUTERI NUR AIZZAT BINTI KAMAL MUSTAPHA

April 2018

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Permainan pembezaan adalah sejenis masalah khas untuk sistem dinamik terutamanya bagi menggerakkan objek. Ramai penyelidik berminat akan kepentingan kawalan dan masalah permainan pembezaan yang diterangkan oleh persamaan pembeza separa parabola dan hiperbolik yang mana masing-masing menggunakan kaedah penghuraian.

Tujuan utama tesis ini adalah untuk mengkaji masalah permainan pembezaan pengejaran dan pengelakkan yang dijelaskan oleh dua-sistem persamaan pembezaan tertib pertama tak terhingga di dalam ruang Hilbert, l_2. Fungsi kawalan pemain berdasarkan kekangan geometri. Pengejaran dianggap selesai, sekiranya keadaan sistem bertemu dengan asalan. Dalam permainan, matlamat pemangsa adalah untuk menamatkan pengejaran manakala secara bertentangan, mangsa perlu mengelak daripada penamatan pengejaran berlaku.

Pertama, kami menyelesaikan sistem persamaan pembezaan bagi tertib pertama bukan homogen untuk mendapatkan penyelesaian am $z_k(t),\ k = 1, 2, \dots$. Kemudian untuk mengesahkan kewujudan dan keunikan penyelesaian am, kami membuktikan bahawa penyelesaian am wujud dalam ruang Hilbert. Seterusnya, kami membuktikan bahawa penyelesaian am adalah berterusan pada selang masa $[0, T]$.

Sumbangan utama kami adalah kami mengkaji permainan dengan menyelesaikan masalah kawalan tambahan, mengesahkan fungsi kawalan dan mencari masa dimana keadaan sistem boleh dikemudikan kepada asalan. Kemudian, kami menyelesaikan masalah pengejaran dengan membina strategi pengejaran dan mendapatkan masa

pengejaran yang terjamin, θ_1, di bawah kelajuan pemangsa $\sum_{k=1}^{\infty}|u_k(t,v(t))|^2 \leq \rho^2$ untuk mana-mana $v(\cdot) \in S(\sigma)$. Walau bagaimanapun, bagi permasalahan persamaan pembezaan pengelakkan, kami membuktikan kemungkinan pengelakkan apabila kelajuan mangsa, σ, adalah lebih besar atau sama berbanding dengan yang mengejar, ρ, pada selang $[0,T]$.

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This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Master of Science. The members of the Supervisory Committee were as follows:

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Declaration by graduate student

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This is to confirm that:

- the research conducted and the writing of this thesis was under our supervision;
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LIST OF ABBREVIATIONS

$\mathbb{R}$	The set of all real numbers
$\mathbb{R}^2$	Two dimesional euclidean space/plane
$\mathbb{R}^n$	The n-dimensional Euclidean space, $\mathbb{R}^n = \{x = (x_1, \dots, x_n \| x_i \in \mathbb{R}, i = 1, \dots, n)\}$
T	Terminal time of a game problem
P_i	The i-th pursuer
E_i	The i-th evader
e	Unit vector
u_i	Control parameter of the i-th pursuer
v_i	Control parameter of the i-th evader
l_2	Hilbert space
$x_i(t)$	State of the i-th pursuer
$y_i(t)$	State of the i-th evader
$C(0,T;l_2)$	Space of continuous functions on [0,T] with value in l_2
$\|z\|$	Norm of z
A^*	The transpose of matrix A
$INSPEM$	Institut Penyelidikan Matematik

CHAPTER 1

INTRODUCTION

1.1 Differential Games

1.1.1 A Brief History

Game theory was originally created to provide a new approach to economic problems. Jon Von Neumann and Oskar Morgenstern felt that the typical problems of economic behavior become strictly identical with the mathematical notions of suitable games of strategy. Friedman (1986) described that the term *game* might suggest that the subject is narrow and unimportant, but this is far from the case. Since the classical work of Von Neuman and Morgenstern was published, the theory of games has proven to be sufficient interest to justify its study as an important discipline.

In business and economics literature, the term *game* refers to the general situation of conflict and competition in which two or more competitors (or participants) are involved in decision-making activities in anticipation of certain outcomes over period of time. The competitors are referred as players. A player may be an individual, a group of individuals or an organization. Dockner et. al (2000) shows a few examples of competitive and conflicting decisions environment involving the interaction between two or more competitors where techniques of theory of games may be used to resolve them are:

- Pricing of products, where a firm's best sales are determine not only by the price levels selected but also by the prices of its competitors set.
- Various TV networks have found that program success is largely depends on what the competitors presents in the same time slot. Therefore, the outcomes of one networks programming decisions have been increasingly influenced by the corresponding decisions made by other networks.
- Success of business tax strategy depends greatly on the position taken by the internal revenue service regarding the expenses that may be disallowed.
- Success of an advertising/marketing campaign depends largely on various types of services offered to the customers, etc.

The competitive situation and outcome not only depend on the decision of one party alone, but rather depends on the interaction between the decision maker and that of a competitor. Therefore, in a world of competitive business, one of the most relevant problems is to study/guess the activities/actions of his rival competitor.

In conjunction with that, the theory of dynamics games raised its concerned with multi-person decision making. The principal characteristics of a dynamic game is that it involves a dynamic decision evolving in time (continuous or discrete), with more than one decision maker, each with its own cost function and possibly having access to different information.

Differential games belong to a subclass of dynamic games called games in the state space. Friedman (1971) indicates that a game in the state space, the modeler introduces a set of variables to describe the state of dynamic system, at any particular instant time in which the game takes place.

Esparza et. al (2013) stated that differential games are a special kind of problems which particularly for moving objects. Differential games began to be studied in 1950's. The notion of the differential game concentrates on such concepts as conflicts, control, optimisation, current information and equilibrium. Differential games are also an attractive mathematical task. Therefore, Theory of Differential Game were intensively developed during 1960's till 1980's.

The origins of differential game theory can be traced back to the late 1940's. Rufus Isaacs (1955) had an appointment with RAND where he formulated missile versus enemy aircraft pursuit schemes in terms of descriptive and navigation (state and control) variables. Further innovation in differential game techniques and basic concepts is followed. However, due to lack of financial support Isaacs's work did not appear in print until 1965.

1.2 Fundamental Concepts

1.2.1 Models in Differential Game Theory

Differential game theory has emerged as a fundamental instrument in pure and applied research. Hajek (1975) divides the game into two different parts which are stochastic and deterministic. The differential game theory can be illustrated in the abbreviated schema where it shows the scope and application of the developed procedures as in Figure 1.1.

In solving a number of practical problems in various fields, it is required to analyse a situation where at the outset, there are two or more opposing parties with conflicting interests and where the action of one depends on the action which the opponent takes. This situation is called as "conflicting situations".

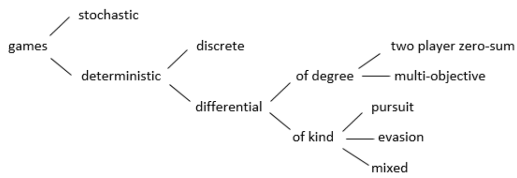

Figure 1.1: Differential game theory scheme

According to Rapoport (1966), numerous examples of conflicting situation can be cited such as situation in the course of military operations. Each of the parties in the operation takes all available measures to prevent the opponent from succeeding. The planning of military operations such as the choice of a armaments system and possible ways of its application in battle are all belong to conflicting situations. Each decision taken in this field must be calculated to ensure that it is least advantageous to the opponent.

The need for analysing such situations is a special calls for mathematical techniques. The theory of differential games, essentially is nothing but a mathematical theory of conflicting situations. The objective of the theory is to analyse and elaborate what constitutes to rational behaviour of each of the opponents in the course of conflicting situation.

The models in Theory of Differential Games described by Rapoport (1966) can be classified depending upon the following factors:

- **Number of Players**: If a game involves only two players (competitors), then it is called a two-person game. However, if the number of players is more, the game is referred to as n-person.
- **Sum of Gains and Losses** : If in a game, sum of the gains to one player is exactly equal to the sum of losses to another player, so that sum of the gains and losses equals zero. Then the game is said to be a zero-sum game. Otherwise it is said to be non-zero sum game.
- **Strategy** : The strategy for a player is the list of all possible actions (moves or course of action) that he will take for every payoff (outcome) that might arise. The particular strategy (or complete plan) by which a player optimizes his gains and losses without knowing the competitor's strategies is called optimal strategy.

Each opposing party or player is known as Pursuer and Evader, where players are denoted as P and E respectively. The pursuer's goal is simply to capture the evader while the evader's goal is certainly to avoid being captured.

1.3 Strategy of Parallel Approach

One of the fundamental concepts in differential game theory is strategy. The enumeration of rules, defining simply the choice for every personal move of a given player depending on the situation arising in the process of the game, is called the strategy of a player. In this section, we will discuss the strategy of parallel approach or known as P-strategy that is often used by many researchers to solve pursuit games.

1.3.1 Control and Trajectory

The equations is govern by the following differential equations

$$\begin{aligned} P &: \dot{x} = u, \;\; x(0) = x_0, \\ E &: \dot{y} = v, \;\; y(0) = y_0, \end{aligned} \tag{1.3.1}$$

where $x, y, x_0, y_0, u, v \in \mathbb{R}^2$, u and v are the control parameters of pursuer, P and evader, E respectively, x_0 and y_0 are initial conditions for the system (1.3.1).

Vectors u and v must satisfy the geometric constraints

$$|u| \leq \rho, \;\; |v| \leq \sigma,$$

where ρ and σ are a given positive number.

Prior to define control functions, we need the following definitions

Definition 1.1 *The set*

$$\{x = (x_1, x_2, \ldots, x_n) |\; a < x_i < b,\; i = 1, \ldots, n\}$$

is called n-cube.

Definition 1.2 *A subset N of $\mathbb{R}^n$ is called a null set (or set of measure zero) in case N can be covered by (is contained) a countable union of n-cubes whose total n-volume is less than an arbitrarily prescribed number $\varepsilon > 0$.*

Example. Any finite or countable infinite set of points in $\mathbb{R}^n$ has measure zero.

Definition 1.3 *Two functions $f(x)$ and $g(x)$ defined on $[a, b]$ that differ in value only* *on null set are said to be equal almost everywhere* $[a,b]$.

Definition 1.4 *Measurable sets of* $\mathbb{R}^n$ *are defined as the members of the smallest of sets of* $\mathbb{R}^n$ *that contains all open sets, all closed sets, all null sets of* $\mathbb{R}^n$*, and also every difference, and countable union, and countable intersection of its members.*

Definition 1.5 *A real-valued function* $h(t)$ *on a real interval* I *is called measurable in case for any real* α *and* β*, the set* $\{t|t \in I,\ \alpha < h(t) < \beta\}$ *is measurable on* I.

Definition 1.6 *A measurable function* $u(t) = (u_1(t), u_2(t)),\ u_1^2(t) + u_2^2(t) \leq \rho^2,\ t \geq 0$ *is called admissible control.*

Thus, equation (1.3.1) has the solution of the form

$$x(t) = x_0 + \int_0^t u(s)ds,$$

where x is the state variable, x_0 is the initial position and u is the control parameter.

1.3.2 P-Strategy

In this section, we study construction of P-strategy. Let the dynamics of pursuer, P and evader, E be described by the differential equations

$$\begin{aligned} P &: \dot{x} = u,\ |u| \leq \rho,\ x(0) = x_0, \\ E &: \dot{y} = v,\ |v| \leq \sigma,\ y(0) = y_0, \end{aligned} \tag{1.3.2}$$

where $\rho > \sigma$, $x, y, x_0, y_0 \in \mathbb{R}^2$, $x_0 \neq y_0$, u and v are control parameters of the pursuer and evader respectively.

Let x_0 and y_0 be any initial positions of players and the unit vector, e, be defined as $e = \dfrac{y_0 - x_0}{|y_0 - x_0|}$. The direction of unit vector, e, is shown in Figure 1.2.

Let

$$v = v_1' + v_2', \tag{1.3.3}$$

where the vector v_2' parallel to e, and v_1' is orthogonal to e.

To define a stratgey for pursuer, set

$$u_1' = v_1'. \tag{1.3.4}$$

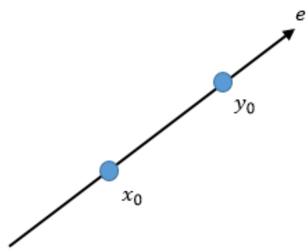

Figure 1.2: Direction of unit vector, *e*

Then, we illustrate the strategy of pursuer

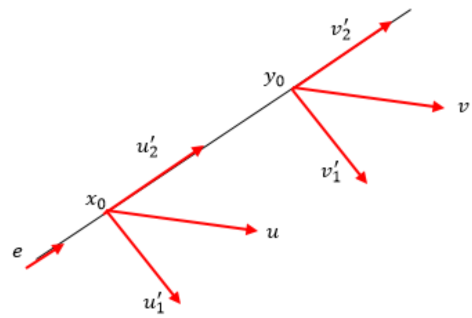

Figure 1.3: Illustration strategy of pursuer

We construct P-strategy as follows

$$u = u_1' + u_2'. \tag{1.3.5}$$

From equation (1.3.3), by applying projection of vector, v, as shown in Figure 1.4, we obtain $v_2' = |v|\cos\alpha.e = (v,e)e$. Then, we have

$$\begin{aligned} v_1' &= v - v_2' \\ &= v - (v,e)e. \end{aligned} \tag{1.3.6}$$

We obtain from equation (1.3.4) that,

$$u_1' = v_1' = v - (v,e)e. \tag{1.3.7}$$

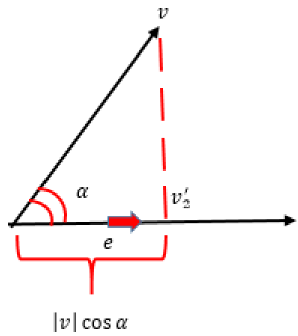

Figure 1.4: Vector projection, v

Assume that the pursuer is moving with maximal speed, ρ. Then

$$\begin{aligned}|u_2'|^2 &= |u|^2 - |u_1'|^2 \\ &= \rho^2 - |v-(v,e)e|^2 \\ &= \rho^2 - \left[|v|^2 - 2(v,(v,e)e) + (v,e)^2\right] \\ &= \rho^2 - |v|^2 + (v,e)^2.\end{aligned}$$

Hence

$$|u_2'|^2 = \sqrt{\rho^2 - |v|^2 + (v,e)^2}.$$

Set

$$u_2' = e\sqrt{\rho^2 - |v|^2 + (v,e)^2}. \tag{1.3.8}$$

Thus, substituting equation (1.3.7) and (1.3.8) into (1.3.5) yields

$$u = v - (v,e)e + e\sqrt{\rho^2 - |v|^2 + (v,e)^2}. \tag{1.3.9}$$

However, if the evader moves with maximal speed, σ, the strategy takes the form

$$u = v - (v,e)e + e\sqrt{\rho^2 - \sigma^2 + (v,e)^2}. \tag{1.3.10}$$

As the application of P-strategy, we prove the following statement.

Theorem 1.1 *If $\rho > \sigma$, then pursuit can be completed in differential game for the time* $T = \dfrac{|y_0 - x_0|}{\rho - \sigma}$.
[Petrosjan (1976)]

Proof.
Let $y_0 \neq x_0$ and $\rho > \sigma$, then

$$\begin{aligned} y(t)-x(t) &= y_0 + \int_0^t v(s)\,ds - x_0 - \int_0^t u(s)\,ds \\ &= y_0 - x_0 + \int_0^t v(s)\,ds - \int_0^t \left[v(s) - (v(s),e)e + e\sqrt{\rho^2 - \sigma^2 + (v(s),e)^2} \right] ds \\ &= y_0 - x_0 - \int_0^t e\left[\sqrt{\rho^2 - \sigma^2 + (v(s),e)^2} - (v(s),e) \right] ds \\ &= e\,d(t), \end{aligned}$$

where $d(t) = |y_0 - x_0| - \int_0^t \left(\sqrt{\rho^2 - \sigma^2 + (v(s),e)^2} - (v(s),e) \right) ds.$

Then, we estimate $d(t)$. To this end, consider the following function

$$f(z) = \sqrt{\rho^2 - \sigma^2 + z^2} - z, \quad -\sigma \leq z \leq \sigma. \tag{1.3.11}$$

It is not difficult to show that $min\ f(z) = f(\sigma) = \rho - \sigma$.

Therefore,

$$\begin{aligned} d(t) &= |y_0 - x_0| - \int_0^t f(z)\,ds \\ &\leq |y_0 - x_0| - \int_0^t f_{min}(z)\,ds \\ &= |y_0 - x_0| - \int_0^t (\rho - \sigma)\,ds \\ &= |y_0 - x_0| - (\rho - \sigma)t. \end{aligned} \tag{1.3.12}$$

Obviously, the right hand side of equation (1.3.12) is $t = T = \dfrac{|y_0 - x_0|}{\rho - \sigma}$. Thus, we obtain $d(T) \leq 0$. Observe, $d(0) = |y_0 - x_0| > 0$. Since $d(t)$ is a continuous function, $d(0) > 0$ and $d(T) \leq 0$, then $d(\tau) = 0$ at some time $\tau \in [0,T]$. Hence $y(\tau) - x(\tau) = 0$ that is $y(\tau) = x(\tau)$. Hence, pursuit is completed.

1.4 Lion and Man Game

In a game of pursuit and evasion, one player (the pursuer) tries to get close to, and possible capture the other (the evader). The evader, in turn, tries to avoid being captured. Consider the task of surveillance, where a guard (pursuer) has to chase and capture an intruder (evader). Another scenario is search-and-rescue, where a rescue worker has to locate a lost hiker. Since the actions of the hiker are not known, worst case pursuit and evasion strategies guarantee that the hiker is found no matter what he does. Many applications related to pursuit-evasion game such as collision-avoidance, search-and-rescue, air traffic control and surveillance. Here, we show lion and man game as an example of pursuit-evasion problem.

A game situation consisting of two players is referred to as two-person game. Based on Rapoport (1966), when there are more than two players, the game situation is known as an *n*-person game. Games are also classified according to their outcomes in terms of each player's gains and losses. If the sum of the players' gains and losses equal zero, the game is referred to as a zero-sum game. The two-person zero-sum game is the most frequently used to demostrate the principles of game theory because it is the simplest mathematically.

Here we introduce Besicovitch (1953) classical two-person zero-sum game (pursuit-evasion) called lion and man game. In this game, it is posed as to determine a strategy for a pursuer (lion), which has a mission to capture evader (man) in a given environment. When the position of the lion and man coincide after a finite time, it means that the lion has successfully capture the man. The aim of the lion is to catch the man and the aim of the man is to avoid being catch by lion, where both have identical motion capabilities. The man wins the game if it can avoid being capture.

In this problem, we are going to show that evasion is possible in differential game of the lion and man. Here, the man constructs strategy to ensure the possibility of not being captured by the lion indefinitely. We say that evasion is possible in the game. Suppose P as a Pursuer (lion) and E as an Evader (man).

The movement of the players are governed by the following equations

$$\begin{aligned} P &: \dot{x} = u, \quad |u| \leq 1, \\ E &: \dot{y} = v, \quad |v| \leq 1, \end{aligned} \tag{1.4.1}$$

where u and v are the control parameters of the pursuer and evader respectively. We denote the radius of the circle by R.

Theorem 1.2 *Evasion is possible in the game of Lion and Man.*
[Besicovitch (1953)]

Proof.
1. Construction of Man (evader's) strategy

In this game, both lion and man have identical motion capabilities and move with maximum speeds equal to 1. The positions of the players are always in a circle for any trajectory, and must not leave the given circle.

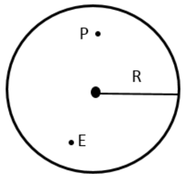

Figure 1.5: State of the players

Both players have perfect information of each other position, but have different goals. Figure 1.5 shows the state of players which are always in a circle.

There are three possible cases for movement of the players in the game as shown in Figure 1.6:

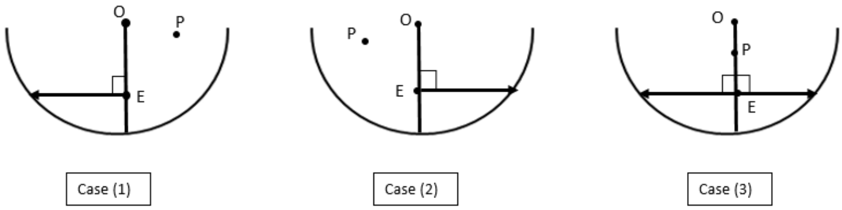

Figure 1.6: The possible movement of Evader (*E*)

In case (1), we can see that P is on the right side of straight line OE, thus E will move to the left perpendicularly to straight line OE. In case (2), E will move to the right perpendicularly to the straight line OE as P is on the left side of straight line OE. In case (3), if P is on the straight line OE, thus E can either move to the left or right. Therefore, we can conclude that movement of E depends on positions of P whether it is on the left or right side of straight line OE where O is the origin of the circle. However, without loss of generality, we assume P is always on the right or on the straight line OE. Thus, E will always moves to the left perpendicularly to line OE.

From Figure 1.7, we denote E_i and P_i for $i \in \{0,1,2,\ldots,n,\ldots\}$ as point of position of E and P respectively at time t_i. At each time, t_i, the distance from E_i to the circumference equal to $\frac{r}{i+1}$. Specifically, at time, t_0, the distance between E_0 to the

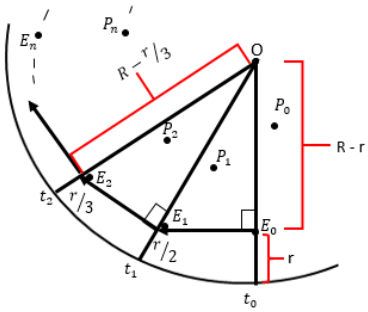

Figure 1.7: Trajectory of P and E inside circular area

circumference is r. If the position of P_0 is on the right or on the line OE_0, then E_0 will move to the left perpendicularly to OE_0 until it reaches at point E_1 at time t_1. The distance between E_1 to circumference is $\frac{r}{2}$. At E_1, if the position of P_1 is on the right or on the line OE_1, then E_1 will move to the left perpendicularly to OE_1 until it reach at point E_2 at time t_2. The distance between E_2 to circumference is $\frac{r}{3}$. This strategy will continue in a similar manner.

2. Evasion is possible on each section, E_iE_{i+1}

Here, we show that evasion is possible on each section E_iE_{i+1}, $i = 0, 1, 2, \ldots$. First, we let $E(t_i) = E_i$ and $P(t_i) = P_i$ is not on the left of OE_i. We assume P_i is moving with speed of $\alpha(t)$ where $0 \leq \alpha(t) \leq 1$.

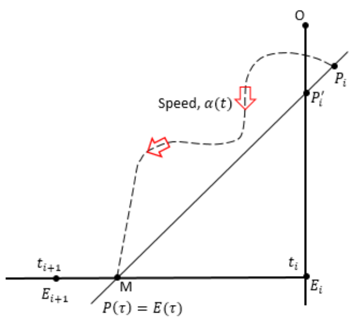

Figure 1.8: Trajectory of P and E on Section E_iE_{i+1}

In order to prove that E can avoid from being captured by P in each section E_iE_{i+1}, we assume the contrary, that is pursuit is completed at some time $\tau \in [t_i, t_{i+1}]$ at point

$M \in E_iE_{i+1}$. Thus,

$$P(\tau) = E(\tau) = M.$$

As all players move with the maximum speed 1, then the time taken of P to reach point M is calculated as follow

$$\widetilde{P_iM} = \int_{t_i}^{\tau} \alpha(t)\, dt \leq \int_{t_i}^{\tau} 1\, dt = \tau - t_i, \qquad (1.4.2)$$

where $\widetilde{P_iM}$ is length of the curve.

On the other hand, the time taken of E to reach point M on E_iE_{i+1} is shown as follows

$$E_iM = (\tau - t_i).1 = \tau - t_i.$$

Hence, we can see from (1.4.2) that

$$E_iM = \tau - t_i \geq \widetilde{P_iM} \geq P_iM \geq P_i^{'}M > E_iM$$

where P_iM and $P_i^{'}M$ are segments of straight lines and $P_i^{'}M$ is the hypotenuse of the right angle triangle $MP_i^{'}E_i$. A contradiction. Hence, our assumption $P(\tau) = E(\tau)$ is wrong. Thus, on each section E_iE_{i+1} evasion is possible.

3. Estimation of total time

Next, we estimate the total time spent by the evader for sections E_iE_{i+1}.

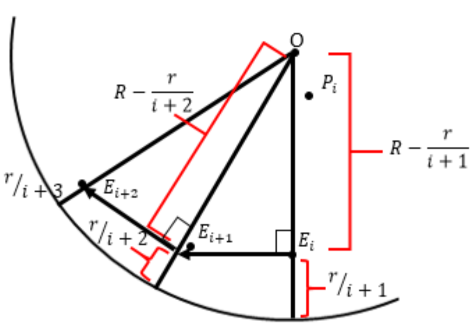

Figure 1.9: Figure of estimation of total time

The time spent by the evader to travel the section E_iE_{i+1} is equal to

$$t_i = \frac{E_iE_{i+1}}{1} = E_iE_{i+1} = \sqrt{\left(R - \frac{r}{i+2}\right)^2 - \left(R - \frac{r}{i+1}\right)^2},$$

where R is the radius of the circle and $i = 0, 1, 2, 3, \ldots, n$. Thus, for sections $E_0E_1, E_1E_2, \ldots, E_nE_{n+1}$, we obtain

$$\begin{aligned}
t_0 &= E_0E_1 = \sqrt{\left(R - \frac{r}{2}\right)^2 - (R-r)^2}, \\
t_1 &= E_1E_2 = \sqrt{\left(R - \frac{r}{3}\right)^2 - \left(R - \frac{r}{2}\right)^2}, \\
&\vdots \\
t_n &= E_nE_{n+1} = \sqrt{\left(R - \frac{r}{n+2}\right)^2 - \left(R - \frac{r}{n+1}\right)^2}.
\end{aligned} \tag{1.4.3}$$

We need to show that $t_n \geq \frac{r}{n+2}$. Indeed,

$$\begin{aligned}
{t_n}^2 &= \left(R - \frac{r}{n+2}\right)^2 - \left(R - \frac{r}{n+1}\right)^2 \\
&= \left(\left(R - \frac{r}{n+2}\right) - \left(R - \frac{r}{n+1}\right)\right)\left(\left(R - \frac{r}{n+2}\right)\right. \\
&\qquad \left. + \left(R - \frac{r}{n+1}\right)\right) \\
&= \left(\frac{r}{n+1} - \frac{r}{n+2}\right)\left(2R - \frac{r}{n+1} - \frac{r}{n+2}\right).
\end{aligned}$$

Since $R \geq r$, then

$$\begin{aligned}
{t_n}^2 &\geq \left(\frac{r}{n+1} - \frac{r}{n+2}\right)\left(2r - \frac{r}{n+1} - \frac{r}{n+2}\right) \\
&= \left(\frac{r}{(n+1)(n+2)}\right)\left(\frac{2rn^2 + 4rn + r}{(n+1)(n+2)}\right).
\end{aligned}$$

Since $2n^2+4n+1 \geq n^2+2n+1$, then

$$\begin{aligned} {t_n}^2 &\geq \left(\frac{r}{(n+1)(n+2)}\right)\left(\frac{rn^2+2rn+r}{(n+1)(n+2)}\right) \\ &= \frac{r^2(n^2+2n+1)}{(n+1)^2(n+2)^2} \\ &= \frac{r^2(n+1)^2}{(n+1)^2(n+2)^2} \\ &= \frac{r^2}{(n+2)^2}. \end{aligned}$$

Hence, we have shown that

$$t_n = \frac{E_nE_{n+1}}{1} \geq \frac{r}{n+2}.$$

Calculating the sum of times spent up to point E_n, we have

$$\begin{aligned} \sum_{i=0}^{n} t_i &\geq \sum_{i=0}^{n} \frac{r}{i+2} \\ &= r\sum_{i=0}^{n} \frac{1}{i+2}. \end{aligned}$$

However, $\sum_{i=0}^{n} \frac{1}{i+2} \to \infty$ as $n \to 0$, then the series $\sum_{i=0}^{\infty} \frac{1}{i+2}$ is divergent. Thus, for the time $t_1+t_2+\cdots+t_n$, the evader will not be captured. Therefore, evasion is possible in the Lion and Man game.

1.5 Objectives of Thesis

The main purpose of this present thesis is to study pursuit and evasion differential game problems in Hilbert space, l_2 with geometric constraints on the control functions of players. Differential game is described by the following infinite system of differential equations:

$$\begin{aligned} \dot{x}_k &= -\alpha_k x_k - \beta_k y_k + u_{k1} - v_{k1}, \quad x_k(0) = x_{k0}, \\ \dot{y}_k &= \beta_k x_k - \alpha_k y_k + u_{k2} - v_{k2}, \qquad y_k(0) = y_{k0}, \end{aligned} \tag{1.5.1}$$

where α_k, β_k are real numbers, $k = 1,2,3,\ldots$, $x_0 = (x_{10}, x_{20}, x_{30}, \ldots) \in l_2$, $y_0 = (y_{10}, y_{20}, y_{30}, \ldots) \in l_2$, $u = (u_{11}, u_{12}, u_{21}, u_{22}, \ldots)$ and $v = (v_{11}, v_{12}, v_{21}, v_{22}, \ldots)$ are control parameters of pursuer and evader respectively.

Objectives of the thesis are organized as follows:

- To suggest and validate a control function for the control problem described by the following infinite system of differential equations:

$$\begin{aligned} \dot{x}_k &= -\alpha_k x_k - \beta_k y_k + \omega_{k1}, \quad x_k(0) = x_{k0}, \\ \dot{y}_k &= \beta_k x_k - \alpha_k y_k + \omega_{k2}, \quad y_k(0) = y_{k0}, \end{aligned} \tag{1.5.2}$$

  where $\omega(\cdot) = (\omega_1(\cdot), \omega_2(\cdot), \omega_3(\cdot), \ldots) \in C(0,T;l_2)$, $\omega_k = (\omega_{k1}, \omega_{k2})$, $k = 1,2,\ldots$.

- To verify a strategy for the pursuer subject to geometric constraints.
- To obtain a guaranteed pursuit time.
- To prove evasion is possible on interval $[0,T]$.

1.6 Outline of thesis

This thesis covers six chapters with the following contents:

Chapter 1 describes briefly about the history of differential games and fundamental concepts of game theory. Number of players, sum of gains and losses and strategy are the fundamental concepts of game theory. Besides, strategy of parallel approach or P-strategy is widely used by many researchers and therefore we discussed it in this chapter. Here, we also discuss on evasion game problem of a classical lion and man game. In this problem, the man (denoted as Evader, E) tries to avoid being captures by the lion (denoted as Pursuer, P). Both P and man E have identical motion capabilities and move with maximum speeds equal to 1. The positions of players are always in a circle. Here, the evader constructs strategy to ensure the possibilty of not being captured by pursuer. Then, it was shown that evasion is possible on each section E_iE_{i+1}. Next, the total time spent by the evader to travel on the section E_iE_{i+1} is estimated.

Chapter 2 focuses on the literature review.

Chapter 3 discusses on how to solve first order non-homogenous linear ordinary differential equations with an initial condition generally. Here, for solving the system, it is divided into two parts. The first part is to obtain solution for first order non-homogenous differential equations and the second part is to solve for homogenous differential equations.

Chapter 4 discusses on the paper of Ibragimov (2013) on a two-person zero-sum pursuit-evasion differential game in Hilbert space, l_2. Here, the pursuer tries to force the state of the system towards the origin of the space, l_2, and the evader tries to avoid this. In this paper, Ibragimov (2013) obtain an equation for the optimal pursuit time and construct optimal strategies for the players in an explicit form. The control functions of the players are subject to integral constraints. It is assumed that the control resource of the pursuer is greater than that evader.

Chapter 5 is devoted to some control and pursuit-evasion differential game problems. These control and differential game problems described by parabolic and hyperbolic partial differential equations can be reduced to ones described by an infinite systems of ordinary differential equations using decomposition method. Besides, in this chapter we also prove for the existence and uniqueness theorem for the infinite system of differential equations in the Hilbert space l_2. First, we prove that the general solution exists in Hilbert space l_2. Next, we prove that the general solution is continous on the interval $[0,T]$. Other than that, for evasion differential game problem, we prove that evasion is possible on the interval $[0,T]$ under geometric constraints on the control functions of players.

Chapter 6 gives a brief conclusion on this thesis and proposes some future studies as an extension to this research.

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