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HYBRID METHODS FOR SOLVING HIGHER ORDER ORDINARY DIFFERENTIAL EQUATIONS

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By

YUSUF DAUDA JIKANTORO

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Doctor of Philosophy

April 2018

DEDICATIONS

I dedicate this work to my late sister, Salamatu Dauda (Lami).

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfillment of the requirement for the degree of Doctor of Philosophy

HYBRID METHODS FOR SOLVING HIGHER ORDER ORDINARY DIFFERENTIAL EQUATIONS

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YUSUF DAUDA JIKANTORO

April 2018

Chairman : **Professor Fudziah Ismail, PhD**
Faculty : **Science**

In this thesis, a class of numerical integrators for solving special higher order ordinary differential equations (ODEs) is proposed. The methods are multistage and multistep in nature. This class of integrators is called ”hybrid methods”, specifically, hybrid methods for directly solving special third order ODEs denoted by HMTD and for directly solving special fourth order ODEs denoted by HMFD are proposed. B-series approach is developed and used in deriving their algebraic order conditions and analyzing the order of convergence of the methods.

Using the algebraic order conditions, a class of explicit HMTD and HMFD are derived. The methods are applied to some test problems alongside some existing integrators in the literature for the purpose of validation. Results obtained show that the proposed methods in this thesis are a better alternatives.

To analyze the methods further, convergence analysis is conducted via consistency and zero stability, where the methods are found to be consistent and zero stable, hence, they are convergent. Absolute stability of the methods is also investigated, where stability polynomials of the methods are presented for obtaining intervals and regions of absolute stability.

Finally, a set of embedded pairs of two-step hybrid methods for solving special second order ODEs are proposed and investigated. The methods are tested on some model problems using different error tolerances. Results obtained are compared with those of existing embedded methods possessing similar properties. From the comparison, it is found that the new embedded methods possess better accuracy and efficiency.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

KAEDAH HIBRID UNTUK MENYELESAIKAN PERSAMAAN PEMBEZAAN BIASA PERINGKAT TINGGI

Oleh

YUSUF DAUDA JIKANTORO

April 2018

Pengerusi : **Professor Fudziah Ismail, PhD**
Fakulti : **Sains**

Dalam tesis ini, satu kelas kaedah pengamiran berangka untuk menyelesaikan persamaan pembezaan biasa (PPB) khas peringkat tinggi dicadangkan. Kaedah ini bersifat multitahap dan multilangkah.Kaedah pengamiranini disebut ”kaedah hibrid”, khususnya, kaedah hibrid untuk menyelesaikan secara langsung PPB khas peringkat ketiga yang dilambangkan sebagai HMTD dan untuk menyelesaikan secara langsung PPB khas peringkat keempat yang dilambangkan sebagai HMFD dicadangkan.Pendekatan siri B dibangunkan dan digunakan untuk menerbitkan syarat peringkat aljabarkaedah tersebut dan untuk menganalisis peringkat penumpuan kaedah yang terhasil.

Dengan menggunakan syarat peringkat aljabar tersebut, satu kelas HMTD dan HMFD yang eksplisit diterbitkan. Kaedah ini digunakan untukmenyelesaikanbeberapa masalah ujian di samping beberapa kaedah pengamiran yang ada dalam literatur untuk tujuan pengesahan. Keputusan yang diperolehi menunjukkan bahawa kaedah yang dicadangkan dalam tesis ini adalah alternatif yang lebih baik.

Untuk menganalisis kaedah tersebut selanjutnya, analisis penumpuan dijalankan melalui kekonsistenan dan kestabilan sifar, di mana kaedah tersebut didapati konsisten dan stabilsifar, oleh itu, ia adalah menumpu. Kestabilan mutlak kaedah juga diselidiki, di mana polinomial kestabilan kaedah dibentangkan untuk mendapatkan selang dan rantau kestabilan mutlaknya.

Akhir sekali, satu set pasangan kaedah hibrid terbenam dua langkah untuk menyelesaikan PPB khas peringkat kedua dicadangkan dan dikaji. Kaedah tersebut diuji pada beberapa masalah model menggunakan toleransi ralat yang berbeza. Hasil yang diperolehidibandingkan dengan kaedah terbenam sedia ada yang mempunyai sifat yang

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I certify that a Thesis Examination Committee has met on 10 April 2018 to conduct the final examination of Dauda Yusuf Jikantoro on his thesis entitled "Hybrid Methods for Solving Higher Order Ordinary Differential Equations" in accordance with the Universities and University Colleges Act 1971 and the Constitution of the Universiti Putra Malaysia [P.U.(A) 106] 15 March 1998. The Committee recommends that the student be awarded the Doctor of Philosophy.

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Declaration by graduate student

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LIST OF ABBREVIATIONS

ODEs	Ordinary Differential Equations
IVPs.	Initial Value Prblems
RK	Runge-Kutta
RKN	Runge-Kutta-Nyström
THM	Two-step Hybrid Method
RKT	Runge-Kutta Method for Third order ODEs
RKFD	Runge-Kutta Method for Fourth Order ODEs
HMTD	Hybrid Method for special Third order ODEs Directly
HMTD5s(6)	5-stage explicit HMTD method of order six
HMTD6s(6)	6-stage explicit HMTD method of order six
HMTD6s(7)	6-stage explicit HMTD method of order seven
RKT3s(5)	3-stage fifth order explicit RKT method for special third order ODEs
Heuns3s(3)	3-stage third order explicit Heuns method
RK4s(4)	4-stage fourth order explicit Runge Kutta method
RK6s(5)	6-stage fifth order explicit Runge Kutta method
RKD3s(5)	3-stage fifth order explicit Runge-Kutta direct method
MAXE	maximum error recorded in a given interval of solution
fun. eval.	total function evaluation for a given step-length.
HMFD	Hybrid Method for special Fourth order ODEs Directly
HMFD4s(4)	4-stage fourth explicit order HMFD method
HMFD5s(5)	5-stage fifth explicit order HMFD method
RKFD3s(5)	3-stage fifth explicit order RKFD method
HM4s(5)	4-stage fifth order hybrid method
THM3s(5)	3-stage fifth order explicit hybrid method
HMFD7s(8)	7-stage explicit HMFD method of order eight
ME	logarithm to base 10 of maximum error
FE	logarithm to base 10 of total function call
LTE	Local Truncation Error
E_{norm}	Error Norm
Tol	Error Tolerance
h_{int}	initial step-size
h_{old}	old step-size
h_{new}	new step-size
SF	Safety Factor
THM5(3)	fifth order embedded two-step hybrid method
THM6(5)	sixth order embedded two-step hybrid method
BRKN5(4)	embedded Runge-Kutta-Nyström algorithm
MRKN5(4)	fifth order embedded Runge-Kutta Nyström Munirah method
NRKN5(4)	embedded Runge-Kutta-Nyström Norazak method
DOPRI5	embedded Runge-Kutta method by Dormand and Prince
SS	Successful Step
FS	Failed Step
FC	total function evaluation

TFHM6(4)	sixth order embedded trigonometrically fitted THM
FFRKN5(4)	optimized embedded Runge-Kutta-Nyström algorithm
FFDOPRI5	phase fitted embedded Runge-Kutta method
CPUT	CPU Time

CHAPTER 1

INTRODUCTION

In science and engineering, differential equations are very important mathematical models, as most physical systems whose state variables vary with time or space are described using differential equations. An equation is said to be a differential equation if it represents a relation between a function f and its derivatives. Depending on the number of variables f depends on, differential equation is broadly divided in to two: ordinary differential equation and partial differential equation. It is ordinary if f depends on only one variable, $f = f(x)$, and partial if f depends on more than one variables, $f = f(x,y,z)$. The highest order of derivative of f present in the expression of a differential equation defines order of the equation. That is, it is of first order if the highest derivative order is one, of second order if the highest derivative order is two and so on. Example of a differential equation is the model that governs the decay of a radioactive substance or the one that governs the growth of a population. Throughout this thesis $y', y'', y''', y^{iv}, ...$, represent first, second, third, fourth,... derivatives of y, respectively.

1.1 Initial Value Problem

Solution of ordinary differential equation (ODE), if it exists, can only be found in its general form, which might not make a complete sense. To achieve more specific solution to an ODE, there is a need for a prior knowledge of what the solution would be at some points. If the solution is specified at some initial points of the solution, then we say initial value conditions are imposed on the ODE. The ODE together with these imposed conditions is called an initial value problem (IVP).

The general form of the IVPs considered in this study is

$$\begin{aligned} y^{(i)} &= f\left(x,y,y',y'',...,y^{(i-1)}\right), \\ y(x_0) &= y_0, y'(x_0) = y'_0, y''(x_0) = y''_0, ..., y^{(i-1)}(x_0) = y_0^{(i-1)}, \end{aligned} \tag{1.1}$$

where $x \in \mathrm{IR}$, $y(x) \in \mathrm{IR}^r$, $f \in \mathrm{IR}^{r+1}$, $i \geq 2$. They occur in many areas of applied sciences such as biology, quantum mechanics, celestial mechanics and chemical engineering, You and Chen (2013).

1.2 Existence and Uniqueness of a Solution

Given an IVP, the first thing that comes to mind is whether it has a solution. If it does, then the next question that arises is how unique is the solution. To answer this question, certain conditions need to be fulfilled.

Definition 1.1 *Refer to Dormand (1996)*
A function $f(x,y):\mathrm{IR}\times\mathrm{IR}^r\rightarrow\mathrm{IR}^r$ is said to satisfy a Lipschitz condition in the variable y on a set D if there exist a constant L>0 such that

$$\|f(x,y_1)-f(x,y_2)\|\leq L\,\|y_1-y_2\|\,, \tag{1.2}$$

whenever (x,y_1), $(x,y_2)\in D$. L is referred to Lipschitz constant.

For example, given that $f(x,y)=\frac{2}{x}y+x^2\mathrm{e}^x$, it can be shown that $f(x,y)$ satisfies Lipschitz condition in y on a set $D=\{(x,t):1\leq x\leq 2\,\text{and}\,-2\leq y\leq 5\}$ with $L=2$. That is

$$\begin{aligned}\|f(x,y_1)-f(x,y_2)\| &= \left\|\left(\frac{2}{x}y_1+x^2\mathrm{e}^x\right)-\left(\frac{2}{x}y_2+x^2\mathrm{e}^x\right)\right\|,\\ &= \left\|\frac{2}{x}\right\|\|y_1-y_2\|,\\ &\leq 2\,\|y_1-y_2\|,\end{aligned}$$

where $L=2$.

Theorem 1.1 *Refer to Butcher (2008b)*
Suppose that $D=\{(x,y):a\leq x\leq b\,\text{and}\,-\infty\leq y\leq\infty\}$ and that $f(x,y)$ is continuous on D. If $f(x,y)$ satisfies Lipschitz condition on D in its second variable y, then an IVP, say

$$y'=f(t,y),\,a\leq t\leq b,\,y(\alpha)=\beta,$$

has a unique solution $y(t)$ for $a\leq t\leq b$.

1.2.1 Well-posedness of a Problem

Definition 1.2 *The IVP*

$$y'=f(x,y),\,a\leq x\leq b,\,y(\alpha)=\beta,$$

is said to be a well-posed problem if:

1. *a unique solution $y(x)$ to the problem exists;*

2. *$\exists$ constants $\varepsilon_0>0$ and $k>0$ such that for any ε with $\varepsilon_0>\varepsilon>0$, whenever $\delta(x)$ is continuous with $|\delta(x)|<\varepsilon\ \forall\ x\in[a,b]$, & when $|\delta_0|<\varepsilon$, the IVP (a perturbed problem associated with the original IVP)*

$$z'=f(x,z)+\delta(x),\,a\leq x\leq b,\,z(\alpha)=\beta+\delta_0,$$

has a unique solution $z(x)$ that satisfies

$$|z(x)-y(x)|<k\varepsilon, \forall x \in [a, b] .$$

1.2.2 Ill-posed problem

The IVP above is said to be an ill-posed problem if it fails any of the well-posed conditions stated above.

1.3 Runge-Kutta Method

An s-stage Runge-Kutta (RK) method is given by

$$\begin{aligned} y_{n+1} &= y_n + h\left(\mathbf{b}^T \otimes \mathbf{I}\right) f(x_n + \mathbf{c}h, \mathbf{Y}), \\ \mathbf{Y} &= y_n + h(\mathbf{A} \otimes \mathbf{I}) f(x_n + \mathbf{c}h, \mathbf{Y}), \end{aligned} \tag{1.3}$$

where $\mathbf{A} = \left[a_{i,j}\right]$, $\mathbf{b} = [b_1, ..., b_s]^T$, $\mathbf{c} = [c_1, ..., c_s]^T$, $\mathbf{Y} = [Y_1, ..., Y_s]^T$ are all real and $\mathbf{I}$ is a real $s \times s$ identity matrix. The table below summarizes the RK coefficients.

$$\begin{array}{c|c} \mathbf{c} & \mathbf{A} \\ \hline & \mathbf{b} \end{array}$$

The method was originally developed towards the end of nineteenth century by Runge and generalized in the twentieth century by Kutta, see Dormand (1996). It is originally a one-step method with multiple stages, that is, it requires only the initial value of a solution to start numerical integration and depends only on one previously computed solution point subsequently, but with multiple functions evaluation per step. It is easier to implement than Taylor method, because there is no formation and evaluation of higher derivatives at each step.

1.3.1 Order Conditions of RK Method

Like many numerical schemes, order conditions determine order of convergence of RK method. It is a relation that exists between the coefficients of RK method that causes annihilation of terms in Taylor series of local truncation error of the method. Local truncation error here means difference between true solution $y(x_{n+1})$ at a point $n+1$ and the approximate solution y_{n+1} obtained by the RK method at the same point $n+1$. This approach for order conditions of RK method remained famous despite its limitations until in the 1960s when a major modern development hit the RK processes by the work of Butcher (2008b), where a more theoretical approach of the famous B-series was introduced to derive order conditions of the RK methods. Presented in Table 1.1 is a set of order conditions of RK methods up to order five.

Table 1.1: Algebraic Order Conditions of RK Method

Order	condition
1	$\sum_i b_i = 1$
2	$\sum_i b_i c_i = \frac{1}{2}$
3	$\sum_i b_i c_i^2 = \frac{1}{6}$
	$\sum_{i,j} b_i a_{i,j} c_j = \frac{1}{6}$
4	$\sum_i b_i c_i^3 = \frac{1}{4}$
	$\sum_{i,j} b_i c_i a_{i,j} c_j = \frac{1}{8}$
	$\sum_{i,j} b_i a_{i,j} c_j^2 = \frac{1}{12}$
	$\sum_{i,j,k} b_i a_{i,j} a_{j,k} c_k = \frac{1}{24}$
5	$\sum_i b_i c_i^4 = \frac{1}{5}$
	$\sum_{i,j} b_i c_i^2 a_{i,j} c_j = \frac{1}{10}$
	$\sum_{i,j,k} b_i a_{i,j} c_j a_{i,k} c_k = \frac{1}{20}$
	$\sum_{i,j} b_i c_i a_{i,j} c_j^2 = \frac{1}{15}$
	$\sum_{i,j} b_i a_{i,j} c_j^3 = \frac{1}{20}$
	$\sum_{i,j,k} b_i c_i a_{i,j} a_{j,k} c_k = \frac{1}{30}$
	$\sum_{i,j,k} b_i a_{i,j} c_j a_{j,k} c_k = \frac{1}{40}$
	$\sum_{i,j,k} b_i a_{i,j} a_{j,k} c_k^2 = \frac{1}{60}$
	$\sum_{i,j,k,l} b_i a_{i,j} a_{j,k} a_{k,l} c_l = \frac{1}{120}$

where

$$\sum_{j=1}^{s} a_{i,j} = c_i.$$

1.4 Hybrid Method for solving Special Second Order ODEs

An s-stage hybrid method for directly solving special second order IVP, denoted by THM, is given by

$$\begin{aligned} y_{n+2} &= y_{n+1} - y_n + h^2 \left(\mathbf{b}^T \otimes \mathbf{I}\right) f(x_n + \mathbf{c}h, \mathbf{Y}), \\ \mathbf{Y} &= (\mathbf{c}+\mathbf{e})\, y_{n+1} - \mathbf{c}y_n + h^2 (\mathbf{A} \otimes \mathbf{I})\, f(x_n + \mathbf{c}h, \mathbf{Y}), \end{aligned} \tag{1.4}$$

where $\mathbf{A} = \left[a_{i,j}\right]$, $\mathbf{b} = [b_1, ..., b_s]^T$, $\mathbf{c} = [c_1, ..., c_s]^T$, $\mathbf{e} = [1, ..., 1]^T$, $\mathbf{Y} = [Y_1, ..., Y_s]^T$ are real and $\mathbf{I}$ is a real $s \times s$ identity matrix. Table 1.2 is a modified Butcher tableau that summarizes the scheme This scheme is a two-step method that directly approximate the solution of special second order IVPs. Unlike Runge-Kutta-Nyström (RKN) method, the integration is independent of approximation of derivative of the solution. It can be seen as RKN method that forsakes its one-step property for improved accuracy and efficiency. The major development of this method is due to the work of Coleman (2003), where he used B-series approach to study order of convergence of the method. Hence, algebraic order conditions analogous to those of RK and RKN methods are presented.

Table 1.2: Coefficients of Hybrid Method for Second Order ODEs

$\mathbf{c}$	$\mathbf{A}$				
-1	0	0	0	$\cdots$	0
0	0	0	0	$\cdots$	0
c_3	$a_{3,1}$	$a_{3,2}$	$a_{3,3}$	$\cdots$	$a_{3,s}$
c_4	$a_{4,1}$	$a_{4,2}$	$a_{4,3}$	$\cdots$	$a_{4,s}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
c_s	$a_{s,1}$	$a_{s,2}$	$a_{s,3}$	$\cdots$	$a_{s,s}$
$\mathbf{b}$	b_1	b_2	b_3	$\cdots$	b_s

This major development prepared the ground for Franco (2006) to derive a class of explicit methods, where the advantages of the methods over RK and RKN are brought to limelight. He also presented analysis of dispersion and dissipation errors, which are the two most important errors to be minimized for any method whose aim is to integrate oscillatory problems.

1.4.1 Order Conditions of Hybrid Method for Second Order ODEs

As previously mentioned above in this section, B-series technique replaces the traditional ad hoc Taylor series technique in the derivation of order conditions of this method. The idea is rather a straightforward one where derivatives of solution of the problem in question, that is, $y'' = f(x,y)$, are associated with rooted trees t.

The following equations were developed and used by Coleman (2003) to generate order conditions of the method:

$$\mathbf{b}^T \psi''(t) = 1 + (-1)^{\rho(t)}, \tag{1.5}$$

$$\psi''_j = \rho(t)(\rho(t)-1)\prod_{i=1}^{m}\psi_j(t_i), \tag{1.6}$$

$$\psi(t) = (-1)^{\rho(t)+1}\mathbf{c} + \psi''(t)\mathbf{A}, \tag{1.7}$$

where t and $\rho(t)$ are the trees and their orders respectively. Table 1.3 shows set of the order conditions generated up to order at least five.

1.4.2 Simplifying Assumption

Simplifying assumptions, as the name connotes, are meant to induce certain relationships between order conditions of a numerical method so that number of independent conditions for a given order are reduced for simplicity of derivation. The simplifying

Table 1.3: Order Conditions of Hybrid Method for Second Order ODEs

$\rho(t)$	Order condition
2	$\sum b_i = 1$
3	$\sum b_i c_i = 0$
4	$\sum b_i c_i^2 = \frac{1}{6}$
	$\sum b_i a_{i,j} = \frac{1}{12}$
5	$\sum b_i c_i^3 = 0$
	$\sum b_i c_i a_{i,j} = \frac{1}{12}$
	$\sum b_i a_{i,j} c_j = 0$
6	$\sum b_i c_i^4 = \frac{1}{15}$
	$\sum b_i c_i^2 a_{i,j} = \frac{1}{30}$
	$\sum b_i c_i a_{i,j} c_j = -\frac{1}{60}$
	$\sum b_i a_{i,j} a_{i,k} = \frac{7}{120}$
	$\sum b_i a_{i,j} c_j^2 = \frac{1}{180}$
	$\sum b_i a_{i,j} a_{j,k} = \frac{1}{360}$
7	$\sum b_i c_i^5 = \frac{1}{15}$
	$\sum b_i c_i^3 a_{i,j} = \frac{1}{30}$
	$\sum b_i c_i^2 a_{i,j} c_j = 0$
	$\sum b_i c_i a_{i,j} a_{i,k} = \frac{1}{30}$
	$\sum b_i c_i a_{i,j} c_j^2 = \frac{1}{72}$
	$\sum b_i c_i a_{i,j} a_{j,k} = -\frac{1}{720}$
	$\sum b_i a_{i,j} a_{i,k} c_k = -\frac{1}{120}$
	$\sum b_i a_{i,j} c_j^3 = 0$
	$\sum b_i a_{i,j} c_j a_{j,k} = -\frac{1}{360}$
	$\sum b_i a_{i,j} a_{j,k} c_k = 0$

equations associated with the order conditions above are:

$$\sum_{j=1}^{s} a_{i,j} c_j^{\lambda} = \frac{c_i^{\lambda+2} + (-1)^{\lambda} c_i}{(\lambda+1)(\lambda+2)}. \quad (1.8)$$

1.5 An Overview of B-series and Rooted Trees

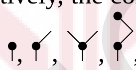
The so called B-series approach to algebraic order conditions of numerical methods is a theoretical formulation of the Taylor series approach where terms of the series of local truncation error are analyzed to formulate theorems that lead to derivation of order conditions of a method irrespective of the order of the method. That is, with this approach, order conditions including those of higher order methods can be derived without having to employ the services of computers and without any form of ambiguity and difficulty.

The terms of the series of the local truncation error contain a combination of derivatives of the true solution. This combination of derivatives are then associated to rooted tree using the concept of graph theory. Hence, with the theorems on ground, only the trees associated to the ODE in question are required to generate the order conditions of a method that solves the ODE.

1.5.1 Rooted Trees

Rooted tree is a simple combinatorial graph with the property of being connected, having no cycles and having a specific vertex designated as root, see Butcher (2008a). Example, suppose that $y'' = f(y) = f$ is differentiated continuously with respect to x we get

$$
\begin{aligned}
y'' &= f, \\
y''' &= f_y\left(y'\right), \\
y^{iv} &= f_{yy}\left(y', y'\right) + f_y f, \\
&\vdots
\end{aligned}
$$

If . and • denote y' and f respectively, the corresponding rooted trees of the derivatives above are respectively given as  ,..., where the two dots are the vertices of the trees.

1.5.2 Order of a Tree

Order of a tree simply refers to the number of vertices possessed by the tree. For instance, two, three, four are the orders of the trees depicted above respectively.

1.5.3 Concept of B-series

Definition 1.3 *Refer to Coleman (2003)*
Let β be a mapping from T_N to set of real numbers IR, *with $\beta(\theta) = 1$. The B_N-series with coefficient function β is a formal series of the form*

$$B(\beta, y) = y + h\alpha(\tau_1)\beta(\tau_1)y' + \ldots = \sum_{t \in T_N} \frac{h^{\rho(t)}}{\rho(t)!}\alpha(t)\beta(t)F(t)\left(y, y', \ldots, y^{N-1}\right).$$

T_N, θ and τ_1 are set of trees associated with ODE of order N, null tree and a tree of order one.

1.6 Problem Statement

A class of RKN methods has proven so efficient for special second order ODEs due to its multistage nature until the emergence of a class of hybrid method (THM) for solving the special second order ODEs by Coleman (2003) and Franco (2006). This class of methods possesses virtually all the properties of the class of RKN methods that makes it more accurate and efficient except one-step property, which is forfeited for more accuracy and efficiency. The idea of RKN was extended recently by You and Chen (2013) and Kasim et al. (2016), where Runge-Kutta type methods, denoted by RKT and RKF, are presented for solving special third and fourth order ODEs in the same fashion as RKN methods. This development triggers the quest to come up with similar hybrid methods like the THM methods for solving special third and fourth order ODEs, which is now a topical research issue within the ranks of researchers in the area of numerical methods for ODEs.

1.6.1 Motivation

The study is mainly motivated by relatively low efficiency of the RKT and RKF methods characterized by dependance of their stages (internal and external) on the derivatives of the solution and excessive memory requirement for implementation of the methods. B-series technique for their order conditions instead of the traditional Taylor series approach is another motivating factor.

1.7 Objectives of the Study

The objectives set to be achieved in this study are:

1. to derive algebraic order conditions of a class of hybrid methods for special third and fourth order ODEs directly (HMTD and HMFD) using B-series approach;
2. to derive and implement a class of explicit HMTD and HMFD using the algebraic order conditions derived in 1;
3. to analyze absolute stability properties of the HMTD and HMFD, and their convergence properties via consistency and zero-stability.
4. to derive and implement variable step size two-step hybrid methods for solving special second order ODEs;

1.8 Scope and Limitation of the Study

The study covers only initial value problems that are based on special second, third and fourth order ODEs. The specialty associated with these problems is nothing more than the independence of the problems on y', y'' and y''' explicitly.

The general form of the problems is out of scope of this study, because the proposed methods possess the non-linear property of RK related methods, which would be computationally inefficient when derived for general case of the ODEs. Boundary value problems based on both general and special case of the ODEs are also not within the scope of the thesis.

1.9 Organization of the Thesis

In Chapter 1, we present background of the study as it relates to the existing numerical methods. Statement of the problem addressed in the study as well as the factors that necessitated the study are presented. In addition, objectives and scope defined for the study are presented.

In Chapter 2, the reviews on higher order ODEs, linear multistep and collocation methods for solving third and fourth order ODEs, Runge-Kutta methods for solving third and fourth order ODEs and hybrid methods for solving second order ODEs are presented.

In Chapter 3, construction of a class of hybrid methods for special third order IVPs is presented and analyzed. B-series technique for the derivation of their order conditions is formulated and discussed. Convergence and absolute stability analysis of the methods are also presented here.

In Chapter 4, derivation of explicit methods of the class above is presented. Application of the methods on thin film flow problems is also presented. Numerical experiment is presented to assess the validity and performance of the new methods.

In Chapter 5, construction of a class of hybrid methods for special fourth order IVPs is presented and analyzed. B-series technique for the derivation of their order conditions is formulated and discussed. Convergence and absolute stability analysis of the methods are also presented here.

In Chapter 6, derivation of some explicit methods of the above class is presented. Numerical experiment is conducted to evaluate the validity and performance of the new fourth order IVPs integrators.

A class of embedded pairs of two-step hybrid methods for solving special second order ODEs is presented in Chapter 7, where 5(3), 6(4) and a trigonometricaly fitted 6(4) methods are derived for solving oscillatory or periodic problems.

Finally, general conclusion of the thesis alongside future work is presented in Chapter 8.

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