UNIVERSITI PUTRA MALAYSIA

THE ALMOST EVERYWHERE CONVERGENCE OF EIGENFUNCTION EXPANSIONS OF ELLIPTIC DIFFERENTIAL OPERATORS IN THE TORUS

EHAB SALEM MATARNEH

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By

EHAB SALEM MATARNEH

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Doctor of Philosophy

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DEDICATIONS

This thesis is dedicated to my loving family,
my father «Salem» and my mother «Fatima»,
my sisters and my brothers.

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Doctor of Philosophy

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March 2018

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Many of the equations of physical sciences and engineering involve operators of elliptic type. Most important among these is non-relativistic quantum theory, which is based upon the spectral analysis of second order elliptic differential operators. Spectral theory of the elliptic differential operators is an extremely rich field which has been studied by many qualitative and quantitative techniques like Sturm-Liouville theory, separation of variables, Fourier and Laplace transforms, perturbation theory, eigenfunction expansions, variational methods, microlocal analysis, stochastic analysis and numerical methods including finite elements. We note here that the applications of second order elliptic operators to geometry and stochastic analysis are also now of great importance.

In the present research we investigated the problems concerning the almost everywhere convergence of multiple Fourier series summed over the elliptic levels in the classes of Liouville functions on Tours. The sufficient conditions for the almost everywhere convergence problems, which are most difficult problems in Harmonic analysis, are obtained in the classes of Liouville. The difficulty is on the obtaining the suitable estimations for the maximal operator of the partial sums of the Fourier series, which guarantees the almost everywhere convergence of Fourier series. The process of estimating the maximal operator involves very complicated calculations which depends on the functional structure of the classes of functions. The main idea on the proving the almost everywhere convergence of the eigenfunction expansions in the interpolation spaces is estimation of the maximal operator of the partial sums in the boundary classes and application of the interpolation Theorem of the family of linear operators. It is well known that the theory of the eigenfunction expansions of the differential operators closely connected with the convergence problems of Fourier series and integrals. The one of the most important summation method which is called spherical summation method connected with the eigenfunction expansions of the Laplace operator, while the ques-

tions on convergence of the multiple Fourier series summed over the elliptic levels can be investigated by using the spectral theory of the elliptic differential operators.

In chapter III and IV of the present thesis maximal operator of spherical and elliptic partial sums are estimated in the interpolation classes of Liouville and the almost everywhere convergence of the multiple Fourier series by spherical and elliptic summation methods are established. The considering multiple Fourier series as an eigenfunction expansions of the differential operators helps to translate the functional properties (for example smoothness) of the Liouville classes into Fourier coefficients of the functions which being expanded into such expansions.

The sufficient conditions for convergence of the multiple Fourier series of functions from Liouville classes are obtained in terms of the smoothness and dimensions. Such results are highly effective in solving the boundary problems with periodic boundary conditions occurring in the spectral theory of differential operators. The investigations of multiple Fourier series in modern methods of harmonic analysis incorporates the wide use of methods from functional analysis, mathematical physics, modern operator theory and spectral decomposition. New method for the best approximation of the square-integrable function by multiple Fourier series summed over the elliptic levels are established in chapter V. Using the best approximation, the Lebesgue constant corresponding to the elliptic partial sums is estimated. The latter is applied to obtain an estimation for the maximal operator in the classes of Liouville.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

PENUMPUAN HAMPIR MERATA-RATA TEMPAT KE PENGEMBANGAN FUNGSI EIGEN UNTUK OPERATOR ELIPTIK PEMBEZAAN DALAM TOURS

Oleh

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Dalam sains fizikal dan kejuruteraan banyak persamaan matematik yang melibatkan operator jenis elips. Yang paling penting di antaranya adalah teori kuantum yang tidak relativistik, yang berdasarkan analisa spektrum bagi operator pembezaan elips peringkat kedua. Aplikasi operator elips peringkat kedua dalam bidang geometri dan analisis stokastik juga sangat penting. Teori spektrum operator pembezaan elips adalah bidang yang telah dikaji oleh banyak teknik kualitatif dan kuantitatif seperti teori Sturm-Liouville, pemisahan pembolehubah, Transformasi Fourier dan Laplace, teori perturbasi, ekspansi eigenfungsi, kaedah variasi, analisis mikrolocal, stokastik analisis dan kaedah berangka termasuk elemen terhingga.

Penyelidikan yang telah dijalankan, kami menyiasat masalah mengenai penumpuan siri Fourier yang dijumlahkan di atas peringkat eliptik dalam kelas fungsi Liouville di atas Torus. Syarat-syarat yang mencukupi untuk masalah-masalah penumpuan, yang merupakan masalah yang paling sukar dalam analisis Harmonik, diperolehi untuk kelas Liouville. Masalah utama adalah untuk menmperolehu anggaran yang sesuai untuk operator maksima bagi penjumlahan separa siri Fourier, yang menjamin penumpuan siri Fourier, melibatkan pengiraan yang sangat rumit dimana ia bergantung kepada struktur fungsi dalam kelas fungsi tersebut. Idea utama yang membuktikan penumpuan pengembangan fungsi eigen dalam ruang interpolasi adalah untuk memperoleh anggaran operator maksima bagi penjumlahan separa dalam kelas sempadan dan penggunaan teorem interpolasi keluarga pengendali linier. Ia adalah jelas bahawa teori tentang pengembangan eigenfungsi operator pembezaan berkait rapat dengan masalah penumpuan siri dan kamiran Fourier. Salah satu kaedah penjumlahan yang penting dikenali sebagai kaedah penjumlahan sfera yang berkaitan dengan pengembangan fungsi eigen untuk operator Laplace, sementara persoalan mengenai penumpuan siri Fourier yang dijumlahkan pada tahap elips boleh disiasat dengan menggunakan teori spektrum buat operator pembezaan elips.

Dalam tesis ini, operator maksima bagi penjumlahan separa sfera dan eliptik diperolehi dalam kelas interpolasi Liouville dan penumpuan siri Fourier dengan kaedah penjumlahan sfera dan eliptik berjaya dibentuk. Dengan mempertimbangkan siri Fourier sebagai pengembangan fungsi eigen bagi operator pembezaan membantu untuk menterjemahkan ciri-ciri fungsional (contohnya kelancaran) kelas Liouville ke dalam pekali Fourier fungsi-fungsi yang diperkembangkan ke dalam pengembangan tersebut.

Syarat-syarat yang mencukupi bagi penumpuan siri Fourier untuk fungsi dari kelas Liouville yang diperoleh dari segi kelancaran dan dimensi. Keputusan sedemikian amat berkesan dalam menyelesaikan masalah persempadanan dengan keadaan sempadan berkala yang timbul dalam teori spektrum operator pembezaan. Kajian mengenai siri Fourier dalam kaedah analisis harmonik moden menggabungkan penggunaan kaedah dari analisis fungsi, fizik matematik, teori operator moden dan penguraian spektrum. Kaedah baru yang terbaik untuk penghampiran fungsi serasi segi empat oleh siri Fourier yang diringkaskan di atas paras eliptik telah dibentukkan. Dengan menggunakan penghampiran terbaik, konstan Lebesgue yang sepadan dengan penjumlahan separa elips berjaya dianggarkan. Yang terakhir ini digunakan untuk mendapatkan anggaran untuk operator maksima dalam kelas Liouville. .

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This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Doctor of Philosophy. The members of the Supervisory Committee were as follows:

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This is to confirm that:

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TABLE OF CONTENTS

LIST OF FIGURES

LIST OF ABBREVIATIONS

N	Dimensional
T^N	N-Torus
$\mathbb{S}^N$	Unit sphere in $\mathbb{R}^{N+1}$
α	Multi-index with size $\lvert\alpha\rvert = \alpha_1 + \alpha_2 + \cdots + \alpha_N$
C^α	The space of continuously differentiable functions of order $\lvert\alpha\rvert$
C^∞	The space of infinitely continuously differentiable functions
Δ	Laplace Operator
$A(D)$	Elliptic Operator
$A(\xi)$	Elliptic Polynomial of order m
$\omega(\delta)$	Modulus of continuity
P_k	Trigonometric polynomials
W_p^k	Sobolev Space
L_p^s	Liouville Space
L_λ	Lebesgue Constant
S_λ	Spherical sum
S_λ^α	Spherical Bochner-Riesz mean sum
E_λ	Elliptic sum
E_λ^α	Elliptic Bochner-Riesz mean sum
$\mathrm{R}_\lambda^\alpha$	The Bochner-Riesz means of Fourier integrals
Θ_λ	Spectral function corresponding to Laplace operator
Ψ_λ	Spectral function corresponding to Elliptic operator

CHAPTER 1

INTRODUCTION

In this study we deal with the almost everywhere convergence problems of the multiple Fourier series where is summed over domains bounded by levels of an elliptic polynomials. The elliptic differential operator with the constant coefficients is unbounded operator and has many applications in engineering sciences. Such operators have self adjoint extension in the Hilbert classes, which can be represented with the help of the spectral decomposition of unity corresponding to the given elliptic differential operator. The latter is closely connected with the multiple Fourier series summed over the levels of elliptic polynomials. The main problem of the harmonic analysis is reconstruction of the function from its Fourier expansion. Obtaining the sufficient conditions for the almost everywhere convergence of the multiple Fourier series of the functions from different classes gives answer to the main problem of the harmonic analysis in the mentioned classes of functions. This research is focused to investigate the problems of the almost everywhere convergence of spectral expansions of the functions with special behavior in terms of the eigenfunctions of the elliptic differential operators. This chapter will focus on main fundamental ideas and concepts from the spectral theory of differential operators to understand modern status of the investigations in the field of convergence and summability of eigenfunction expansions related to differential operators. Besides background on our subject matter, this chapter will entail descriptions on our research objectives, motivation and a brief outline of this thesis.

1.1 Motivation

The theory of differential equations is one of the outstanding discovers of the human mind. Its influence upon the development of physical science would be hard to exaggerate. Much of the subject matter in this thesis is confined not only to second order differential operators but even to elliptic differential operators of order m. One justification for concentrating on this topic is that many of the equations which have proved important in the physical sciences and engineering involve operators of this type. Most important among these is non-relativistic quantum theory, which is based upon the spectral analysis of Laplace operator. Applications of second order elliptic operators to geometry and stochastic analysis are also now of great importance. The importance of study the Fourier series with respect to elliptic polynomial is coming from the fact of its relative to elliptic partial differential operators (by Fourier transform). Many problems of mathematical physics can be solved by separation methods of partial differential equations. When separation method is applied then finding a solution of the partial differential equation will be reduced to the problems of convergence of the eigenfunction expansions of elliptic operators. This connection between the theory of multiple Fourier series and the theory of partial differential equations is considered a sub-domain of the spectral theory. It is known that the theory of the spectral decompositions of the differential operators closely connected with the convergence problems of Fourier series and integrals. Moreover elliptic differential operators play an important role in modern mathematical physics. Such operators are highly effective in solving the eigenvalue problems occurring in the spectral theory of differential operators. Spectral theory is an extremely rich field which has been studied by many qualitative and quantitative techniques - for example Sturm-Liouville theory, separation of variables, Fourier and Laplace transforms, perturbation theory, eigenfunction expansions, variational methods, microlocal analysis, stochastic analysis and numerical methods including finite elements (Alimov et al. (1976), Alimov et al. (1992), Reed and Simon (1979), Hörmander (1985), Stein and Weiss (1971), Taylor (1996), Shubin (2001) and Titchmarsh (1958)).

1.2 Multiple Fourier Series and Spectral Theory of Elliptic Differential Operators in the Torus

The main mathematical objects we work with throughout this thesis are several variable functions which are 2π-periodic in each variable defined on N-torus T^N. N is reserved for dimension.

1.2.1 Functions on Torus

We define Torus as a cube $[-\pi,\pi]^N$:

$$T^N = \{x = (x_1, x_2, \cdots, x_N) \in \mathbb{R}^N : -\pi < x_i \leq \pi, i = 1, \cdots, N\},$$

which naturally isomorphic to $\mathbb{R}^N/\mathbb{Z}^N$. By this we mean, for $x, y \in \mathbb{R}^N$ we say that

$$x \equiv y,$$

if $x - y \in 2\pi\mathbb{Z}^N$. Here $\equiv$ is an equivalence relation that partitions $\mathbb{R}^N$ into equivalence classes, where $2\pi\mathbb{Z}^N$ is the additive subgroup of $\mathbb{R}^N$ and $\mathbb{Z}^N$ is integer coordinates.

Example 1.1 *let $n = (n_1, n_2, \cdots, n_N)$ an element belongs to $\mathbb{Z}^N$ then*

$$(x_1, x_2, \cdots, x_N) \equiv (x_1 + 2\pi n_1, x_2 + 2\pi n_2, \cdots, x_N + 2\pi n_N),$$

when $N = 2$, we have $(-\frac{2}{3}\pi, -\frac{1}{2}\pi) \equiv (\frac{4}{3}\pi, \frac{3}{2}\pi)$, or, equivalently, $(-\frac{2}{3}\pi, -\frac{1}{2}\pi) = (\frac{4}{3}\pi, \frac{3}{2}\pi)(mod 2\pi)$.

By this setting one can visualize $T = [-\pi,\pi]$ as a circle by bringing together the endpoints of line segment $[-\pi,\pi]$. For T^2, one can again bring together the right and left sides of the square $[-\pi,\pi]^2$ and then the top and bottom sides as well, which is shaped as a 2−dimensional manifold embedded in $\mathbb{R}^3$. See Figure 1.2.1. The T^N in this way can be indicated of as Cartesian product of N copies $S^1 \times \cdots \times S^1$ of the circle.

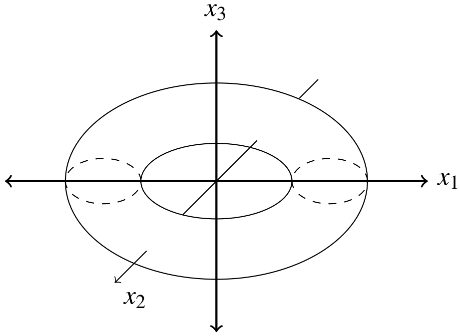

Figure 1.1: 2-dimensional torus T^2.

The T^N is an additive group and then zero $0 = (0_1, 0_2, \cdots, 0_N)$ is the identity element of the group. Therefore, the inverse of $x \in T^N$ is denote by $-x = (-x_1, -x_2, \cdots, -x_N)$, see Example 1.1.

The T^N can be thought of as subset of $\mathbb{C}^N$ such that

$$(e^{ix_1}, e^{ix_2}, \cdots, e^{ix_N}) \in \mathbb{C}^N, \quad (x_1, x_2, \cdots, x_N) \in [-\pi, \pi]^N,$$

this mean the interval $[-\pi, \pi]$ can be visualized as the unit circle in $\mathbb{C}$ once $-\pi$ and π are identified. Now, we say that a function f is $2\pi-$periodic in every coordinate, if

$$f(x_1 + 2\pi n_1, x_2 + 2\pi n_2, \cdots, x_N + 2\pi n_N) = f(x_1, x_2, \cdots, x_N),$$

for all $x \in \mathbb{R}^N$ and $n \in \mathbb{Z}^N$. Hence, such a function is defined on torus T^N. The $N-$dimensional Lebesgue measure (i.e volume in $\mathbb{R}^N$) is restricted to the set $T^N = [-\pi, \pi]^N$ and denoted by dx. By translation invariance of the Lebesgue measure (i.e. $m(E + a) = m(E)$, $E \subset \mathbb{R}^N$, $a \in \mathbb{R}^N$) and the periodicity of functions on T^N, we have

$$\int_{T^N} f(x)\,dx = \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} f(x_1, x_2, \cdots, x_N)\,dx_1 \cdots dx_N = \int\limits_{[-\pi,\pi]^N} f(x)\,dx = \int\limits_{[y, y+2\pi n]^N} f(x)\,dx,$$

for all f on T^N and $y \in \mathbb{R}^N$. Finally, the L_P spaces on T^N are nested such that

$$L_1 \supset \cdots \supset L_2 \supset \cdots \supset L_\infty.$$

1.2.2 Multiple Fourier Series

As is well known, the classical trigonometric series in one dimensional takes the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \tag{1.1}$$

where $x \in \mathbb{R}$ and the coefficients $a_0, a_1, b_1, \cdots$ are real. The terms of this series are all of period 2π, therefore, it is sufficient to study the trigonometric series in $T = [-\pi, \pi]$.

The following partial sum

$$S_\nu(x) = \frac{a_o}{2} + \sum_{n=1}^{\nu}(a_n \cos nx + b_n \sin nx), \quad x \in \mathbb{R},$$

is a finite trigonometric sum so-called *trigonometric polynomial* of order ν, because of fact that $S_\nu(x)$ is the real part of an ordinary polynomial $P(z)$ of degree ν, where $z = e^{ix}$. If $|a_\nu| + |b_\nu| \neq 0$, $S_\nu(x)$ is said to be strictly of order ν. The complex form of this partial sum takes the form

$$S_\nu(x) = \sum_{-\nu}^{\nu} c_n e^{inx}$$

by putting $c_n = \frac{1}{2}(a_n - ib_n)$, $c_{-n} = \bar{c}_n$, $\forall n = 0, 1, 2, \cdots$. This complex form is νth symmetric partial sum of the Laurent series, i.e., the sum of the $2\nu + 1$ central terms of

$$\sum_{-\infty}^{\infty} c_n e^{inx}. \tag{1.2}$$

The series (1.2) may be written in the form (1.1) with a_n and b_n are real, it will be a cosine series if and only if the c_n are real, while it is a sine series if and only if the c_n are purely imaginary. The form (1.2) gives advantage to leave the coefficients unrestricted and often suggests a complex method of summation.

Let consider multiple trigonometric series

$$\sum_{n \in \mathbb{Z}^N} c_n e^{i(n,x)} = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \cdots \sum_{n_N=-\infty}^{\infty} c_{n_1 n_2 \cdots n_N} e^{i(n_1 x_1 + n_2 x_2 + \cdots + n_N x_N)}, \tag{1.3}$$

with arbitrary complex coefficient c_n. We denote by expression (n, x) the inner product. Similar to one dimension case the series (1.3) is called trigonometric polynomial, whenever $\{c_n\}_{\mathbb{Z}^N}$ is a finitely supported sequence in $\mathbb{Z}^N$. The degree of series (1.3) is the large number $|b_1| + \cdots + |b_N|$ such that $c_b \neq 0$, where $b = (b_1, b_2, \cdots, b_N)$.

In the multidimensional case, there will be different concept of partial sum. For example, in two dimensions one needs to find in what sense will expand the set of integer numbers (i.e. $(n_1, n_2) \in \mathbb{Z}^2$) and exhausting $\mathbb{R}^2$, thus one may take the set as circle, cube or ellipsoid, it leads to take the sum in what it is called principle value sense, thus will face various partial sums of a multiple series, for instant.

$i)$ The rectangular partial sum

$$S_\nu(x) = \sum_{|n_1| \le \nu_1} \sum_{|n_2| \le \nu_2} \cdots \sum_{|n_N| \le \nu_N} c_{n_1 n_2 \cdots n_N} e^{i(n_1 \cdot x_1 + n_2 \cdot x_2 + \cdots + n_N \cdot x_N)}, \tag{1.4}$$

where the vector ν is belong to $\mathbb{N}^N \cup \{0\}$.

ii) The cubic partial sum takes the form

$$S_l(x) = \sum_{|n_1|\le l} \sum_{|n_2|\le l} \cdots \sum_{|n_N|\le l} c_{n_1 n_2 \cdots n_N} e^{i(n_1 \cdot x_1 + n_2 \cdot x_2 + \cdots + n_N \cdot x_N)}, \qquad (1.5)$$

where l is a positive integer.

iii) The spherical partial sum is defined by

$$S_\lambda(x) = \sum_{\sqrt{n_1^2+n_2^2+\cdots+n_N^2}\le\lambda} c_{n_1 n_2 \cdots, n_N} e^{i(n_1 x_1 + n_2 x_2 + \cdots + n_N x_N)}.$$

If we use the notation $|n|^2 = n_1^2 + n_2^2 + \cdots + n_n^2$, then the spherical partial sums will take the similar form as in one dimensional Fourier series:

$$S_\lambda(x) = \sum_{|n|\le\lambda} c_n e^{i(n,x)}. \qquad (1.6)$$

One can visualize this sum as an analogy with multiple integral on spherical domain with radius λ, (circular in $\mathbb{R}^2$) see Figure1.2.2.

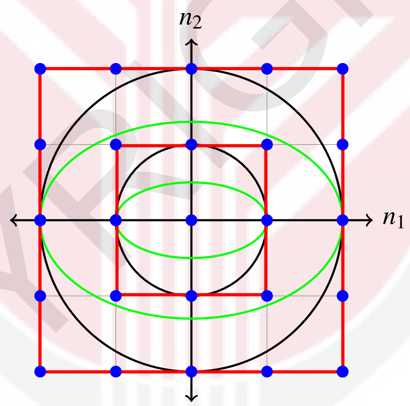

Figure 1.2: Plot Lattices $(n_1, n_2) \in \mathbb{Z}^2$, enclosed by circles, squares and ellipses.

Remark that the cases of spherical and cubical sums form a family of enclosing sets, but this is not the case in rectangular sum. However, we tend to restrict the summation methods for the multiple partial sum into two type, as following:
A) If $\Omega = \{G\}$ is a family of finite subsets of $\mathbb{Z}^N$ such that,
1) $\forall G', G'' \in \Omega \quad \exists G \in \Omega : G' \cup G'' \subset G$;
2) $\bigcup_{G\in\Omega} G = \mathbb{Z}^N$,
then one may define the partial sum with respect to the family Ω by

$$S_G(x) = \sum_{n\in G} c_n e^{i(n,x)}. \qquad (1.7)$$

It is clear that if Ω is taken as a family of rectangular, cubes or balls, then one has the rectangular, cubic and spherical partial sums, respectively. The concept of convergence can be defined as follows:

If $\forall \varepsilon > 0,\ \exists G_\varepsilon \in \Omega$ such that

$$\forall G \in \Omega : (G_\varepsilon \subset G) \Rightarrow |S_G(x) - f(x)| < \varepsilon,$$

we say that the series (1.3) is Ω-convergent at the point x to $f(x)$.

B) If Q is any bounded subset of $\mathbb{R}^N$ containing the origin and Ω is a family of all sets of the form $\{n \in \mathbb{Z}^N : n \in \lambda Q\}$, where $\lambda > 0$. Then the corresponding partial sums have the form

$$S_{\lambda Q}(x) = \sum_{n \in \lambda Q \cap \mathbb{Z}^N} c_n e^{i(n,x)}. \tag{1.8}$$

In the case of type B) the series (1.3) is Ω-convergent at the point x, if $\lim_{\lambda \to \infty} S_{\lambda Q}(x)$ exists. In particular, when Q is the unit cube or ball, respectively, the $S_{\lambda Q}(x)$ is cubic or spherical partial sum.

Note that the rectangular (1.4) convergence implies cubic (1.5) convergence, but the converse is not true. On the other hand, the spherical(1.6) convergence dose not imply cubic convergence, and thus not rectangular convergence, the converse is not true either.

Example 1.2 *1. Let*

$$\sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} (n_1^2 \delta_{n_2 0} - n_2^2 \delta_{n_1 0}) e^{i(n_1 x_1 + n_2 x_2)},$$

where $\delta_{jk} = 1$, if $j = k$ and zero otherwise. Take point $(0,0)$, then the series is not rectangularly convergent but quadratically convergent.

2. The series

$$\sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} (n_1^2 (\delta_{n_2 0} - \delta_{n_2 2})) e^{i(n_1 x_1 + n_2 x_2)}$$

is rectangularly convergent at the point $(0,0)$ when $n_2 \geq 2$, but not circularly convergent, since

$$S_R(0,0) = \sum_{n_1^2 + n_2^2 \leq R^2} n_1^2 (\delta_{n_2 0} - \delta_{n_2 2}) = 2R^2 \longrightarrow \infty, \quad R > 2.$$

3. Let

$$\sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} c_{n_1 n_2} e^{i(n_1 x_1 + n_2 x_2)}$$

with $c_{5 \cdot 2^k, 0} = -c_{4 \cdot 2^k, 3 \cdot 2^k} = k$, $k = 1, 2 \ldots$, and other coefficients are zero. We have

$$S_R(0,0) = \sum_{n_1^2 + n_2^2 \leq R^2} c_{n_1 n_2} = \sum_{l=0}^{R} \sum_{n_1^2 + n_2^2 = l^2} c_{n_1 n_2} = 0$$

for all $R > 0$, because if the circle of radius l contains the point $(5\cdot 2^k, 0)$ then will contain a point $(4\cdot 2^k, 3\cdot 2^k)$ as well. However, the sequence

$$Q_{4\cdot 2^k} = \sum_{n_1=-4\cdot 2^k}^{4\cdot 2^k} \sum_{n_2=-4\cdot 2^k}^{4\cdot 2^k} c_{n_1 n_2} = -k$$

tends to $-\infty$ as $k \longrightarrow \infty$. Here we used the fact that the point $(5\cdot 2^k, 0)$ does not belong to the square

$$\{(n_1, n_2) : |n_1| \le 4\cdot 2^k, |n_2| \le 4\cdot 2^k\}.$$

In order to focus on the Fourier series, one may determine the coefficient c_n in the following manner, assume that the series (1.3) converge to a function $f(x)$ in the sense that allows one to integrate the series term by term, as it is the case of uniform convergence and convergence in $L_p(T^N)$, then we have

$$f(x) = \sum_{n\in\mathbb{Z}^N} c_n e^{i(n,x)},$$

now by multiplying the both side with $e^{-i(n,x)}$ and integrating over T^N we obtain

$$\hat{f}(n) = (2\pi)^{-N} \int_{T^N} f(x) e^{-i(n,x)} dx, \tag{1.9}$$

where $\hat{f}(n) = c_n$ is called the Fourier coefficients of $f(x)$. The *nth* Fourier coefficients are defined if a complex-valued function $f \in L_1(T^N)$, moreover $\hat{f}(n) \to 0$ as $|n| \to 0$, this is because of Riemann-Lebesgue lemma.

In this point we introduce some elementary properties of Fourier coefficients (see Grafakos (2008)). Therefore, we denote by $\bar{f}$ the complex conjugate of the function f, by $\tilde{f}(x) = f(-x)$ the reflection of f, and by $\tau^y(f)(x) = f(x-y)$, $\forall y \in T^N$ the translation of f. Let $f, g \in L_1(T^N)$, then for all $n, h \in \mathbb{Z}^N$, $\gamma \in \mathbb{C}$, $y \in T^N$ and a multi-indices $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_N) \in \mathbb{N}^N \cup \{0\}$ with size $|\alpha| = |\alpha_1| + \cdots + |\alpha_N|$ we have

(1) $\widehat{(f+g)}(n) = \hat{f}(n) + \hat{g}(n)$,

(2) $\widehat{\gamma f}(n) = \gamma \hat{f}(n)$,

(3) $\widehat{\tilde{f}}(n) = \hat{f}(-n)$,

(4) $(\widehat{\tau^y(f)})(n) = \hat{f}(n) e^{-i(n,y)}$,

(5) $\widehat{\bar{f}}(n) = \bar{\hat{f}}(-n)$,

(6) $\hat{f}(0) = \int_{T^N} f(x) dx$,

(7) $(\widehat{e^{i(h,\cdot)} f})(n) = \hat{f}(n-h)$,

(8) $\widehat{(f * g)}(n) = \hat{f}(n)\hat{g}(n)$,

(9) $\sup_{n\in\mathbb{Z}^N} |\hat{f}(n)| \le \|f\|_{L_1(T^N)}$,

(10) $\widehat{(\partial^\alpha f)}(n) = (in)^\alpha \hat{f}(n)$, $f \in C^\alpha$ space of the continuously differentiable functions of order $|\alpha|$,

where $f * g = \int_{T^N} f(x-y)g(y)dy$ is convolution function.

Hence, one can associate each function $f \in L_p(T^N)$, $1 \le p \le \infty$ with a multiple trigonometric series (Fourier series) such that

$$\sum_{n\in\mathbb{Z}^N} \hat{f}(n)e^{i(n,x)}, \tag{1.10}$$

so-called the Fourier series of the function $f(x)$. The main and natural questions arise here: does the Fourier series converge to f? If it does, in what sense that convergence is done and with respect to which family $\{G\}$ as mentioned above?. What is the effect of dimensions on behavior of the Fourier series? Also under which classes of functions the convergence may be true?. The study of these questions release an important field so-called Fourier analysis a sub-domain of the harmonic analysis.

An interesting type of a family $\{G\}$ is when $G = \{\xi \in \mathbb{R}^N : A(\xi) < \lambda\}$ and $A(\xi)$ is an algebraic polynomial of even degree m with constants a_α defined

$$A(\xi) = \sum_{|\alpha|\le m} a_\alpha \xi^\alpha, \tag{1.11}$$

where $\xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^N$, $\xi^\alpha = \xi_1^{\alpha_1}\xi_2^{\alpha_2}\dots\xi_N^{\alpha_N}$, $\alpha = (\alpha_1, \dots, \alpha_N)$ is multi-index with α_j- nonnegative integers, and the size of α is defined as follows $|\alpha| = \alpha_1 + \cdots + \alpha_N$. If $|A(\xi)| > 0, \forall\, \xi \ne 0$, then $A(\xi)$ is called elliptic polynomial, for instance, if $m = 2, N = 2$ and a_α are real then we have:

$$A(\xi_1, \xi_2) = a_{00} + a_{10}\xi_1 + a_{01}\xi_2 + a_{20}\xi_1^2 + a_{11}\xi_1\xi_2 + a_{02}\xi_2^2.$$

This family is relative to the elliptic partial differential operator as we will see in section (1.2.3).

Unfortunately, the set G with (1.11) does not refer to summation method of type $B)$ as it is mentioned above, the reason for that it is not similar for different values of λ. However, if we take a homogeneous elliptic polynomial of order m (i.e. $A_h(t\xi) = t^m A_h(\xi)$) defines

$$A_h(\xi) = \sum_{|\alpha|=m} a_\alpha \xi^\alpha, \tag{1.12}$$

then we can have a summation method of type $B)$ by putting $Q = \{\xi \in \mathbb{R}^N : A_h(\xi) < 1\}$ and $\lambda = t^{1/m}, t > 0$, to which the sphere belong (i.e. $A_h(\xi) = |\xi|^2$). We note that if m tend to infinity, then the set Q takes a hyper-rectangular shape. Thus, the partial sum of the Fourier series of function f summed over domains bounded by level surfaces of an elliptic polynomial takes form

$$E_\lambda f(x) = \sum_{A(n)<\lambda} \hat{f}(n)\, e^{i(n,x)}, \tag{1.13}$$

where $A(n) = \{A(\xi) < \lambda\} \cap \mathbb{Z}^N$, and $\lambda > 0$ (see Figure 1.2.2).

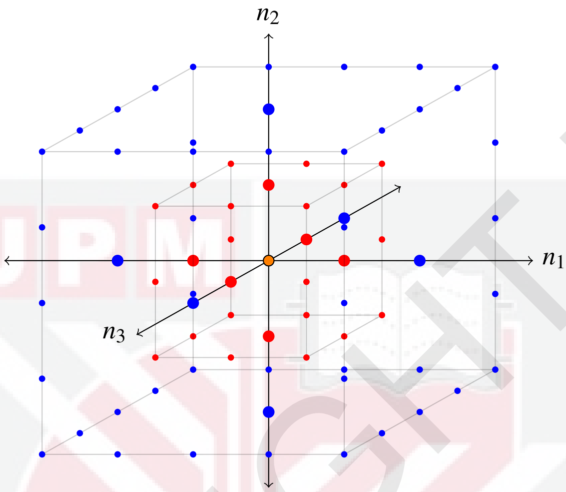

Figure 1.3: Lattices in 3-dimensional.

For functions which are not very smooth the partial sum $E_\lambda f(x)$ does not approach the function $f(x)$ as $\lambda \to \infty$ but oscillates around it. However, that is the reason to study the convergence for averages of $E_\lambda f(x)$, then by taking the averages of (1.13) one obtains

$$\frac{1}{\lambda}\int_0^\lambda E_t f(x)dt = \sum_{A(n)<\lambda} \left(1 - \frac{A(n)}{\lambda}\right) \hat{f}(n)\, e^{i(n,x)},$$

this expression is called spherical Cesàro means (or *spherical Fejér means*). It is natural to expect that the behavior of convergence of series will be better as many as possible it is integrated, this motivated to study more general operators for complex number α with $\Re(\alpha) > 0$ defined as follows

$$E_\lambda^\alpha f(x) = \sum_{A(n)<\lambda} \left(1 - \frac{A(n)}{\lambda}\right)^\alpha \hat{f}(n)\, e^{i(n,x)}, \tag{1.14}$$

is known as Bochner-Riesz means of order α with respect to elliptic polynomial A. It turns out that for fixed $\Re(\alpha)$, the series (1.14) may failed to converge as the dimension increases (Stein (1958)).

We say the series (1.13) converges to $f(x)$ by Bochner-Riesz method of order α if

$$\lim_{\lambda\to\infty} E_\lambda^\alpha f(x) = f(x).$$

As an example we note that the spherical partial sums takes the following form

$$S_\lambda^\alpha f(x) = \sum_{|n|^2<\lambda} \left(1 - \frac{|n|^2}{\lambda}\right)^\alpha \hat{f}(n)\, e^{i(n,x)}. \tag{1.15}$$

It is known that there are various forms of convergence such as pointwise convergence, uniform convergence, almost everywhere convergence and convergence in $L_p(T^N)$. We define them as follows:

Let $\{f_j\}_{j\in\mathbb{N}}$ be a sequence of functions from $L_p(T^N)$, and $B \subset T^N$. We say that $f_j(x)$ converges pointwise to the function $f(x)$ on set B if

$$\lim_{j\to\infty} f_j(x) = f(x) \quad \text{for each } x \in B.$$

Unfortunately, this type of convergence does not guarantee to make a sequence of continuous functions converge to continuous function as well as be differentiable, for example consider the sequence $f_j(x) = e^{-jx}$, $j = 0,1,2,\cdots$, on $[0,1]$, then $f_j(x) \to 0$ as $j \to \infty$ for each $x \in (0,1]$, but $f_j(0) = 1$ for all j.

Therefore, we introduce the uniform convergence, for that we define L_p-norm by

$$\|f\|_{L_p(T^N)} = \left(\int_{T^N} |f(x)|^p \, dx\right)^{\frac{1}{p}}, \; 1 \le p < \infty,$$

and for $p = \infty$ by

$$\|f\|_{L_\infty(T^N)} \overset{def}{=} ess.\sup|f| \overset{def}{=} \inf\{C > 0 : \mu(\{x \in T^N : |f(x)| > C\}) = 0\},$$

where μ is a positive measure on T^N.

Thus, the sequence $f_j(x)$ converge uniformly to function f if

$$\lim_{j\to\infty} \|f_j - f\|_{L_\infty} = 0.$$

In general we have the following convergence in norm, it is said that f_j converge to $f \in L_p(T^N)$ in L_p-norm, if

$$\lim_{j\to\infty} \|f_j - f\|_{L_p} = 0, \quad 1 \le p < \infty.$$

We pass forward for definition of the almost everywhere convergence. We said that the sequence f_j almost everywhere converges to $f \in L_p(T^N)$, if

$$\mu\{x : \lim_{j\to\infty} f_j(x) \neq f(x)\} = 0.$$

In this study we focus on the almost everywhere convergence of the multiple Fourier series related to the functions from classes of Liouville.

1.2.3 Spectral Theory of the Elliptic Differential Operators

Let consider an arbitrary differential operator with constant coefficients:

$$A(D) = \sum_{|\alpha| \le m} a_\alpha D^\alpha, \tag{1.16}$$

where $D^\alpha = D_1^{\alpha_1} \dots D_N^{\alpha_N}$, $D_j = \frac{1}{i}\frac{\partial}{\partial x_j}$, and as above, α is a multi-index.

The polynomial (1.11) is associated with differential operator $A(D)$ by replaced D with $\xi \in \mathbb{R}^N$ and it is called a symbol of operator $A(D)$, the homogeneous polynomial (1.12) is called its principle symbol. The operator $A(D)$ is said to be elliptic of order m if its principle symbol satisfies $|A_h(\xi)| > 0$ for all $\xi \in \mathbb{R}^N$, $\xi \ne 0$.

Example 1.3 *1. Cauchy-Riemann operator* $m = 1, N = 2$

$$A(D_1, D_2) = \frac{1}{2}(D_1 + iD_2)$$

is elliptic. Since $|A(\xi_1,\xi_2)| = \frac{1}{2}\sqrt{\xi_1^2+\xi_2^2} > 0$, *if* $(\xi_1,\xi_2) \ne (0,0)$.

2. Bitsadze operator

$$A(D_1, D_2) = D_1^2 + 2iD_{12} + D_2^2$$

also elliptic.

The operator $A(D)$ is considered in the Hilbert space $L_2(T^N)$ as an unbounded operator with domain $C^\infty(T^N)$ the class of infinitely differentiable functions on T^N. In case the coefficients are real, the $A(D)$ will satisfy the symmetric condition:

$$(Au, v) = (u, Av), \ \forall\, u, v \in C^\infty(T^N). \tag{1.17}$$

In addition, since the operator $A(D)$ is elliptic, then by Gårding's inequality the operator $A(D)$ is semi-bounded (see Hörmander (1985))

$$(Au, u) \ge \lambda_A(u, u), \quad \forall\, u \in C^\infty(T^N), \tag{1.18}$$

where nonnegative constant λ_A is called lower bound of A. Hence, Friedrichs's theorem asserts that for every symmetric semi-bounded operator there are at least one self-adjoint extension with the same lower bound, then there is a self-adjoint extension $\bar{A}$ in $L_2(T^N)$ of operator $A(D)$ which, indeed, its closure, and they are coincided on the domain of definition i. e. $\bar{A}u = Au$, $u \in C^\infty(T^N)$. By von Neumann's spectral theorem, the operator $\bar{A}$ has a spectral decomposition of unity $\{E_\lambda\}$, and then it can be represented in the following form (see Alimov et al. (1976), Alimov et al. (1977))

$$\bar{A} = \int_{\lambda_A}^{\infty} \lambda \, dE_\lambda,$$

the projections E_λ increase monotonically, and continuous on the left, moreover

$$\lim_{\lambda\to\infty} \|E_\lambda u - u\|_{L_2(T^N)} = 0, \quad u \in L_2(T^N).$$

The operator $\bar{A}$ has a complete orthonormal system (see Definition 1.6.1) of eigenfunctions $\{(2\pi)^{-N/2}e^{i(n,x)}\}$ in $L_2(T^N)$ corresponding to the eigenvalues $A(n)$, $n \in \mathbb{Z}^{\mathbb{N}}$. Thus, the spectral decomposition of $f \in L_2(T^N)$ coincides with (1.13) partial sums of the multiple Fourier series of function f related to $A(n)$. An interesting fact that the lower order coefficients $a_\alpha, |\alpha| < m$ of $A(D)$ do not influence the convergence of the spectral decomposition $E_\lambda f$ provided the function f is sufficiently smooth. Then one can reduce the study of convergence for partial sum (1.13) to the study of simpler case, that is, applying the summation over expanding its principle symbol $A_h(n)$.

The spectral decomposition E_λ can be written as an integral operator:

$$E_\lambda f(x) = \int_{T^N} \Psi_\lambda(x,y) f(y)\, dy,$$

where the kernel

$$\Psi_\lambda(x,y) = (2\pi)^{-N} \sum_{A(n)<\lambda} e^{i(n,x-y)} \tag{1.19}$$

is called the spectral function of operator $\bar{A}$. Indeed, the operator E_λ is a convolution operation $\Psi_\lambda * f$, such that

$$E_\lambda f(x) = \int_{T^N} \Psi_\lambda(x-y) f(y)\, dy. \tag{1.20}$$

In the case of second order differential operator, a Laplace operator

$$\Delta = \sum_{k=1}^{N} \frac{\partial^2}{\partial x_k^2} \tag{1.21}$$

is considered in the Hilbert space $L_2(T^N)$ as an unbounded operator with domain $C^\infty(T^N)$. For $u, v \in C^\infty(T^N)$, and by inner product in $L_2(T^N)$ one can see the operator $-\Delta$ symmetric and nonnegative, indeed

$$(\Delta u, v) = (u, \Delta v)$$
$$(-\Delta u, u) = (\nabla u, \nabla u) \geq 0,$$

where $\nabla = (\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_N})$ is the gradient of u, and it is a simplest elliptic operators since its symbol $|\xi|^2 > 0, \forall \xi \neq 0$. Then, as above, by Friedrichs's theorem it has a nonnegative self-adjoint extension, again, denoted by $\bar{A}$ and coincides with the closure of $-\Delta$ in $L_2(T^N)$, such that

$$\bar{A} = \int_0^\infty \lambda\, dE_\lambda,$$

with a complete orthonormal system of eigenfunctions $\{(2\pi)^{-N/2}e^{i(n,x)}\}$, corresponding to the eigenvalues $\{|n|^2\}$, $n \in \mathbb{Z}^{\mathbb{N}}$. Therefore a spectral decomposition of a function $f \in L_2(T^N)$ have the form

$$S_\lambda f(x) = \sum_{|n|^2<\lambda} \hat{f}(n)\, e^{i(n,x)},$$

where it shows that the spectral expansion $E_\lambda f$ coincides with the spherical partial sums of the Fourier series.

1.3 Functional Spaces on the Torus

We deal in this study with functions which are termed as a Liouville spaces L_p^s where $s > 0$ (coincide for s integers with Sobolev spaces). They study the differentiability and smoothness of functions. A fundamental fact is that smoothness can be measured and fine-tuned by using the Fourier transform. Indeed, the investigation of the subject is based on this point. We commence by the Sobolev spaces.

1.3.1 Sobolev Spaces $W_p^k(T^N)$

The main idea of introducing of the Sobolev spaces is measuring the smoothness of a given function in terms of the integrability of its derivatives. We give the classical definition of Sobolev spaces.

Definition 1.3.1 *Let k be a nonnegative integer and let $1 \leq p < \infty$. The Sobolev space $W_p^k(T^N)$ is defined as the space of functions f in $L_p(T^N)$ all of whose distributional derivatives $\partial^\alpha f$ are also in $L_p(T^N)$ for all multi-indices α that satisfy $|\alpha| \leq k$. This space is normed by the expression*

$$\|f\|_{W_p^k} = \sum_{|\alpha|\leq k} \|\partial^\alpha f\|_{L_p} \tag{1.22}$$

where $\partial^{(0,\cdots,0)} f = f$.

The index k indicates the degree of smoothness of a given function in W_p^k. These spaces form a decreasing sequence (see Grafakos (2009))

$$L_P \supset W_p^1 \supset W_p^2 \supset \cdots,$$

meaning that each $W_P^{K+1}(T^N)$ is a subspace of $W_P^k(T^N)$ in view of the Sobolev norms.

The Sobolev space $W_p^k(T^N)$ is complete. To see this, let f_j a Cauchy sequence in W_p^k, then $\partial^\alpha f_j$ is a Cauchy sequence in L_p for all $|\alpha| \leq k$. By the completeness of L_p, there exist functions f_α such that $\partial^\alpha f_j \to f_\alpha$ in L_p, then clearly

$$(-1)^{|\alpha|}\langle f_j, \partial^\alpha \varphi\rangle = \langle \partial^\alpha f_j, \varphi\rangle \to \langle f_\alpha, \varphi\rangle, \tag{1.23}$$

for all $\varphi \in C^\infty(T^N)$. Here, $\langle f, \varphi\rangle = f(\varphi)$ represents the action of a distribution f on a test function $\varphi \in C^\infty(T^N)$ (see Hörmander (1983)). For instance, the Dirac mass at the origin δ_0 is delta distribution defined by

$$\langle \delta_0, \varphi\rangle = \varphi(0)$$

Since the left hand side of (1.23) converges to

$$(-1)^{|\alpha|}\langle f, \partial^\alpha \varphi\rangle,$$

it follows that the distributional derivative $\partial^\alpha f$ of f is f_α. This implies that $f_j \to f$ in W_p^k.

1.3.2 Liouville Spaces $L_p^s(T^N)$

The Liouville space is considered as generalization of Sobolev Spaces, where the classes $W_p^k(T^N)$ can also be described in terms of Fourier transform. By that one can extend the definition of Sobolev space to the case in which the index k is real. We explain the idea by putting $p = 2$. By using the Plancherel's identity we obtain that the norm (1.22) for $f \in W_2^k$ can be introduced in the form

$$\|f\|^2_{W_2^k} = \sum_{n\in\mathbb{Z}^N} \sum_{|\alpha|\le k} |n^\alpha|^2 |\hat{f}(n)|^2.$$

Taking into account the following inequality

$$c_k(1+|\xi|^2)^k \le \sum_{|\alpha|\le k} |\xi^\alpha|^2 \le C_k(1+|\xi|^2)^k,$$

which holds for all $\xi \in \mathbb{R}^N$ with some positive constants c_k, C_k. This gives a motivation for the following definition (see Alimov et al. (1992)).

Definition 1.3.2 *Let s be a real number and let $1 \le p < \infty$. The Liouville space $L_p^s(T^N)$ is defined as the space of all functions f in $L_p(T^N)$, such that*

$$\sum_{n\in\mathbb{Z}^N} (1+|n|^2)^{\frac{s}{2}} \hat{f}(n) e^{inx} \in L_p(T^N). \tag{1.24}$$

Thus the norm of f in $L_p^s(T^N)$ has a form

$$\|f\|_{L_p^s(T^N)} = \left\| \sum_{n\in\mathbb{Z}^N} (1+|n|^2)^{\frac{s}{2}} \hat{f}(n) e^{inx} \right\|_{L_p(T^N)}.$$

We observe that when $s = 0$ then $L_p^s = L_p$. The space L_p^s coincides with the space W_p^k when $s = k$ a positive integer number. We also note that when $s \geq 0$ the elements of L_p^s are always L_p, i.e. $L_p^s \subseteq L_p$.

If $0 < s_1 < s$ then $L_p^s \subset L_p^{s_1}$. Moreover we have the embedding

$$L_p^s \to L_q^{s_1}, \quad L_p^s \subset L_q^{s_1}$$

provided

$$s - \frac{N}{p} = s_1 - \frac{N}{q}.$$

Example 1.4 *Let $f(x) = |x|^t, t > 0$ and $x \in T^N$. Let $s = k$ a positive integer, then for all α with $|\alpha| = k$ the derivative $\partial^\alpha f$ has the singularity $|x|^{t-k}, t < k$. Thus, $f \in L_p^k$, if $|x|^{t-k} \in L_p$, provided $k - \frac{N}{p} < t$.*

1.4 Interpolation Theorem of the Family of Linear Operators

There is two classical interpolation theorems which form the foundation to whole theory. In brief, they are the Marcinkiewicz interpolation theorem which is well known as real method, it is applicable for non-linear operators and the endpoint estimates are of weak type(p,q). And the Riesz-Thorin interpolation theorem which is known as the complex method. On contrary of the previous theorem, it is mostly applicable for linear operators with strong endpoint estimates . The Riesz-Thorin interpolation theorem was the motivation for new result, namely Stein's theorem on interpolation of analytic families of linear operators. Therefore, we demonstrate the Riesz-Thorin theorem by the next example in order to have clear picture about next subsections (see Stein and Weiss (1971)).

Example 1.5 *Let the operator $\mathfrak{T}$ acting on L_p by convolution operation such that $\mathfrak{T}(f) = f * g$ for fixed $g \in L_p(\mathbb{R}^N)$, where*

$$(f * g)(x) = \int_{\mathbb{R}^N} f(y)g(x-y)\, dy.$$

One can use the Minkowski's integral inequality to see that the operator $\mathfrak{T}$ is bounded on $L_1(\mathbb{R}^N)$ and having values in $L_p(\mathbb{R}^N)$ with operator norm $\|\mathfrak{T}\| \leq \|g\|_{L_p(\mathbb{R}^N)}$. On the other hand, one can apply Holder's inequality to show that the operator $\mathfrak{T}$ is bounded on space $L_{p'}(\mathbb{R}^N)$ and taking values in space $L_\infty(\mathbb{R}^N)$ whenever $1/p + 1/p' = 1$, again we have $\|\mathfrak{T}\| \leq \|g\|_{L_p(\mathbb{R}^N)}$. These endpoint estimates for operator $\mathfrak{T}$ led to a natural question that whether $\mathfrak{T}$ maps $L_r(\mathbb{R}^N)$, $1 \leq r \leq p'$ boundedly into some space $L_q(\mathbb{R}^N)$, a positive answer one can easily have by applying the Riesz-Thorin interpolation theorem and the result is so-called Young's inequality:

$$\|f * g\|_{L_q} \leq \|g\|_{L_p} \|f\|_{L_r},$$

where $\frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1$.

The Stein's theorem is considered a generalization of the Riesz-Thorin theorem in which the single operator $\mathfrak{T}$ is replaced by a family of operators $\{\mathfrak{T}_z\}_z$ that depend analytically on a parameter $z \in S = \{z \in \mathbb{C} : 0 \le Re(z) \le 1\}$.

Let (X, μ) and (Y, ν) be measure space. Suppose that to each $z \in S$ there is assigned a linear operator $\mathfrak{T}_z$ on the space of simple function on X and taking values in the space of ν-measurable function on Y in such a way that $(\mathfrak{T}_z f)g$ is ν-integrable on Y whenever f and g are simple function on X and Y. The family $\{\mathfrak{T}_z\}_z$ is said to be an *admissible growth* if the mapping

$$z \to \int_Y (\mathfrak{T}_z f) g \, d\nu$$

is analytic (i.e. it can be expanded as a convergent Taylor series for every point in the domain) in the interior of S, continuous on S and there exists a constant $C_{f,g}$ such that

$$\ln \left| \int_Y (\mathfrak{T}_z f) g \, d\nu \right| \le C_{f,g} e^{a|\Im(z)|}, \quad a < \pi,$$

for all $z = x + iy$ in the strip S.

Theorem 1.4.1 (Stein's Theorem) *Let $\{\mathfrak{T}_z\}_{z \in S}$ be an admissible growth family of linear operators satisfying*

$$\|\mathfrak{T}_{iy} f\|_{L_{q_0}(T^N)} \le M_0(y) \|f\|_{L_{p_0}(T^N)}$$
$$\|\mathfrak{T}_{1+iy} f\|_{L_{q_1}(T^N)} \le M_1(y) \|f\|_{L_{p_1}(T^N)}$$

for all simple functions f on T^N, where $1 \le p_j, q_j \le \infty$, for $j = 0, 1$ and $M_j(y)$ are positive functions on the real line such that

$$\sup_{-\infty < y < \infty} e^{-b|y|} \ln M_j(y) < \infty$$

for some $b < \pi$. Let $0 \le t \le 1$ satisfy

$$\frac{1}{p} = \frac{1-t}{p_0} + \frac{t}{p_1} \qquad \text{and} \qquad \frac{1}{q} = \frac{1-t}{q_0} + \frac{t}{q_1}.$$

Then

$$\|\mathfrak{T}_t f\|_{L_q(T^N)} \le M_t \|f\|_{L_p(T^N)}$$

for all simple functions f on T^N, where for $0 \le t \le 1$,

$$\ln M_t = \frac{\sin(\pi t)}{2} \int_{-\infty}^{\infty} \left[\frac{\ln M_0(y)}{\cosh(\pi y) - \cos(\pi t)} + \frac{\ln M_1(y)}{\cosh(\pi y) - \cos(\pi t)} \right] dy.$$

1.5 Fractional Powers of Elliptic Operators

Let Λ be a subset of the complex plane (in the applications this will, as a rule, be an angle with the vertex at the origin). In spectral theory it is useful to consider operators depending on a parameter $\lambda \in \Lambda$. An example of such an operator is the resolvent $(A - \lambda I)^{-1}$.

We assume that the differential operator (1.16) is elliptic, symmetric and semi-bounded, by that we mean respectively, its principle symbol (1.12) is positive and satisfies both equality (1.17) and inequality (1.18).

We denote by A an operator acting in the Hilbert space $L_2(T^N)$ with the domain of definition $D(A) = C^\infty(T^N)$:

$$Au = A(D)f(x), \quad f \in C^\infty(T^N).$$

It is proven that (see Theorem 9.3 Shubin (2001)) there exists $\lambda_0 > 0$ such that the resolvent $R_\lambda = (A - \lambda I)^{-1}$ is defined for $|\lambda| \geq \lambda_0$. The spectrum $\mathscr{P}(A)$ of A is a discrete subset of the complex plane. We choose a number ρ satisfying the condition that the disk $|\lambda| < 2\rho$ does not intersect with the spectrum $\mathscr{P}(A)$. Now we select a contour of the form $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3$:

$$\Gamma_1 = \{\lambda \in \mathbb{C} : \lambda = re^{i\pi}, +\infty > r > \rho\}$$
$$\Gamma_2 = \{\lambda \in \mathbb{C} : \lambda = \rho e^{i\phi}, \pi > \phi > -\pi\}$$
$$\Gamma_3 = \{\lambda \in \mathbb{C} : \lambda = re^{-i\pi}, \rho < r < +\infty\}.$$

Consider the integral

$$A_z = \frac{i}{2\pi} \int_\Gamma \lambda^z (A - \lambda I)^{-1} d\lambda, \tag{1.25}$$

where λ^z is defined as a analytic function of $\lambda \in \mathbb{C} \backslash (-\infty, 0]$. Note that the integral (1.25) converges in the operator norm on $L_2(T^N)$ for $\Re(z) < 0$ and also A_z is a bounded operator on $L_2(T^N)$.

With the help this operator we can define fractional power of the elliptic operator as follows: Let $z \in \mathbb{C}$ and $k \in \mathbb{Z}$ be such that $\Re(z) < k$. Put, on $C^\infty(T^N)$

$$A^z = A^k A_{z-k}.$$

Such defined operator is independent of the choice of integer k. And for arbitrary $k \in \mathbb{Z}$, $s \in \mathbb{R}$, the function A^z is analytic operator function of z in the half-plane $Re(z) < k$ with values in the Banach space $\mathscr{L}(L_2^s(T^N), L_2^{s-mk}(T^N))$ of bounded linear operators from $L_2^s(T^N)$ to $L_2^{s-mk}(T^N)$ (see Theorem 10.1 Shubin (2001)).

Let the distribution $f \in \bigcup_s L_2^s(T^N)$ and let $f(x) = \sum_{n \in \mathbb{Z}^N} \hat{f}(n) e^{i(n,x)}$ be the Fourier series of f. We denote the eigenvalues of A by λ_n. Then

$$A^z f(x) = \sum_{n\in\mathbb{Z}^{\mathbb{N}}} \lambda_n^z \hat{f}(n) e^{i(n,x)},$$

which follows from the following identity by putting $f(x) = e^{i(n,x)}$

$$\begin{aligned} A^z(e^{i(n,x)}) &= \frac{i}{2\pi}\int_\Gamma \lambda^z (A-\lambda I)^{-1} e^{i(n,x)} d\lambda \\ &= \frac{ie^{i(n,x)}}{2\pi}\int_\Gamma \lambda^z(\lambda_n-\lambda)^{-1} d\lambda = \lambda_n^z e^{i(n,x)}, \end{aligned}$$

which it is obtained by note that $A = \sum \lambda_n(\cdot, e^{-i(n,x)})e^{i(n,x)}$, $I = \sum(\cdot, e^{-i(n,x)})e^{i(n,x)}$, where $(\cdot,\cdot)$ is inner product in $L_2(T^N)$ and by using the Cauchy formula.

Example 1.6 *The fractional powers of Laplace operator (1.21). Let I denotes identity operator, then it is easy to see that*

$$(\overline{I-\Delta})e^{i(n,x)} = (1+|n|^2)e^{i(n,x)}.$$

For any $\beta \in R$ we define a fractional power of the operator $\overline{I-\Delta}$ as follows

$$(\overline{I-\Delta})^\beta \phi(x) = \sum_{n\in Z^N} (1+|n|^2)^\beta \hat{\phi}(n) e^{i(n,x)}, \quad \forall\, \phi \in C^\infty(T^N),$$

where by $\hat{\phi}(n)$ denotes Fourier coefficients of the function ϕ:

$$\hat{\phi}(n) = (2\pi)^{-N}\int_{T^N} \phi(x) e^{-i(n,x)}\, dx, \quad n \in Z^N.$$

1.6 Orthonormal Basis in T^N and Menchoff-Rademacher Theorem

In this section we deal with the system of functions $\{\phi_k(x)\}_{k=0}^\infty$ from $L_2(T^N)$.

Definition 1.6.1 *If a sequence of functions $\{\phi_k(x)\}_{k=0}^\infty$ satisfy conditions*

$$\int_{T^N} \phi_n(x)\overline{\phi_k(x)}\, dx = \begin{cases} 1 & when \;\; n = k, \\ 0 & when \;\; n \neq k, \end{cases}$$

then $\{\phi_k(x)\}_{k=0}^\infty$ is said to be an orthonormal system of functions on T^N.

Let $\{\phi_k(x)\}$ be the orthonormal system of functions in $L_2(T^N)$ and let define the following numbers

$$c_k = \int_{T^N} f(x)\overline{\phi_k(x)}\, dx, \quad k = 0,1,2,...., \tag{1.26}$$

so-called the Fourier coefficients of f related to $\phi_k(x)$, then we write

$$f(x) \sim \sum_{k=0}^{\infty} c_k \phi_k(x),$$

as a Fourier series of f relative to the orthonormal system $\{\phi_k(x)\}_{k=0}^{\infty}$. The symbol $\sim$ says nothing about convergence of the series, it only means that the coefficients are given by (1.26).

The following theorem is important tool to prove almost everywhere convergence of Fourier series by general orthonormal system. We refer for more facts on the general orthonormal systems to Zygmund (1959a).

Theorem 1.6.2 (Menchoff-Rademacher) *Let $\{\phi_k(x)\}_{k=0}^{\infty} \subset L_2(T^N)$ be an orthonormal system of functions and let $\{c_k\}_{k=0}^{\infty}$ be an arbitrary sequence of complex numbers such that*

$$\sum_{k=0}^{\infty} |c_k|^2 \log^2(k+1) < \infty.$$

Then the series $\sum_{k=0}^{\infty} c_k \phi_k$ converges almost everywhere. Moreover

$$\left\| \sup_{\nu} \left| \sum_{k=0}^{\nu} c_k \phi_k(x) \right| \right\|_{L_2(T^N)}^2 \leq C \sum_{k=0}^{\infty} |c_k|^2 \log^2(k+1).$$

The proof of this fact can be found in Alexits (1961). The Menchoff-Radimacher Theorem will be used in chapter III to establish the almost everywhere convergence and estimation of the maximal operator in $L_2(T^N)$.

1.7 Best Approximation by Trigonometric Polynomials

We denote by $C_{2\pi}$ a class of 2π-periodic and continuous functions on whole $\mathbb{R}$. Let us recall some classical results from the theory of approximations of the functions from $C_{2\pi}$ by trigonometric polynomials:

$$P_k(x) = \sum_{n=0}^{k} (a_n \cos nx + b_n \sin nx), \quad k = 0, 1, 2, \ldots$$

It is well known that any function from $C_{2\pi}$ can be approximated by trigonometric polynomials. In fact if $f \in C_{2\pi}$, then for any $\varepsilon > 0$ there exists a trigonometric polynomial $P_k(x)$ such that

$$|P_k(x) - f(x)| < \varepsilon, \quad \forall x \in \mathbb{R},$$

for the proof of this fact we refer to Bari (1964).

Our method of estimation of the maximal operator is based on the best approximation by trigonometric polynomials of several variables. Here we give the main idea of the best approximation by trigonometric polynomials of single variable, which will be extended to several variables in chapter III and IV.

Let $\mathfrak{P}_k$ denotes the set all of trigonometric polynomials of degree not exceeding k, by taking an arbitrary trigonometric polynomial $P(x) \in \mathfrak{P}_k$ we put

$$\mathfrak{M}(P) = \max_{x \in \mathbb{R}} |P(x) - f(x)|.$$

We define a best approximation of $f(x)$ in $\mathfrak{P}_k$ as follows

$$\mathfrak{B}_k(f) = \inf_{P \in \mathfrak{P}_k} \mathfrak{M}(P).$$

For any k there exists a polynomial $P \in \mathfrak{P}_k$, such that

$$\mathfrak{M}(P) = \mathfrak{B}_k(f).$$

For the function $f \in C_{2\pi}$ we define the modulus of continuity by the following

$$\omega(\delta) = \sup_{|x-y| \le \delta} |f(x) - f(y)|, \tag{1.27}$$

where it assumed that the number δ is positive. The module of continuity of the function f has following properties

1. The $\omega(\delta)$ is increasing function.
2. If a function $f(x)$ is uniformly continuous, then

$$\lim_{\delta \to 0} \omega(\delta) = 0.$$

3. (a) If k is natural number, then

$$\omega(k\delta) \le k\omega(\delta).$$

   (b) If λ is arbitrary positive number, then

$$\omega(\lambda\delta) \le (\lambda + 1)\omega(\delta). \tag{1.28}$$

The following Theorem is known as Jackson type theorem (see Natanson (1961)) and will be extended to $N-$dimensional case in chapter III for trigonometric polynomials summed over spherical levels and in chapter IV for trigonometric polynomials summed over elliptic levels.

Theorem 1.7.1 *For any function f from $C_{2\pi}$ we have*

$$\mathfrak{B}_k(f) \le 12\omega\left(\frac{1}{k}\right).$$

From this fact we easily derive the estimation for partial sums of Fourier series. In fact let $f \in C_{2\pi}$ satisfies the inequality

$$|f(x)| \le M, \quad \forall x \in \mathbb{R},$$

then the partial sums of the Fourier series of the function f satisfy the inequality

$$|S_k f(x)| \le M(2 + \ln k). \tag{1.29}$$

From the estimation 1.29 we conclude the following

Theorem 1.7.2 *Let $\mathfrak{B}_k(f)$ be the best approximation of $f \in C_{2\pi}$ by trigonometric polynomials of order not exceeding k. Then for all $x \in \mathbb{R}$ one has*

$$|S_k f(x) - f(x)| \le (3 + \ln k)\mathfrak{B}_k(f). \tag{1.30}$$

For the proof of this fact we refer to Natanson (1961). The idea of estimation 1.30 will be used in chapter III and IV.

1.8 Research Objectives

Here, we introduce the goals of this research:

1. To obtain the sufficient conditions for the almost everywhere convergence of the multiple Fourier series summed by spherical method in the classes of Liouville on Torus.
2. To prove almost everywhere convergence of the elliptic partial sums of the multiple Fourier series in the classes of Liouville on Torus.
3. To investigate the conditions for the best approximation of the summable functions by multiple Fourier series summed over elliptic levels.
4. To generalize the Menchoff-Rademacher Theorem to the spectral decompositions of the self adjoint differential operators.

1.9 Outline of the Thesis

In this thesis, we have organized the chapter as following.

- Chapter 1 - Introduction: In this chapter we provide background of tools used for our subject matter, the objective and methods applied to solve the problems in hand and the factors responsible for our motivation behind the idea.
- Chapter 2 - Literature review: This chapter deals with researches by previous authors that dealt with issues related to our topic.
- Chapter 3 - we prove almost everywhere convergence of the spherical partial sums of Fourier series of functions from the classes of Liouville by estimation the maximal operators corresponding to the spherical partial sums.
- Chapter 4 - we obtain the sufficient conditions for the almost everywhere convergence of the multiple Fourier series related to elliptic differential operator using the best approximation of summable functions by Fourier series summed over domains bounded by levels of the elliptic polynomial.
- Chapter 5 - In this chapter we obtain the sufficient conditions for the almost everywhere convergence of the eigenfunction expansions of the square-integrable functions, related to the spectral decompositions of the self-adjoint elliptic differential operators.
- Chapter 6 - Summary and Future Work: In the last section of the thesis we will present the outcome of this thesis. We also discuss the possibility for future research and give suggestions on open problems for future researchers of the subject.

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