UNIVERSITI PUTRA MALAYSIA

PARAMETERIZATION OF NICE POLYNOMIALS

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IPM 2018 15

PARAMETERIZATION OF NICE POLYNOMIALS

By

HOZJEE ANTON

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Master of Science

July 2018

DEDICATIONS

To

My beloved parents:
Anton Siridon and Helena Frigillana
My supportive husband:
Jefres Justin
My amazing sisters:
Erracholl Anton
Clarra Ann Anton
Capreesandy Anton
Evvy Olivia Anton
Claire Verra Anton
...
Gadis-Gadis INSPEM

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Master of Science

PARAMETERIZATION OF NICE POLYNOMIALS

By

HOZJEE ANTON

July 2018

Chairman : **Siti Hasana Sapar, PhD**
Faculty :

A univariable polynomial $p(x)$ is said to be nice if all of its coefficients as well as all of the roots of both $p(x)$ and its derivative $p'(x)$ are integers. $p(x)$ is called Q-nice polynomial if the coefficients, roots, and critical points are rational numbers.

This research concentrates on finding parameterized families of symmetric polynomial with four, five, and seven roots. The relations between the roots and critical points of polynomials with four, five, and seven roots are considered respectively. By using the technique of parameterization and substitution, the pattern of solutions of the polynomials in the field of integer, rational, and $Q(\sqrt{x})$ are observed. Then, based on the pattern of solutions, theorems will be constructed.

Parameterized families of symmetric polynomials with four and five roots in the field of integral and rational numbers are obtained. Meanwhile, the roots and critical points for symmetric polynomials with seven roots are studied in the field of $Q(\sqrt{x})$. Hence, parameterized families of symmetric polynomials with seven roots are found.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Master Sains

PEMPARAMETERAN POLINOMIAL YANG ELOK

Oleh

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Julai 2018

Pengerusi : **Siti Hasana Sapar, PhD**
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Polinomial dengan satu pembolehubah dikatakan elok jika kesemua pekalinya termasuk juga punca bagi kedua-dua $p(x)$ dan terbitannya, $p\prime(x)$ adalah integer. $p(x)$ dikatakan polinomial Q-elok sekiranya pekali, punca, dan titik genting adalah nombor nisbah.

Kajian ini tertumpu kepada mencari keluarga berparameter bagi polinomial simetri dengan punca-punca empat, lima, dan tujuh. Hubungan antara punca-punca dan titik-titik genting polinomial punca-punca empat, lima, dan tujuh masing-masing dipertimbangkan. Dengan menggunakan kaedah parameterisasi dan penggantian, corak penyelesaian-penyelesaian untuk polinomial-polinomial dalam medan integer, nisbah, dan $Q(\sqrt{x})$ diperhatikan. Kemudian, berdasarkan kepada corak penyelesaian-penyelesaian tersebut, teorem-teorem akan dibina.

Keluarga parameterisasi bagi polinomial-polinomial simetri untuk punca-punca empat dan lima dalam medan integer dan nombor-nombor nisbah diperolehi. Sementara itu, punca-punca dan titik-titik genting untuk polinomial-polinomial simetri dengan punca tujuh dikaji dalam medan $Q(\sqrt{x})$. Oleh itu, keluarga parameterisasi bagi polinomial-polinomial simetri dengan punca tujuh diperolehi.

ACKNOWLEDGEMENTS

First and foremost, praises and thanks to God for His showers of blessings throughout my research work to complete this thesis completely. I would like to express my sincere and deepest gratitude to the chairman of the supervisory comitte, Assoc. Prof. Dr. Siti Hasana Sapar for her continuous support, courage, guidance, and assistance from the very beginning till the completion of my studies.

My sincere appreciation also goes to the member of the supervisory comitte, Dr. Mohamat Aidil Mohamat Johari for his valueable support and courage. I am grateful and thankful to the Institute of Mathematical Research (INSPEM) and all the staff for providing me with excellent and comfortable atmosphere for completing this research. Thanks also to all my friends who are willing to help me especially in terms of sharing ideas, suggestions, advices, and experiences on every aspects. I really appreciate all of that and your help means a lot to me.

Last but not least, a million thanks to all of my family members and my beloved husband who always give me moral support and encouragement in completing this research. I am very lucky to have all of you in my life that always give me the strongest support in everything I do.

This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Master of Science. The members of the Supervisory Committee were as follows:

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- the research conducted and the writing of this thesis was under our supervision;
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CHAPTER 1

INTRODUCTION

1.1 Preliminary

In this chapter, we introduce some background of nice and Q-nice polynomials and also polynomials in the field of $Q(\sqrt{x})$. This research concentrates on the problem of finding, constructing, and classifying parametrized family of nice, Q-nice polynomials and polynomials with roots and coefficients in the field of $Q(\sqrt{x})$. Next, we state the objectives and the methodology of this research. We also discuss some literature reviews on nice, Q-nice polynomials and polynomials in the field of $Q(\sqrt{x})$.

1.2 Background

An expression consisting of variables (indeterminates) and coefficients, that involves the operations of addition, subtraction, multiplication, and non-negative integer exponents of variables is called a polynomial. Polynomials appear in a wide variety of areas of mathematics and science. For example, they are used to form polynomial equations, which encode a wide range of problems, from elementary word problems to complicated problems in the sciences; they are used to define polynomial functions, which appear in settings ranging from basic chemistry and physics.

There are numerous mathematical problems in number theory that can be understood and expressed in simple terms, yet give challenges to the mathematical investigations to find the solutions for so many years. The existence of a variety of methods and tools to solve these simple form problems explain the attraction of number theory for mathematicians. The never-ending individual contributions, calculations and conjectures become an evidence to the continuing interest in this area over the centuries.

A polynomial in a single indeterminate x can always be written (or rewritten) in the form of

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + ... a_2 x^2 + a_1 x + a_0, \tag{1.1}$$

where $a_0, ..., a_n$ are constants and x is the indeterminate. A number a is a root of a polynomial $p(x)$ if and only if the linear polynomial $x - a$ divides $p(x)$, that is there is another polynomial $q(x)$ such that $p(x) = (x-a)q(x)$. There is a chance that $x - a$ divides $p(x)$ more than once. If $(x-a)^2$ divides $p(x)$, then a is called a multiple root of $p(x)$, otherwise a is called simple root of $p(x)$. If $p(x)$ is a nonzero polynomial, there is highest power of m such that $(x-a)^m$ divides $p(x)$, which is called the multiplicity of the root a in $p(x)$. When $p(x)$ is the zero polynomial, the corresponding polynomial equation is trivial and this case is usually excluded when considering roots, as, with the above definitions, every number is a root of the zero polynomial, with an undefined multiplicity. With this exception made, the number of roots of $p(x)$, even counted with their respective multiplicities, cannot exceed the degree d of $p(x)$ Leung et al. (1992).

A polynomial $p(x)$ of degree d is totally nice if the roots of $p(x), p\prime(x), ..., p^{d-1}(x)$ are integers. The known example of nice polynomials with distinct roots are limited to quadratic, cubic, quartic, quintic, and sextic polynomials. Oftenly, we work with nice polynomials $p(x)$. However, this is not always the case. We will have to encounter many polynomials with roots that are rational or even irrational numbers. Polynomials with rational coefficients, roots, and critical points are called Q-nice polynomials.

Transformations alter a polynomial while maintaining the original characteristics of that polynomial. Common types of transformations include rotations, translations, reflections, and scalling (also known as stretching or shrinking). Two nice polynomials that are obtained from transforming a nice polynomial into other nice polynomial are called equivalent. Shifting a nice polynomial $p(x)$ to the left or right by $k \in \mathbb{N}$ units results in an equivalent nice polynomial. Besides, stretching or shrinking $p(x)$ by a factor $k \in \mathbb{N}$ also results in an equivalent nice polynomial. If $p(x)$ is Q-nice, shifting $p(x)$ to the left or right by $k \in \mathbb{Q}$ units, stretching or shrinking $p(x)$ by a factor of $k \in \mathbb{Q}$, or reflecting $p(x)$ over one of the coordinate axes result in an equivalent Q-nice polynomial. Since the stretch or shrink is an equivalence transformation, we now see that the existence of nice and Q-nice polynomials is equivalent. This means, for every Q-nice polynomial, there exists an equivalent nice polynomial and vice versa. Furthermore, raising a nice polynomial $p(x)$ to the nth power also results in an equivalent nice (or Q-nice) polynomial. Groves (2004) states this in the following definitions and theorems:

Definition 1.1 *A univariable polynomial $p(x)$ is said to be nice if all of its coefficients as well as the roots of both $p(x)$ and its derivative $p\prime(x)$ are integers.*

Some examples of nice polynomials are as follows:

Example 1.1 *Let*

$$\begin{aligned} p_1(x) =& x^4 - 14886134x^2 - 1131126696x + 12096252681037, \\ p_1(x) =& (x-887)(x+973)(x-3787)(x+3701). \end{aligned}$$

The roots of $p_1(x)$ *are element of* $\mathbb{Z}$. *Given the first derivative of* $p_1(x)$ *is*

$$p_1\prime(x) = 4x^3 - 6944x^2 - 10960972x - 406270312,$$
$$p_1\prime(x) = 4(x-2747)(x+973)(x+38).$$

The roots of $p_1\prime(x)$ *are also element of* $\mathbb{Z}$. *Thus,* $p_1(x)$ *is a nice polynomial.*

Example 1.2 *Let*

$$p_2(x) = x^4 - 556x^3 + 83380x^2 - 2904000x,$$
$$p_2(x) = x(x-50)(x-176)(x-330).$$

The roots of $p_2(x)$ *are element of* $\mathbb{Z}$. *Given the first derivative of* $p_2(x)$ *is*

$$p_2\prime(x) = 4x^3 - 1668x^2 + 166760x - 2904000,$$
$$p_2\prime(x) = 4(x-22)(x-120)(x-275).$$

The roots of $p_2\prime(x)$ *are also element of* $\mathbb{Z}$. *Thus,* $p_2(x)$ *is a nice polynomial.*

Definition 1.2 *A univariable* $p(x)$ *is said to be Q-nice if all of its coefficients as well as the roots of both* $p(x)$ *and its derivative* $p(x)$ *are rational numbers.*

Some examples of *Q*-nice polynomials are as follows:

Example 1.3 *Let*

$$p_3(x) = x^4 + \frac{13936x^3}{81} + \frac{7075904x^2}{729} + \frac{391104512x}{2187},$$
$$p_3(x) = x(x+\frac{3496}{81})(x+\frac{608}{9})(x+\frac{184}{3}).$$

The roots of $p_3(x)$ *are element of* $\mathbb{Q}$. *Given the first derivative of* $p_3(x)$ *is*

$$p_3\prime(x) = 4x^3 + \frac{13936x^2}{27} + \frac{14151808x}{729} + \frac{391104512}{2187},$$
$$p_3\prime(x) = 4(x+\frac{1748}{27})(x+\frac{152}{3})(x+\frac{368}{27}).$$

The roots of $p_3\prime(x)$ *are also element of* $\mathbb{Q}$. *Thus,* $p_3(x)$ *is a Q-nice polynomial.*

Example 1.4 *Let*

$$\begin{aligned} p_4(x) =& x^7 + \frac{20072521x^6}{416} + \frac{1425940083225x^5}{1664} + \frac{2744947255168125x^4}{416} \\ &+ \frac{12249327126187757812 5x^3}{6656}, \\ p_4(x) =& x^3(x+\frac{457317}{32})(x+\frac{26775}{2})^2(x+\frac{373625}{52}). \end{aligned}$$

Given the first derivative of $p_4(x)$ is

$$\begin{aligned} p_4\prime(x) =& 7x^6 + \frac{60217563x^5}{208} + \frac{7129700416125x^4}{1664} + \frac{2744947255168125x^3}{104} \\ &+ \frac{36747981378563273437 5x^2}{6656}, \\ p_4\prime(x) =& 7x^2(x+\frac{26775}{2})(x+\frac{457317}{104})(x+\frac{19125}{2})(x+\frac{224175}{16}). \end{aligned} \tag{1.2}$$

The roots of $p_4(x)$ and its first derivative, $p_4\prime(x)$ are rational numbers. Thus, $p_4(x)$ is Q-nice polynomial.

In nice polynomials, the roots and critical points of the polynomials are integer numbers. Meanwhile, for *Q*-nice polynomials, the roots and critical points are rational numbers.

Theorem 1.1 *Let $p(x)$ be a polynomial with integer coefficients. Then, for all natural numbers n, $p(x)$ is nice (or $Q-$nice) iff $[p(x)]^n$ is nice (or $Q-$nice).*

While working with nice and *Q*-nice polynomials, we need the relations between the roots and critical points of polynomial as stated by Evard (2004) in the following lemma:

Lemma 1.1 *Let $p(x)$ be a polynomial of degree d with rational coefficients and d rational roots $r_1, r_2, ..., r_d$. Then $p(x)$ is $Q-$nice iff there exist rational numbers $c_1, c_2, ..., c_{d-1}$ such that*

$$(d-k)S_k(r_1, r_2, ..., r_d) = dS_k(c_1, c_2, ..., c_{d-1}),$$

for all $k \in \mathbb{N}$ such that $1 \leq k \leq d-1$ and S_k, the kth elementary symmetric polynomial, is defined as follows:

$$S_k(r_1, ..., r_d) = \sum_{1 \leq i_1 < i_2 < ... < i_k \leq d} r_{i_1} r_{i_2} ... r_{i_k}.$$

Groves (2004) gives the relations between roots and critical points of all polynomials with four, five, and seven roots in the following lemmas:

Lemma 1.2 *A polynomial*

$$p(x) = x^{m_0}(x-r_1)^{m_1}(x-r_2)^{m_2}(x-r_3)^{m_3} \tag{1.3}$$

of degree d with integer coefficients and with four integer roots is nice iff there exist integers c_1, c_2, *and* c_3 *such that*

$$(d-m_1)r_1 + (d-m_2)r_2 + (d-m_3)r_3 = d(c_1+c_2+c_3), \tag{1.4}$$

$$(m_0+m_3)r_1r_2 + (m_0+m_2)r_1r_3 + (m_0+m_1)r_2r_3 = d(c_1c_2+c_1c_3+c_2c_3), \tag{1.5}$$

$$m_0r_1r_2r_3 = dc_1c_2c_3. \tag{1.6}$$

Lemma 1.3 *A polynomial*

$$p(x) = x^{m_0}(x^2-r_1^2)^{m_1}(x^2-r_2^2)^{m_2} \tag{1.7}$$

of degree d with five integer roots is nice iff there exist integers c_1 *and* c_2 *such that*

$$(m_0+2m_2){r_1}^2 + (m_0+2m_1){r_2}^2 = d{c_1}^2 + d{c_2}^2, \tag{1.8}$$

$$m_0{r_1}^2{r_2}^2 = d{c_1}^2{c_2}^2. \tag{1.9}$$

Lemma 1.4 *If*

$$p(x) = x^{m_0}(x^2-r_1^2)^{m_1}(x^2-r_2^2)^{m_2}(x^2-r_3^2)^{m_3} \tag{1.10}$$

is a symmetric or antisymmetric polynomial with integer coefficients and with seven integer roots, then $p(x)$ *is nice iff there exist integers* c_1, c_2, *and* c_3 *such that*

$$D_1r_1^2 + D_2r^2 + D_3r_3^2 = d(c_1^2+c_2^2+c_3^2), \tag{1.11}$$

$$D_{12}r_1^2r_2^2 + D_{13}r_1^2r_3^2 + D_{23}r_2^2r_3^2 = d(c_1^2c_2^2+c_1^2c_3^2+c_2^2c_3^2), \tag{1.12}$$

$$m_0r_1^2r_2^2r_3^2 = dc_1^2c_2^2c_3^2. \tag{1.13}$$

where $D_i = d-2m_i$ *and* $D_{ij} = d-2m_i-2m_j$.

1.3 Objective and Methodology

The main objectives of this research are:

(i) to construct and classify parameterized family of nice and Q-nice polynomials with four roots;

(ii) to construct and classify parameterized family of nice and Q-nice polynomials with five roots;

(iii) to construct and classify parameterized family of polynomials with seven roots in the field of $\mathbb{Q}(\sqrt{x})$.

In this research, we use parameterization technique and substitution. A parameterization technique is the process of expressing equations in terms of parameters. The procedures of this technique are:

- Step 1: Consider the relations between the roots and critical points of the polynomials.
- Step 2: Find the characteristics of these relations and construct parameterized family of polynomials in terms of parameters u and v.
- Step 3: Observe the pattern of solutions of the polynomials in the field of $\mathbb{Z}, \mathbb{Q}$, and $\mathbb{Q}(\sqrt{x})$.
- Step 4: Construct the theorems based on the steps above.

Then, we give an illustration for Steps 1-3 to get more understandings of parameterization technique. The solutions of polynomials with four and five roots are found in the field of $\mathbb{Z}$ and $\mathbb{Q}$. However, no solutions of polynomials with seven roots are found in these fields. Thus, we consider this case in the field of $\mathbb{Q}(\sqrt{x})$.

1.4 Literature Review

The mathematical study of polynomials has been studied by many researchers. Caldwell (1990) found that if $p(x)$ is a nice polynomial of degree d, and a, b, c are arbitrary integers such that $a \neq 0, c \neq 0$, then $g(x) = a^d[p(\frac{x+b}{a})]$ is also a nice polynomial of degree d. Thus, given a single nice polynomial $p(x)$ of degree d, we readily obtain infinitely many nice polynomials of degree d. Any two nice polynomials that are obtained in this manner from the same nice polynomial will be considered 'equivalent'. Buddenhagen et al. (1992) gave a systematic description of all nice cubics polynomial. They found that if a cubic polynomial with rational roots has a double or triple roots, its derivative will necessarily have rational roots. Thus, they assumed that by multiplication, the roots are integers and by translation, the middle root is zero. In their paper, they looked for nonnegative integers a and b so that the derivative of

$$y = (x+a)x(x-b) \tag{1.14}$$

which is

$$y\prime = 3x^2 + 2(a-b)x - ab$$

has rational roots. This happens when the discriminant is a perfect square. If, instead, the leftmost root is made as zero, (1.14) can be stated as

$$y = x(x-a)(x-b).$$

The problem of finding properties, characterizations, and methods of construction of polynomials with coefficients, roots, and critical points in the ring of rational integers is on the list of unsolved problems published by Nowakowski (1999). Bucholz and MacDougall (2000) considered the problem of classifying all univariate polynomials, defined over a domain k, with the property that they and their derivatives have all their roots in k. From a number of theoretic perspective, the most interesting cases for this problem are $k = \mathbb{Z}$ or $k = \mathbb{Q}$. In this thesis, we want the polynomials to have roots and critical points in $\mathbb{Z}, \mathbb{Q}$, and $\mathbb{Q}(\sqrt{x})$.

Evard (2004) considered the relations between roots and critical points of polynomials to present equivalences of nice polynomials which reduce the search of just one representant in each equivalence class. A key relations to deal with nice polynomials, namely, the system of equations has been established. Also, some of the results have been generalized to $\mathbb{Z}$ domain. For simplicity, $p(x) \in \mathbb{Z}$.

Groves (2004) presented the relations between the roots and critical points for the general four roots case and the symmetric five, six, and seven roots cases. Besides, using the relations for the general three and four roots cases, the author suggested pattern for the relations for the general N root case. Groves (2008) and Groves (2007b) considered the problem of finding, constructing, and classifying nice polynomials of three and four roots. In these papers, complete solutions to the symmetric three and four roots cases and also the general three roots case have been presented. However, the general four roots case has not been solved. Groves (2007a) investigated polynomials with four roots over integral domains D of any characteristic. The author extended the earlier results in Z to all integral domains D of any characteristic. The roots and critical points of a polynomial can be considered in field of interest depends on the context in which we are working. For instance, roots and critical points of a polynomial possibly lie in extension of $\mathbb{Q}$.

Choudry (2015) found nice nonsymmetric quartic, quintic, and sextic polynomials whose coefficients are given in parametric terms and whose roots are all distinct. This yields infinitely many distinct nice quintic and sextic polynomials whose roots are all distinct.

1.5 Organization of Thesis

In this research, we concentrate on finding and constructing parameterized polynomials. In Chapter 2, we construct parameterized families of symmetric polynomials with four roots in $\mathbb{Z}$ and $\mathbb{Q}$. The general four roots case will be discussed in detail through this chapter.

In Chapter 3, we focus on finding parameterized families of symmetric polynomials with five roots in $\mathbb{Z}$ and $\mathbb{Q}$. Lastly, in Chapter 4, we consider the roots and critical points of parameterized family polynomials with seven roots in $\mathbb{Q}(\sqrt{x})$. We present some examples based on the results obtained.

In Chapter 5, we will give the summary and the conclusion of this research. Finally, we give some suggestions for the future research.

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