UNIVERSITI PUTRA MALAYSIA

ON SOME PACKING AND PARTITION PROBLEMS IN GEOMETRIC GRAPHS

HAZIM MICHMAN TRAO

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ON SOME PACKING AND PARTITION PROBLEMS IN GEOMETRIC GRAPHS

By

HAZIM MICHMAN TRAO

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Doctor of Philosophy

July 2018

DEDICATION

To the memory of my late mother, for her love and support throughout the years of her life.

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Doctor of Philosophy

ON SOME PACKING AND PARTITION PROBLEMS IN GEOMETRIC GRAPHS

By

HAZIM MICHMAN TRAO

July 2018

Chairman : Professor Adem Kilicman, PhD
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Graph packing problem refers to the problem of finding maximum number of edge-disjoint copies of a fixed subgraph in a given graph G. A related problem is the partition problem. In this case, the edge-disjoint subgraphs are sought but require that the union of subgraphs in this packing is exactly G. It is often useful to restrict the subgraphs of G to a certain graph or property.

In this thesis, packing and partition problems are studied for different properties such as matching, tree and cycle, and these problems are considered in various geometric graphs. The families of geometric graphs that are investigated include the complete geometric graphs, triangle-free geometric graphs, and complete bipartite geometric graphs.

First, the problem of partitioning the complete geometric graph into plane spanning trees is investigated, giving sufficient conditions, which generalize the convex case and the wheel configuration case.

The problem of packing plane perfect matchings is studied in triangle-free geometric graphs where we establish lower and upper bounds for the problem. An algorithm for computing such plane perfect matchings is also presented. Moreover, a sufficient condition is provided for the existence the set of edge-disjoint plane perfect matchings whose union is a maximal triangle-free geometric graph.

Furthermore, the problem of partitioning the complete bipartite geometric graphs into plane perfect matchings is studied and sufficient conditions for the problem are presented.

Finally, the problem of packing 1-plane Hamiltonian cycles into the complete geometric graph is studied in order to establish lower and upper bounds for the problem. An algorithm for computing such 1-plane Hamiltonian cycles is also presented.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia
sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

MENGENAI BEBERAPA MASALAH PEMBUNGKUSAN DAN PEMBAHAGIAN DALAM GRAF GEOMETRIK

By

HAZIM MICHMAN TRAO

Julai 2018

Pengerusi : Profesor Adem Kilicman, PhD
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Masalah pembungkusan graf merujuk kepada masalah mencari bilangan maksimum salinan pinggir-sisihan pada sub graf tetap dalam graf G yang diberikan. Masalah yang berkaitan ialah masalah pembahagian. Dalam kes ini, sub graf pinggir-sisihan dicari tetapi penyatuan sub graf dalam pembungkusan ini perlu tepat dengan G. Adalah sering berguna untuk menyekat sub graf G kepada tertentu graf atau sifat tertentu.

Dalam tesis ini, masalah pembungkusan dan pembahagian dikaji untuk sifat yang berbeza seperti padanan, pokok dan kitaran, dan masalah ini dipertimbangkan dalam pelbagai graf geometrik. Keluarga graf geometrik yang diselidiki termasuk graf geometrik yang lengkap, graf geometrik bebas segitiga, dan graf geometrik bipartit lengkap.

Pertama, masalah membahagikan graf geometrik lengkap kepada satah yang merangkumi pokok disiasat dan dengan memberikan keadaan yang mencukupi akan mengitlakkan kes cembung dan kes konfigurasi roda.

Masalah padanan sempurna pembungkusan satah telah dikaji dalam graf geometrik bebas segitiga dengan batas bawah dan atas untuk masalah ini ditetapkan. Satu algoritma untuk mengira padanan sempurna satah sebegini juga dibentangkan. Selain itu, keadan yang mencukupi disediakan untuk memperihalkan kewujudan set padanan sempurna satah pinggir-sisihan, yang penyatuannya merupakan graf geometrik bebas segitiga maksimum.

Tambahan pula, masalah pembahagian graf geometrik bipartit lengkap kepada padanan satah sempurna telah dikaji dan keadaan yang mencukupi untuk masalah tersebut dibentangkan.

Akhirnya, masalah pembungkusan kitaran Hamiltonian 1-satah ke dalam graf geometrik lengkap juga telah dikaji untuk menghasilkan batas bawah dan atas untuk masalah ini. Satu algoritma untuk mengira kitaran Hamiltonian 1-satah ini juga dibentangkan.

ACKNOWLEDGEMENTS

I wish to express my sincere gratitude to my supervisor, Professor Dr. Adem Kilicman, for his guidance and constructive criticism, throughout the research the preparation of this thesis. This helped me to shape the person I am today both academically and personally.

Of course, none of this would have been possible without my associate supervisor, Professor Dr. Chia Gek Ling, who was patient and gentle with me throughout the period of this thesis. He has dedicated his valuable time to show me the nature of graph theory. I am truly grateful for his involvement in my work.

I would also like to relay my gratitude to the rest of my committee members, Associate Professor Dr. Mohamad Rushdan Md Said and Associate Professor Dr. Siti Hasana Bt Sapar, for their co-operations.

My thanks are also extended to the Universiti Putra Malaysia for practical support.

During the course of this long journey, I cannot imagine a better company than that of Niran Abbas, who has always given me solid support in various aspects of my life. She is not only my wife and companion but also a colleague who has offered me valuable many advice for my thesis throughout.

Last but not least, I would like to thank my lovely daughters, Lujain and Hawraa, who brings warmth and peace to my heart, thus giving me spiritual peace.

I certify that a Thesis Examination Committee has met on 4 July 2018 to conduct the final examination of Hazim Michman Trao on his thesis entitled "On Some Packing and Partition Problems in Geometric Graphs" in accordance with the Universities and University Colleges Act 1971 and the Constitution of the Universiti Putra Malaysia [P.U.(A) 106] 15 March 1998. The Committee recommends that the student be awarded the Doctor of Philosophy.

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This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Doctor of Philosophy. The members of the Supervisory Committee were as follows:

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Declaration by graduate student

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LIST OF ABBREVIATIONS

$\lfloor x \rfloor$	The largest integer not greater than x.
$\lceil x \rceil$	The smallest integer not less than x.
$G = (V, E)$	The graph G with vertex set V and edge set E.
$V(G)$	The vertex set of a graph G.
$E(G)$	The edge set of a graph G.
$\|V\|$	The order of a set V.
$\|E\|$	The size of a set E.
$d(v)$	The degree of a vertex v in G.
K_n	The complete graph of n vertices.
$K_{n,m}$	The complete bipartite graph on two sets of order n and m.
P_n	The path of length n.
C_n	The cycle of length n.
$N(v)$	The set of vertices adjacent to the vertex v.
$CH(P)$	The convex hull of a point set P.
1-PHC	The one plane Hamiltonian cycle.
1-PHP	The one plane Hamiltonian path.

CHAPTER 1

INTRODUCTION

Recently, two related types of graph theoretical problems, packing and partition problems, have garnered a lot of interest among those in the field of graph theory. These two problems offer important applications in various distinct areas of science (e.g. mathematics, physics and computer science). In this thesis, these problems are considered in various types of geometric graphs.

This chapter includes necessary basic definitions, followed by a scop of our study, a description of problem statements, research objectives, and finally, a review of the obtained results.

1.1 Definitions and Notation

In order to understand the terminology of the packing and partition problems, the notation and definitions which will be used throughout this thesis are first stated. For more graph theoretic terminology, the work of Chartrand and Lesniak (2005) may be consulted.

A *graph* G is a finite, non-empty set $V(G)$ together with a (possibly empty) set $E(G)$ (disjoint from $V(G)$) of two-element subsets of disjoint elements of $V(G)$. If the vertex pairs of the edge set are ordered, the graph is referred to as *directed* and otherwise *undirected*. The terms *order* and *size* are used to refer to the number of elements in the vertex and edge set, respectively. A graph may be naturally represented in the plane as points with simple curves among them.

Two vertices u and v within V are said to be *adjacent*, if $uv \in E$, and the degree of a vertex v is the number of adjacent vertices of v. In the case that all vertices of G have degree d, G is called a *d-regular* graph or a regular graph of degree d.

Two edges in a graph are said to cross each other if they have a common point which is interior to both of them. A graph is said to be *non-crossing* or *plane* if its edges do not cross each other.

Given a graph $G = (V, E)$, a graph $G' = (V', E')$ is termed a *subgraph* of G if $V' \subseteq V$ and $E' \subseteq E$. A subgraph G' is an *induced subgraph* of G if every edge of G contained in V' belongs to E'. A subgraph is termed *spanning* if $V' = V$. Two subgraphs H_1 and H_2 of a graph G are said to be *edge-disjoint* if $E(H_1) \cap E(H_2) = \emptyset$.

A length represents a number of edges. A *path* of a certain length n in a graph, denoted by P_n, is an ordered list of distinct vertices $v_0, v_1, \ldots, v_n$ such that $v_{i-1}v_i$ is an edge for all $1 \leq i \leq n$. The first and last vertices of a path are its endpoints, while the rest are interior points. Likewise, a *cycle* of length n in a graph, denoted by C_n, is an ordered list of distinct vertices $v_1, v_2, \ldots, v_n$ such that $v_{i-1}v_i$, $2 \leq i \leq n$, and $v_n v_1$ are edges.

In the case that G contains an induced subgraph isomorphic to a graph H, then G is said to contain H. The graph G is said to be H-free if it does not contain H. In particular, a graph is *triangle-free* if it does not contain a cycle of length three.

In extremal graph theory, many problems stated in the form: For a graph on n vertices satisfying a certain property P, how many edges have? A special case if P is H-free property for some fixed graph H, Turán solved the following problem:

Theorem 1.1.1 *(Turán 1941) (Felsner, 2004)*
The number of edges of a K_{m+1}-free graph on n vertices is at most $(1 - \frac{1}{m})\frac{n^2}{2}$.

A wheel W is a graph with n vertices, obtained from a cycle C_{n-1} whose vertices are joined to an additional vertex by edges.

A *tree* is a connected graph with no cycle. Let T denote a tree with m vertices. It is easy to see that for any two vertices in T, there is a unique path joining these two vertices. If T has only two vertices of degree 1, then T is called an *m-path*. Let T' be the tree obtained from T by deleting all vertices of degree 1. If T' is a path, then T is called a *caterpillar*. A *double star* is a caterpillar in which T' is a 2-path.

A *matching* in a graph is a set of edges without common vertices. A *perfect matching* is a matching in which every vertex of the graph is incident to exactly one edge of the matching. In other words, perfect matchings are only possible on graphs with an even number of vertices.

A *k-factor* is a regular graph of degree k. Since a perfect matching is a regular graph of degree one, it is a *1-factor*.

A graph G_1 is *isomorphic* to a graph G_2, denoted $G1 \cong G2$, if there exists a one-to-one mapping referred to as *isomorphism*, denoted by ϕ, from $V(G_1)$ onto $V(G_2)$ such that $uv \in E(G_1)$ if and only if $\phi(u)\phi(v) \in E(G_2)$.

The *union* of graphs G and H, written $G \cup H$, consist of the vertex set $V(G) \cup V(H)$ and the edge set $E(G) \cup E(H)$. In order to specify the edge-disjoint union with $E(G) \cap E(H) = \emptyset$.

For $n \geq 1$, the *complete graph*, denoted by K_n, consists of n vertices with all possible edges. Thus, K_n contains $n(n-1)/2$ edges; consequently, every pair of vertices form an edge.

A *bipartite graph* is a graph whose vertices can be divided into two disjoint sets, A and B, such that every edge connects a vertex in A to one in B. A *complete bipartite graph* is a special kind of bipartite graph where every vertex of A is connected to every vertex of B.

A point set P in the plan is said to be in *general position* if no three points are collinear. Through out this thesis, all point sets will be considered to be in the general position.

The *convex hull* of P is a smallest convex set that contains all points of P and is denoted by $CH(P)$. A set P is said to be in the *convex position* in the plane if all points are on the boundary of $CH(P)$. A *convex geometric graph* is a graph with the vertices in convex position.

A set P of m points is said to be in *regular wheel configuration* in the plane, if $m-1$ of its points are regularly spaced on a circle C and a point at the center of C. It is to be noted that those points in C are the convex hull of P and the ordering of the points of C around the convex hull.

The *double chain* of P of $2m$ points consists of two sets of m points, one forming a convex chain (the lower chain) and another forming a concave chain (the upper chain). Furthermore, each straight-line, defined by two points from the upper chain, leaves all the points from the lower chain on the same side, and vice versa.

An interesting topic of study within combinatorial geometry is halving lines, which plays an important role in the analysis of many algorithms in combinatorial geometry (Edelsbrunner, 1987).

A *halving line* of a set P of $2m$ points in general position in the plane, is a line passing through two points in P, and cutting the remaining set of $2m-2$ points in half, that is, leaving $m-1$ points of P on each side.

Given an arbitrary line passing through any point of P, it can be rotated by at most 180 degrees where it hits some other point of P which makes it a halving line. Thus, the minimum number of the halving lines is m while the maximum number is unknown.

On other hand, a line L is said to *bisect* a set P of m points in general position in the plane, if both open half spaces defined by L contain precisely $\frac{m}{2}$ points. To be more specific, L does not passes through any point of P. There is no loss of generality to assume m is odd, since otherwise, any point v may be removed and any line that bisects $P - \{v\}$ also bisects P.

In the case that P is a disjoint union of two point sets P_1 and P_2, the ham-sandwich cut theorem guarantees the existence of a line that simultaneously bisects P_1 and P_2 (Edelsbrunner, 1987) and (Lo et al., 1994).

Theorem 1.1.2 *(Ham Sandwich Theorem) (Lo et al., 1994)*
For a point set P in general position in the plane that is partitioned into sets R and B, there exists a line that simultaneously bisects R and B.

1.2 Scope of Study

This study particularly focuses on the counting of edge-disjoint plane subgraphs in graphs, which is known as packing or partition problem. This problem is one of the problems in combinatorial computational geometry and which is particularly studied in geometric graphs. So a brief background of combinatorial computational geometry and geometric graphs are introduced in the next two sections.

1.2.1 Combinatorial Computational Geometry

The origins of the term "computational geometry" are disputed among researchers, but many computing scientists point to the doctoral thesis of (Shamos, 1978) or his earlier paper on geometric complexity (Shamos, 1975). Yet, there are those who dispute that the term was concocted ten years earlier with the doctoral thesis of (Forrest, 1968) or his later paper on computational geometry (Forrest, 1971).

There are even, others who claim that it began with the formal investigation of (Minsky and Papert, 1969) in which geometric properties of a figure can and cannot be computed with a variety of computational neural network models available at the time.

Berg reported that the birth of computational geometry, as a discipline, was credited to the discipline of algorithms design and analysis in the late 1970s (Berg et al., 2008). Since then, the field has garnered recognition as a stand-alone discipline supported by a large community of avid researchers, journals and conferences. The emergence of computational geometry as a research discipline stems from two main reasons. Firstly, from the aesthetic nature of related problems and solutions, and secondly, from the many applications which include computer graphics, integrated circuit design (ICD), computer vision and image processing, robotics, and other applications rooted in geometric algorithms.

One of the main branches of computational geometry is "combinatorial computational geometry", which treats geometric objects as discrete entities. A vital book on the topic was written by (Preparata and Shamos, 1985), who dates the first usage of the term with combinatorial by 1975. Ever since, this phrase has been used to refer to algorithmic study on discrete and combinatorial geometric structures and can also be regarded as the geometric version of "Algorithm Design and Analysis".

Geometric settings, such as networks of roads or wires in the plane, along with restrictions in computational graph theory were of the both two problems influential in the development of computational geometry. This field in modern times is considered the basis of robotics, geographic information systems (GIS) and computer-aided design and manufacturing (CAD/CAM) among others.

This thesis is devoted to studying one of the fundamental questions in combinatorial computational geometry. This question involves the computing of a number of non-crossing configurations in the plane realized with straight line segments and connected by pairs of points from a finite ground set. Graph classes of interest realized in this way include spanning trees, perfect matchings and spanning cycles.

1.2.2 Geometric Graphs

Geometric applications often lead to questions which may be unable to be answered by traditional graph theory. Thus, there is a demand for new techniques which could appropriately answer such demands.

In this section, the definitions surrounding the geometric properties of graphs are given. A *geometric graph* is a graph whose vertex set is a set of points in general position in the plane (i.e., no three points lie on a common line) and whose edge set contains straight-line segments (possibly crossing) connecting the corresponding points. A geometric graph is *plane* or *non-crossing* if its edges do not cross each other. It is to be noted that the abstract graph corresponding to a plane geometric graph is always planar, but some geometric realizations of a planar graph may have crossing edges. A geometric graph is said to be *1-plane* if every edge has at most one crossing.

Upon the realization that many classical questions in extremal graph theory have natural analogues (Bollobás, 1978), researchers attempted to initiate the systematic study of geometric graphs. Among the first researchers to do this were (Avital and Hanani, 1966), (Kupitz, 1979) and (Alon and Erdos, 1989).

There are many applications of geometric graphs from circuit design, mapping, wireless networks (Frieze et al., 2009), the web graph (Aiello et al., 2009; Fu et al., 2010), and on-line social networks (Bonato et al., 2012). Within these stochastic models and a feasible chosen metric space, the network vertices are characterised by points. On the other hand, edges are chosen by a mixture of relative proximity of the vertices and probabilistic rules.

Edge crossings are generally undesirable in most applications of geometric graphs. By definition or by nature, numerous structures which are the topic of study for computational geometry are non-crossing. Such examples include minimum spanning trees, Delaunay triangulations or shortest-path of traveling salesman. Another example is of the shortest spanning tree and shortest matching on n given points in which the plane is automatically non-crossing and can be computed in polynomial time (Eppstein, 2000) and (Mitchell, 2000).

1.3 Problem Statements

One of the fundamental questions in combinatorial computational geometry is finding the maximum number of non-crossing geometric graphs on a set of points in the plane. Packing and partition problems have a rich history in graph theory, with many of the oldest and most intensely studied topics within this area (Pyber, 1992).

Beineke, in a survey, defined both the general packing and covering problem as stated in the following problems (Beineke, 1969):

Problem 1.3.1 *The packing problem is the problem of asking for the maximum number of graphs with a certain property and having as their union (being edge-disjoint subgraphs of) a given graph G.*

While, the covering problem involves finding the minimum number of graphs with a certain property whose union is a given graph G (Harary, 1970). A related problem is the partition problem,

Problem 1.3.2 *The partition problem is the packing problem such that the union of subgraphs is precisely a given graph G.*

In other words, a partition of the edge set of a graph is both a packing and a covering problem.

Graph packing and partition problems have been studied in many contexts, including from the algorithmic complexity perspective. In general, the problem of partitioning a given graph and the packing problem proved are both NP-complete in (Dor and Tarsi, 1997).

Radoiĉiĉ proved that detecting a non-crossing perfect matching and spanning tree in a given geometric graph is NP-complete (Radoiĉiĉ, 2004).

The packing problem may be considered in which it is to be determined whether a graph contains a spanning tree or a cycle that does not share edges. In this case, the decision problem proved is NP-complete (Bernáth and Király, 2015). They also established the NP-completeness of the partition problem of determining whether the edge set of a graph can be partitioned into a spanning tree.

1.4 Research Objectives

The contributions to this thesis are organized as four objectives below:

A well-known problem concerning geometric graphs asks whether every complete geometric graph on $2n$ vertices has a partition of its edge set into n plane spanning trees?

In the case that the given graph is a convex complete graph on $2n$ vertices, an affirmative answer to the above question follows from a result of (Bernhart and Kainen, 1979). For the case of geometric graphs, a characterization for the partitions of the complete convex graph into plane spanning trees is presented in (Bose et al., 2006). They presented a sufficient condition for partitioning the edge set of complete geometric graphs into plane spanning double stars. So our first objective is:

1. **(i)** To determine whether any complete geometric graph on $2n$ vertices has a partition of its edge set into n plane spanning caterpillars or non-caterpillar.

   (ii) To give a complete characterization for the partitions of the complete geometric graph in wheel configuration into plane spanning trees.

The problem of packing non-crossing perfect matchings into the complete geometric graph is considered in (Biniaz et al., 2015). They proved a lower bound for the problem and presented tight bounds for some special configurations of point sets. In the case that the maximum number of edge-disjoint non-crossing perfect matchings is packed into a set of an even number of points in convex position, the result is a maximal triangle-free geometric graph, this lead to the following objective:

2. To find the maximum number of edge-disjoint non-crossing perfect matchings on a set of $2n$ points in the plane such that their union is a triangle-free geometric graph.

For same above reasons, it is natural to consider the following objective:

3. To determine whether any complete bipartite geometric graph $K_{n,n}$ have a partition of its edge set into n plane perfect matchings.

Finding a Hamiltonian cycle in a graph is **NP**-complete even if the graph is known to be planar (Garey et al., 1976). Moreover, counting the number of Hamiltonian cycles that are contained in a graph is **#P**-complete even if the graph is known to be planar (Liśkiewicz et al., 2003).

Many past researchers have attempted the problem of counting the number of plane Hamiltonian cycles on a given graph, which may not necessarily be edge-disjoint (Motzkin, 1948), (Akl, 1979), (Dumitrescu, 1999), (Sharir and Welzl, 2006) and (Sharir et al., 2013).

It is not always possible to find more than one edge-disjoint plane Hamiltonian cycle for a given set of points. A relaxation is considered in order to solve it as a 1-plane, rather than a plane problem. Thus, the following objective is considered:

4. To find the number of 1-plane Hamiltonian cycles can be packed into a complete geometric graph K_n, for any set of n points in the plane.

1.5 Organization of Thesis

This thesis falls into seven chapters. Since a short introduction of the content is given at the beginning of each chapter, we shall give only a brief outline of the thesis.

In Chapter 1, the background of graph theory problems within the context of current study is given. This is followed by objectives set in order to address current packing and partition problems.

In Chapter 2, past studies which address packing and partition problems are presented. These studies are organized into appropriate sections, each addressing a sub-problem arising depending on the type of graph.

In Chapter 3, the first objective will be considered for a given set P of $2n$ points in the plane for the partitioning of complete geometric graphs into plane spanning trees. In the case that P is in general position, sufficient conditions are given to generalize two cases: the wheel configuration and the convex position. For the case that P is regular wheel configuration, all such partitions are characterized.

In Chapter 4, the second objective will be considered for the packing of edge-disjoint plane perfect matchings for triangle-free geometric graphs for a given set P of $2n$ points in the plane where $2n$ points, $n = 2^k + h$ and $0 \leq h < 2^k$. For the case P in general position, the problem is approached in two directions. The first direction shows that there exist at least k edge-disjoint non-crossing perfect matchings whose union is a triangle-free geometric graph. The second direction presents a sufficient condition for the existence of n edge-disjoint non-crossing perfect matchings whose union is a maximal triangle-free geometric graph. For the other two special positions for P in both convex position and regular wheel configuration, a tight upper bound will be given.

In Chapter 5, the third objective will be considered for partitioning complete bipartite geometric graphs on a given set P of $2n$ points in the plane into n non-crossing perfect matchings. The problem will be solved by considering three special cases of P: convex position, double chain position and general position. For each cases, sufficient conditions will be presented for a complete bipartite geometric graph having a partition into plane perfect matchings. For the obtained graphs, the constructions will be investigated.

In Chapter 6, the fourth and final problem will be considered for the packing of 1-plane Hamiltonian cycles into complete geometric graphs, on a given set P of n points in the plane. In the case that P is in general position, an algorithm will be presented for computing such 1-plane Hamiltonian cycles. Moreover, the lower bound, $k - 1$, will be proved, where $n = 2^k + h$ and $0 \leq h < 2^k$ makes use of the previous algorithm. On the other hand, for the other two special positions for P (convex position and regular wheel configuration), a tight upper bound will be given.

In Chapter 7, a summary of the results of the present thesis is presented with the recommendations for some open problems.

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