UNIVERSITI PUTRA MALAYSIA

G-ANGULABILITY OF CONVEX GEOMETRIC GRAPHS

NIRAN ABBAS ALI AL-HAKEEM

FS 2018 77

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By

NIRAN ABBAS ALI AL-HAKEEM

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Doctor of Philosophy

September 2018

DEDICATIONS

To Prophet Muhammad and Al-Imam Al-Mahdy peace be upon them.
To the memory of my late dear father
for his love, encouragement, and support over the years.
To my dear mother
for her sacrifice and great patience.

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Doctor of Philosophy

G-ANGULABILITY OF CONVEX GEOMETRIC GRAPHS

By

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September 2018

Chairman: Professor Adem Kilicman, PhD
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In this thesis, we consider the g-angulation existence problem of a convex geometric graph G. A *triangulation* on n points in convex position is a plane graph on the convex hull in which each face is a triangle except the exterior face. A *g-angulation* on n points in convex position is a plane graph in which each face is a g-cycle except the exterior face. In particular, the g-angulation is a triangulation if $g = 3$. We say that T_n is a triangulation of a graph $G(V,E)$ if $E(T_n) \subseteq E$. On a given graph G, deciding whether G has a triangulation or not is known as the *Triangulation Existence Problem*.

Since Triangulation Existence Problem is known to be an NP-complete problem, we consider the problem on a convex geometric graph G. In order to decide whether G admits a triangulation, we determine necessary and sufficient conditions on a subgraph F of complete convex graph K_n with $|E(F)| \leq n-1$ for which $G = K_n - F$ admits a triangulation. For $|E(F)| \geq n$, we investigate the possibility of placing F in K_n for certain families of graphs F such that G admits a triangulation. These results are then applied to determine the convex skewness of G. The *skewness* of a graph G, denoted $sk(G)$, is the minimum number of edges to be deleted from G such that the resulting graph is planar.

Finally, we extend the triangulation existence problem to the *g-angulation existence problem* for a convex geometric graph G. For any $g \geq 3$ we present a complete characterization of a subgraph F of K_n with $|E(F)| \leq n-1$ for which $G = K_n - F$ admits a g-angulation. For $|E(F)| \geq n$, we investigate the possibility of placing 2-regular graphs F in K_n such that G admits a g-angulation and the possibility of placing 3-regular graphs F in K_n such that G admits a 4-angulation.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

KAJIAN KEBOLEH G-ANGULASI GRAF GEOMETRIK CEMBUNG

Oleh

NIRAN ABBAS ALI AL-HAKEEM

September 2018

Pengerusi: Profesor Professor Adem Kilicman, PhD
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Tesis ini memberi perhatian kepada masalah kewujudan g-angulasi pada graf geometrik cembung G. Suatu triangulasi pada n titik dalam kedudukan cembung merupakan graf satah di badan cembung, yang setiap muka adalah segi tiga kecuali muka luar. Angulasi-g pada n titik dalam kedudukan cembung merupakan graf satah yang setiap muka merupakan kitaran g, kecuali muka luar. Khususnya, g-angulasi merupakan triangulasi jika $g = 3$. T_n boleh dinyatakan sebagai triangulasi graf $G(V,E)$ jika $E(T_n) \subseteq E$. Pada suatu graf G yang diberikan, untuk memutuskan sama ada G mempunyai triangulasi atau tidak dikenali sebagai Masalah Kewujudan Triangulasi.

Oleh kerana Masalah Kewujudan Triangulasi diketahui sebagai masalah NP, masalah tersebut dipertimbangkan pada graf geometrik cembung G. Untuk meventukan sama ada G mempunyai triangulasi atan tidak, syorat-syarat perlu dan mencukupi telah ditentukan pada subgraf F dari graf cembung lengkap K_n dengan $|E(F)| \leq n-1$, dan $G = K_n - F$ merupakan triangulasi. Untuk $|E(F)| \geq n$, kemungkinan meletak F di dalam K_n bagi keluarga graf F supaya G merupakan triangulasi disiasat. Keputusan ini kemudiannya digunakan untuk menentukan *skewness* cembung G. *Skewness* sesuatu graf G, yang dilambangkan sebagai $sk(G)$, merupakan bilangan minimum tepi yang perlu dipadamkan dari G supaya graf yang dihasilkan adalah satah.

Akhirnya, masalah kewujudan triangulasi dilanjutkan kepada masalah kewujudan g-angulasi untuk graf geometrik cembung G. Untuk mana-mana nilai $g \geq 3$, pencirian lengkap telah dibentangkan untuk subgraf F bagi K_n dengan $|E(F)| \leq n-1$, yang mana $G = K_n - F$ merupakan g-angulasi. Untuk $|E(F)| \geq n$, kemungkinan meletakkan 2 graf biasa F di dalam K_n supaya G merupakan g-angulasi dan kemungkinan meletakkan 3 graf biasa F di dalam K_n supaya G merupakan 4-angulasi juga telah disiasat.

ACKNOWLEDGEMENTS

First and foremost, all praise to the almighty ALLAH for His blessings and mercy that enable me to learn.

I wish to express my most sincere appreciation to my supervisor, Professor Dr. Adem Kilicman for his excellent guidance, sharing of experience, valuable support and constant encouragements.

My profound gratitude to my co-supervisor Professor Dr. Chia Gek Ling, who I am proud to be one of his students. I am sincerely grateful to him, for his positive attitude, generous sharing of knowledge, and invaluable advices. I will be always indebted to him, for making sure that I stayed on track when the crunch was on.

My thanks also extended to the members of a supervisory committee Associate Professor Dr. Mohamad Rushdan Md Said and Associate Professor Dr. Siti Hasana Sapar and to all those who have taught me mathematics.

I cannot end without thanking my family. My thanks to my dear husband, and to my lovely daughters, Hawraa and Lujain, who had to get less mama than they needed during this long journey.

I certify that a Thesis Examination Committee has met on 26 September 2018 to conduct the final examination of Niran Abbas Ali on her thesis entitled "G-Angulability of Convex Geometric Graphs" in accordance with the Universities and University Colleges Act 1971 and the Constitution of the Universiti Putra Malaysia [P.U.(A) 106] 15 March 1998. The Committee recommends that the student be awarded the Doctor of Philosophy.

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LIST OF ABBREVIATIONS

$G=(V,E)$	The graph G with vertex set V and edge set E.
$V(G)$	The vertex set of a graph G.
$E(G)$	The edge set of a graph G.
$\|V\|$	The order of a set V.
$\|E\|$	The size of a set E.
$d_G(v)$	The degree of a vertex v in G.
K_n	The complete graph of n vertices.
C_n	The cycle of length n.
$N_G(v)$	The set of vertices adjacent to the vertex v.
$CH(S)$	The convex hull of a point set S.
$\lfloor x \rfloor$	The largest integer not greater than x.
$\lceil x \rceil$	The smallest integer not less than x.
$P(n,k)$	The generalized Petersen graph.
$\mathscr{F}_n(^*)$	The characterization of size $n-2$ that forbids any possible triangulations of $K_n - \mathscr{F}_n(^*)$.
$\mathscr{J}_n(^*)$	The characterization of size $n-1$ that forbids any possible triangulations of $K_n - \mathscr{J}_n(^*)$.
$\mathscr{F}_{n,g}(^*)$	The characterization of size $n-g+1$ that forbids any possible g-angulations of $K_n - \mathscr{F}_{n,g}(^*)$.
$\mathscr{J}_{n,g}(^*_\beta)$	The characterization of size at least $n-g+2$ and at most $n-1$ that forbids any possible g-angulations of $K_n - \mathscr{J}_{n,g}(^*_\beta)$ where $\beta \in \{1,2,\ldots,2g-3\}$.

CHAPTER 1

INTRODUCTION

In this thesis, we review a problem regarding geometric graphs that are plane. We deal with the decision problem whether a given convex geometric graph has a g-angulation, which is a very important problem within the field of graph theory.

One of the basic tasks in computational geometry and its applications is decomposing simple polygon into simpler components. The most important simple polygon decomposition is triangulation. For representing geometries and other information appearing in a huge variety of applications, triangulations are widely used as a basis.

In this chapter, we present some definitions and notations followed by the problem of triangulating polygon, a description of problems, research objectives, and finally, a review of the obtained results.

1.1 Definitions and Notations

Graphs are mathematical structures used to model pairwise relations between objects and *Graph theory* is the study of graphs. The objects (which are correspond to mathematical abstractions) are called *vertices*, and each of the related pairs of vertices is called an *edge*. If the pairs of vertices of the edge set are ordered, the graph is referred to as *directed graph* and otherwise *undirected graph*.

In the most common sense of the term, a *graph* is an ordered pair (V,E) including a set of *vertices* V together with a set of *edges* E. In a more generalized concept, E is a multi-set of unordered pairs of vertices which are not necessarily distinct. This type of object is called a *multigraph* or *pseudograph*. *Multigraph* allows multiple edges (the edge that has the same end vertices). Some authors allow multigraphs to have loops (an edge that connects a vertex to itself) while others reserving the term multigraph for the case with multiple edges and they call the graphs that have loops pseudographs.

The two sets V and E are usually taken to be finite and the graphs are called *finite graphs*. It should be noted that many well-known results are not true (or somewhat different) for *infinite graphs* because many arguments fail in the infinite state.

In this thesis, we consider only finite and simple graphs which has no multiple edges nor loops. The *order* of G is the number of its vertices and denoted by n. The *size* of G is the number of its edges and denoted by m. The vertices belonging to an edge are called *end vertices* of the edge.

A graph $H = (V', E')$ is a *subgraph* of $G = (V, E)$ if V' is a subset of V and all edges of E' are also present in E. A subgraph is called *spanning* if $V' = V$.

For subset S of vertices of V, the *induced subgraph* $G[S]$ is the graph whose vertex set is S and whose edge set consists of all of the edges in E that have both end vertices in S. The induced subgraph $G[S]$ may also be called the subgraph induced in G by S.

The deletion of a vertex v in G is to obtain a subgraph induced in G by $V - \{v\}$. Two vertices u and v within V are said to be *neighbors* of each other, if $uv \in E$. $N_G(v)$ denotes the set of neighbors of v in G, and $d_G(v)$ denotes the *degree* of v in G which is the size of $N_G(v)$. u is *adjacent* to v if $u \in N_G(v)$.

A graph in which every vertex has the same degree is called a *regular graph*. A *complete graph* is a graph, in which the vertices are pairwise adjacent. A vertex which is not incident to any edge is called an *isolated vertex*.

A *path* of length ℓ is a graph with $\ell + 1$ distinct vertices, call them $v_0, \ldots, v_\ell$, whose edge set consists of the pairs $v_i v_{i+1}$ with $0 \leq i \leq \ell - 1$. A path is *closed*, if $v_0 = v_\ell$ and open otherwise. If a path is closed, it is a *cycle*. The *length* of a cycle is the number of vertices in it. The graph is *connected* if there is a path in G between any two vertices in G.

For two graphs G_1 and G_2, if there is a bijection $h : V(G_1) \to V(G_2)$ such that $h(u)h(v)$ is an edge of G_2 if and only if uv is an edge in G_1; then G_1 and G_2 are *isomorphic* and we write $G_1 \cong G_2$. We use K_n, P_n and C_n for the complete graphs, paths and cycles with n vertices.

If a graph G can be drawn in the plane without any pair of crossing edges then G is *planar* graph; such a drawing is called a *planar embedding* of G. Planar embedding of G divides the plane into regions; such regions are called *faces* of G. A finite graph has an unbounded face, also called the *exterior face* or *outer face*. A graph with a planar embedding where all vertices are on the outer face is an *outerplanar* graph.

A graph $H = (V, E')$ is a *maximum planar subgraph* of a graph $G = (V, E)$ if H is a planar subgraph of G such that there is no planar subgraph $H' = (V, E'')$ of G with $|E''| > |E'|$.

A graph $H = (V, E')$ is a *maximal planar subgraph* of a graph $G = (V, E)$ if H is a planar subgraph of G such that every graph $H' \in \{(V, E' \cup \{e\}) | \, e \in E \backslash E'\}$ is nonplanar.

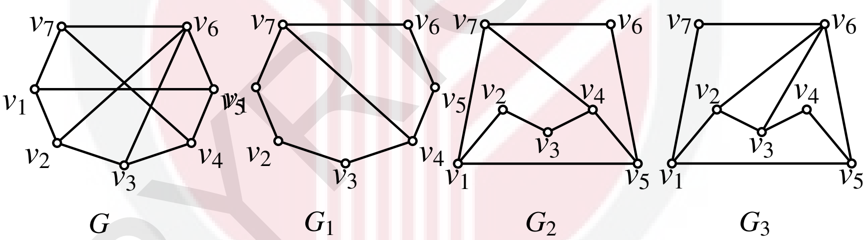

Figure 1.1: Maximal planar subgraph and Maximum planar subgraph.

In Figure 1.1, G is a nonplanar graph. G_1 is a planar subgraph of G but not a maximal since adding the edge v_1v_5 to G_1 does not destroy the planarity where $G_2 \cong G_1 \cup v_1v_5$. G_2 is a maximal planar subgraphs of G, while G_3 is both a maximal and a maximum planar subgraphs of G.

The skewness of a graph G, denoted by $sk(G)$, is the minimum number of edges in G whose deletion results in a planar graph.

A subset of the plane is called *convex set* if and only if for any pair of its points p, q the line segment pq is completely contained in it. Given a finite set of points S in the plane, the *convex hull* $CH(S)$ of a set S is the smallest convex set that contains S. To be more precise, it is the intersection of all convex sets that contain S. A set S is in *convex position* if no point of S lies in the interior of $CH(S)$.

A set of points S is in *general position* in the plane if no three points lie on a common line. A *triangulation* of a set S of points in general position in the plane is a partition of $CH(S)$ into triangles whose vertices are all the points in S.

A *geometric graph* is a graph whose vertex set is a set of points in general position in the plane and whose edge set contains straight-line segments. A *convex geometric graph* is a geometric graph whose vertex set is a set of points in convex position. A geometric graph is plane or non-crossing if its edges do not cross each other.

1.2 Polygon Triangulation

A *polygon* is a region of the plane bounded by a finite collection of line segments forming a closed curve. A polygon is *simple* if its edges cross only in their end vertices. An *n-gon* is a polygon with n sides; for example, a triangle is a 3-gon.

A *convex polygon* is a simple polygon in which no line segment between two points on the boundary ever goes outside the polygon. A *diagonal* is a line segment between two vertices which does not intersect the polygon.

A *planar straight-line graph* is a term used for an embedding of a planar graph in the plane such that its edges are mapped into straight line segments (Berg et al., 2008). Fáry's theorem (1948) states that every planar graph has this kind of embedding. Triangulations may be viewed as special cases of planar straight-line graphs.

A *triangulation* of a polygon is a partition of the interior of the polygon into triangles whose edges are non-crossing diagonals. A polygon with n vertices can always be triangulated and will have $n-2$ triangles and will require the introduction of $n-3$ diagonals.

In how many ways can a plane convex polygon of n sides be divided into triangles by diagonals? Leonhard Euler posed this problem in 1751 to the mathematician Christian Goldbach.

Euler, Goldbach, and Segner proved in 1758 that the number of triangulations of a convex $(n+2)$-gon is

$$\frac{1}{n+1}\binom{2n}{n}.$$

Recently, this value is called Catalan's number. Catalan's name eventually stuck with the problem, despite his modest contributions which were the observing that $\frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n-1}$ and interpreting triangulations as "bracketed sequences".

1.3 Applications of Triangulation

In computational geometry a triangulation of a finite planar set is a well studied structure (O'Rourke, 1994), (Preparata and Shamos, 1985).

In engineering, the most important applications are (1) mesh generation for finite element methods (Srinivasan et al., 1992), (Zienkiewicz and Taylor, 1989), (Ho-Le, 1988), (Sapidis and Perucchio, 1989), (Bern and Eppstein, 1992), and (2) scattered data interpolation (Quak and Schumaker, 1990).

Polygons are very convenient for computer representation of the boundary of the objects from the real world. Computing the triangulation of a polygon is a fundamental algorithm in computational geometry. Triangulation of a simple polygon and its applications in geographic information systems and finite element mesh generation form a significant task in computational geometry .

Triangulations are frequently used in three important fields which are terrain modeling, finite-element methods, and social science research.

In the first field, samples from a terrain is represented by points, and a bivariate interpolating surface is provided by the triangulation, providing an elevation model of the terrain (see (Cohen and Koss, 1993) and (Silveira, 2009)).

In the second field, the complex domain is subdivided using triangulations by creating a mesh of triangles, over which a system of differential equations can be solved more easily (see (Briechle and Hanebeck, 2004), (Esteves et al., 2006) and (Font-Llagunes and Batlle, 2009)).

In the third field, the concept of triangulation refers to a process by which a researcher wants to verify a finding by showing that independent measures of it agree with or, at least, do not contradict it (see (Miles and Huberman, 1994) and (Nordberg, 2009)).

The shapes of the triangles has serious consequences on the result with respect to all these fields. For example, in finite-element methods, since elements of large aspect ratio can lead to poorly conditioned systems, the aspect ratio of the triangles is particularly important (see (Byrod et al., 2007)).

Triangulations have also been heavily applied to other application areas such as pattern recognition, computer graphics, solid modelling and geographic information systems (Okabe et al., 1999), (Schroeder and Shephard, 1988), (Wang, 1992), (Wang and Aggarwal, 1986).

In geographical information systems (GIS), triangulations are used to represent terrain surfaces and relations between geographical objects. Systems for modeling geological structures in the oil and gas industry use triangulations for representing surfaces that separate different geological structures, and for representing properties of these structures.

Computer-aided design (CAD) systems with triangulation features are common in the manufacturing industry and in particular within the automotive industry, which has been a driving force for this research for many decades. We also find applications within engineering fields that simulate physical phenomena using finite element methods (FEM) which use triangulation. Also, the visualization and computer graphics are among the huge number of applications that use triangulation.

However, the quadrangulations (4-angulations) of polygons have been investigated in the computational geometry literature ((Everett et al., 1992), (Kahn et al., 1983), (Lubiw, 1985), (Sack and Toussaint, 1981), (Sack and Toussaint, 1988), (García et al., 2009)). In the study of finite element methods and scattered data interpolation (Lai, 1996), it has been shown that quadrangulations may be more desirable objects than triangulations. These applications provide new motivation to confine our attention to the study of g-angulations (triangulations, quadrangulations, pentagulations, etc) of point sets from the computational geometry point of view.

We first give precise definitions for g-angulation, convex g-angulation and potentially g-angulable.

Definition 1.1 *Let S be a set of n points in general position in the plane.*

1. *A* g-angulation *of S is a plane graph in which each face interior to the convex hull of S is a g-cycle.*
2. *A* convex g-angulation *is a g-angulation on S of n points in convex position in the plane.*

When $g = 3$, g-angulation is usually called a triangulation. We say that G admits a g-angulation if it contains a spanning plane subgraph in which each interior face of $CH(V(G))$ is a g-cycle. That is, G admits a g-angulation G_n if G_n is a g-angulation of $V(G)$ with $E(G_n) \subseteq E(G)$. Which means a geometric graph G admits a triangulation if it contains a trianglation on $V(G)$ as a subgraph.

Definition 1.2 **:** *Let K_n be a convex complete graph with n vertices. F is said to be potentially g-angulable in K_n where $g \geq 3$ if there exists a configuration of F in K_n such that $K_n - F$ admits a g-angulation.*

1.4 Problem statement

The g-angulation existence problem is, On a given geometric graph G, decide whether there exists a g-angulation of G. In computational complexity theory, an *NP*-complete decision problem is one belonging to both the *NP* and the *NP*-hard complexity classes. The triangulation existence problem is *NP*-complete (see (Lloyd, 1977), (Schulz, 2006)).

This thesis considers the following problem.

Problem 1.4.1 *On a given convex geometric graph G, decide whether there exists a g-angulation of G.*

1.5 Objectives

The contributions to this thesis are the following research objectives:

1. To find the maximum number of edges can be removed from a convex complete graph so that the resulting graph still contains a convex g-angulation.

2. To characterize F_n such that $K_n - F_n$ admits no g-angulations.

3. To find the necessary and sufficient condition on F_n such that $K_n - F_n$ admits a g-angulation.

4. To precise the graphs F_n that are potentially g-angulable.

1.6 Organization of Thesis

This thesis falls into six chapters. Since a short introduction of the content is given at the beginning of each chapter, we shall give only a brief outline of the thesis.

Chapter 2 is the literature review on triangulation and quadrangulation.

We devote Chapter 3 to consider the triangulation existence problem of a convex geometric graph $K_n - F_n$ where F_n is a spanning subgraph of a convex complete graph K_n with $|E(F_n)| \leq n-1$. We define two configurations $\mathscr{F}_n(^*)$

and $\mathscr{J}_n(*)$ and show that $K_n - \mathscr{F}_n(*)$ and $K_n - \mathscr{J}_n(*)$ admit no triangulation. If $|E(F_n)| \leq n-2$, we show that $K_n - F_n$ admits a triangulation if and only if $F_n \neq \mathscr{F}_n(*)$. If $|E(F_n)| = n-1$, we show that $K_n - F_n$ admits a triangulation if and only if $F_n \neq \mathscr{J}_n(*)$. For $|E(F_n)| \geq n$, we investigate the possibility of placing F_n in K_n such that $K_n - F_n$ admits a triangulation for certain families of graphs F_n. These results are then applied to determine the convex skewness of the convex graphs of the form $K_n - F_n$.

Chapter 4 investigates the basic combinatorial property of a given set S of n points in general position in the plane and states the necessary and sufficient condition for S to admit a g-angulation. It also determines the number of edges of the convex g-angulation and the number of the inner faces. Two results are obtained in this chapter and then applied to get new short proofs for some well-known results in graph theory.

Chapter 5 extends the problem of Chapter 3 to the g-angulation existence problem of a convex geometric graph $K_n - F_n$ where F_n is a spanning subgraph of a convex complete graph K_n with $|E(F_n)| \leq n-1$. We define two configurations $\mathscr{F}_{n,g}(*)$ and $\mathscr{J}_{n,g}(^*_\beta)$ ($\beta \in \{1,2,\ldots,2g-3\}$) and show that $K_n - \mathscr{F}_{n,g}(*)$ and $K_n - \mathscr{J}_{n,g}(^*_\beta)$ admit no g-angulation. If $|E(F_n)| \leq n-g+1$, we show that $K_n - F_n$ admits a g-angulation if and only if $F_n \neq \mathscr{F}_{n,g}(*)$. If $n-g+2 \leq |E(F_n)| \leq n-1$, we show that $K_n - F_n$ admits a g-angulation if and only if $F_n \neq \mathscr{J}_{n,g}(^*_\beta)$. For $|E(F_n)| \geq n$, we investigate the possibility of placing (i) 2-regular graphs F_n in K_n such that $K_n - F_n$ admits a g-angulation (ii) 3-regular graphs F_n in K_n such that $K_n - F_n$ admits a 4-angulation.

In Chapter 6, a summary of the results of the present thesis is presented together with some open problems.

REFERENCES

Aichholzer, O., Cabello, S., Monroy, R. F., Flores-Peñaloza, D., Hackl, T., Huemer, C., Hurtado, F., and Wood, D. R. (2010). Edge-removal and non-crossing configurations in geometric graphs. *Discrete Mathematics & Theoretical Computer Science*, 12(1):75–86.

Ali, N. A., Chia, G. L., Trao, H. M., and Kilicman, A. (2016). Triangulability of Convex Graphs and Convex Skewness. *Discrete Mathematics*. arXiv:1611.09033v1 [math.CO].

Battista, G. D. and Tamassia, R. (1989). Incremental Planarity Testing. In *Proceedings of the 30th Annual Symposium on Foundations of Computer Science, (FOCS'89)*, pages 436–441.

Berg, M., Cheong, O., Van Kreveld, M., and Overmars, M. (2008). Polygon Triangulation. In *Computational Geometry, Algoritms and Applications*, Guarding an Art Gallery. Springer. Chapter 3.

Bern, M. and Eppstein, D. (1992). Mesh generation and optimal triangulation. In Hwang, F. K. and Du, D. Z., editors, *Computing in Euclidean Geometry*, volume 4. World Scientific Publishing. Chapter 3.

Bernhart, F. R. (1975). A Three and Five Color Theorem. *Proceedings of The American Mathematical Society*, 52(1):493–493.

Boltyanskil, V. G. (1963). *Equivalent and equidecomposable figures*. Translated from the Russian, Heath, Boston.

Bondy, J. A. and Murty, U. R. S. (1976). *Graph theory with applications*. North-Holland.

Bose, P. and Toussaint, G. (1995). No Quadrangulation is Extremely Odd. In *Algorithms and Computations*, volume 1004 of *Lecture Notes in Computer Science*, pages 372 – 381. Springer.

Briechle, K. and Hanebeck, U. D. (2004). Localization of a mobile robot using relative bearing measurements. *IEEE Transactions on Robotics and Automation*, 20(1):36–44.

Byrod, M., Josephson, K., and Astrom, K. (2007). Fast optimal three view triangulation. In Yagi, Y., Kang, S. B., Kweon, I. S., and Zha, H., editors, *Asian Conference on Computer Vision (ACCV)*, volume 4844, pages 549–559, Heidelberg. Springer.

Cai, L., Han, X., and Tarjan, R. E. (1993). An $O(m\log n)$-time algorithm for the maximal planar subgraph problem. *SIAM Journal on Computing*, 22(85):1142–1162. Published online: 26 May 2009.

Chazelle, B. (1991). Triangulating a simple polygon in linear time. *Discrete and Computational Geometry*, 6:485–524.

Chia, G. L. and Lee, C. L. (2003). Crossing Numbers and Skewness of Some Generalized Petersen Graphs. In *Proceedings of the Indonesia-Japan joint conference on Combinatorial Geometry and Graph Theory (IJCCGGT)*, pages 80–86, Bandung, Indonesia.

Chia, G. L. and Lee, C. L. (2009). Skewness and crossing numbers of graphs. *Bulletin of the Institute of Combinatorics and its Applications*, 55:17–32.

Chia, G. L. and Lee, C. L. (2012). Skewness of generalized Petersen graphs and related graphs. In *Frontiers Mathematics in China*, volume 7, pages 427–436, China. Higher Education Press.

Chia, G. L. and Lee, C. L. (2015). Crossing Numbers of Nearly Complete Graphs and Nearly Complete Bipartite Graphs. *Ars Combinatoria*, 121:437–446.

Chia, G. L. and Lee, C. L. (2016). Regular graphs with small skewness and crossing numbers. *Bulletin of the Malaysian Mathematical Sciences Society*, 39:1687–1693.

Chia, G. L. and Sim, K. A. (2013). On the skewness of the join of graphs. *Discrete Applied Mathematics*, 161:2405–2409.

Chiba, T., Nishioka, I., and Shirakawa, I. (1979). An algorithm of maximal planarization of graphs. In *Proceedings IEEE Symposium on Circuits & Systems, (ISCAS)*, pages 649–652.

Cimikowski, R. J. (1992). Graph planarization and skewness. *Congressus Numerantium*, 88:21–32.

Cohen, C. and Koss, F. (1993). A comprehensive study of three object triangulation. *Mobile Robots VII*, 1831:95 – 106.

Conway, J. H. and Coxeter, H. S. M. (1973). Triangulated polygons and frieze patterns. *Mathematical Gazette*, 57:88 – 94.

Djidjev, H. N. (1995). A Linear Algorithm for the Maximal Planar Subgraph Problem. In Akl, S. G., Dehne, F., Sack, J. R., and Santoro, N., editors, *Proceedings 4th International Workshop on Algorithms and Datastructures, (WADS'95)*, volume 955 of *Lecture Notes in Computer Science*, pages 369–380. Springer-Verlag.

Djidjev, H. N. (2006). A linear-time algorithm for finding a maximal planar subgraph. *SIAM Journal on Discrete Mathematics*, 20(2):444–462.

Esteves, J., Carvalho, A., and Couto, C. (2006). Position and orientation errors in mobile robot absolute self-localization using an improved version of the generalized geometric triangulation algorithm. In *IEEE International Conference on Industrial Technology (ICIT)*, volume I, pages 830–835, Mumbai, India.

Everett, H., Lenhart, W., Overmars, M., Shermer, T., and Urrutia, J. (1992). Strictly convex quadrilateralizations of polygons. In *Proceedings of the 4th Canadian Conference on Computational Geometry*, pages 77–83.

Farr, G. and Eades, P. (2003). Skewness of Graphs with Small Cutsets. In *Graphs and Combinatorics*, volume 19, pages 177–194, Japan. Springer. Chapter 3.

Fáry, I. (1948). On straight-line representation of planar graphs. *Acta Scientiarum Mathematicarum (Szeged)*, 11:229—233.

Font-Llagunes, J. and Batlle, J. (2009). Consistent triangulation for mobile robot localization using discontinuous angular measurements. *Robotics and Autonomous Systems*, 57(9):931–942.

García, A., Hurtado, F., Huemer, C., Tejel, J., and Valtr, P. (2009). On triconnected and cubic plane graphs on given point sets. *Computational Geometry: Theory and Applications*, 42(4):913—-922.

Giuseppe, D. B., Eades, P., Tamassia, R., and Tollis, I. G. (1999). *Graph Drawing. Algorithms for the Visualization of Graphs*. Prentice Hall.

Gutwenger, C. and Chimani, M. (2009). Non-Planar Core Reduction of Graphs. *Discrete Mathematics*, 309(7):1838–1855.

Gutwenger, C. and Mutzel, P. (2001). A Linear Time Implementation of SPQR-Trees. In *Proceedings of the 8th International Symposium on Graph Drawing*, volume I, pages 77–90, Berlin Heidelberg. Springer-Verlag.

Harary, F. (1969). *Graph Theory*. Addison-Wesley, Boston.

Heggernes, P. (2006). Minimal triangulations of graphs: A survey. *Discrete Mathematics*, 306:297–317.

Ho-Le, K. (1988). Finite element mesh generation methods: A review and classification. *Computer Aided Design*, 22:27–38.

Hopcroft, J. E. and Tarjan, R. E. (1974). Efficient planarity testing. *Journal of the Association for Computing Machinery*, 21:549–568.

Houle, M. E., Hurtado, F., Noy, M., and Rivera-Campo, E. (2005). Graphs of triangulations and perfect matchings. *Graphs and Combinatorics*, 21:325–331.

Hsu, W. L. (1995). A linear time algorithm for finding maximal planar subgraphs. In Staples, J., Eades, P., Katoh, N., and Moffat, A., editors, *Algorithms and Computation, 6th International Symposium, (ISAAC'95)*, volume 1004 of *Lecture Notes in Computer Science*, pages 352–361. Springer-Verlag.

Jayakumar, R., Swamy, M. N. S., and Thulasiraman, K. (1989a). $O(n^2)$ Algorithms for Graph Planarization. In Leeuwen, J. v., editor, *Proceedings 14th International Workshop on Graph-Theoretic Concepts in Computer Science*, volume 344, pages 352–377. Springer-Verlag.

Jayakumar, R., Thulasiraman, K., and Swamy, M. N. S. (1989b). $O(n^2)$ Algorithms for Graph Planarization. *IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems*, 8:257–267.

Jungnickel, D. (2013). *Graphs, Networks and Algorithms*. Springer-Verlag, Berlin Heidelberg.

Kahn, J., Klawe, M., and Kleitman, D. (1983). Traditional galleries require fewer watchmen. *SIAM Journal on Algebraic Discrete Methods*, 4:194–206.

Kainen, P. C. (1974). A generalization of the 5-color theorem. *Proceedings of the American Mathematical Society*, 45:450–453.

Kainen, P. C. (1975). Chromatic number and skewness. *Journal of Combinatorial Theory Series B*, 18:32–34.

Kant, G. (1993). *Algorithms for Drawing Planar Graphs*. PhD thesis, University of Utrecht.

Kotzig, A. (1955). On certain decompositions of a graph. *Matematicko-fyzikálny Časopis*, 5:144–151.

Lai, M. (1996). Scattered data interpolation and approximation by using C^1 piecewise cubic polynomials. *Computer Aided Geometric Design*, 13:81–88.

Liebers, A. (2001). Planarizing graphs A survey and annotated bibliography. *Journal of Graph Algorithms and Applications*, 5:1–74.

Liu, P. C. and Geldmacher, R. C. (1979). On the Deletion of Nonplanar Edges of a Graph. *Congressus Numerantium*, 24:727–738.

Lloyd, E. L. (1977). On triangulations on a set of points in the plane. *Proc. of IEEE Symposium on Foundations of Computer Science (FOCS), IEEE*, pages 228–240.

Lubiw, A. (1985). Decomposing polygonal regions into convex quadrilaterals. In *Proceedings of the 1st ACM Symposium on Computational Geometry*, pages 97–106, Baltimore, Maryland, USA.

Maehara, H. (2003). Plane graphs with straight edges whose bounded faces are acute triangles. *Journal of Combinatorial Theory Series B*, 88(1):237–245.

Mendonca, C. G. X., Schaffer, K., Xavier, E. F., Stolfi, J., Faria, L., and Figueiredo, C. M. H. (2002). The splitting number and skewness of $C_n \times C_m$. *Ars combinatoria*, 63:193–205.

Mezzini, M. and Moscarini, M. (2009). Simple algorithms for minimal triangulation of a graph and backward selection of a decomposable Markov network. *Theorectical Computer Science*. doi: 10.1016/j.tcs.2009.10.004.

Miles, M. and Huberman, A. (1994). *Qualitative Data Analysis: An Expanded Sourcebook*. Thousand Oaks. Sage Publications, California, 2nd edition.

Nordberg, K. (2009). The Triangulation Tensor. *Computer Vision and Image Understanding*, 113:935 – 945.

Okabe, A., Boots, B., and Sugihara, K. (1999). *Patial Tessellations: Concepts and Applications of Voronoi Diagrams*. John Wiley & Sons, Chichester, England.

O'Rourke, J. (1994). *Computational Geometry in C*. Cambridge University Press.

Ozawa, T. and Takahashi, H. (1981). A graph-planarization algorithm and its application to random graphs. In Saito, N. and Nishizeki, T., editors, *Proceedings of the 17th Symposium, Research Institute of Electrical Communication*, volume 108 of *Lecture Notes in Computer Science*, pages 95–107, Sendai, Japan. Tohoku University, Springer-Verlag.

Pahlings, H. (1978). On the chromatic number of skew graphs. *Journal of Combinatorial Theory, Series B*, 25(3):303–306.

Poutré, J. A. L. (1994). Alpha-Algorithms for Incremental Planarity Testing. In *Proceedings of the 26th Annual ACM Symposium on Theory of Computing, (STOC'94)*, pages 706–715.

Preparata, F. P. and Shamos, M. I. (1985). *Computational Geometry - An Introduction*. Springer-Verlag, 1st edition.

Quak, E. and Schumaker, L. L. (1990). Cubic spline fitting using data dependent triangulations. *Computer-Aided Geometric Design*, 7:293–301.

Sack, J. and Toussaint, G. (1981). A linear-time algorithm for decomposing rectilinear star-shaped polygons into convex quadrilaterals. In *Proceedings of the 19th Annual conference on Communications, Control and Computing*, pages 21–30, Urbana, Illinois, USA.

Sack, J. and Toussaint, G. (1988). Guard placement in rectilinear polygons. In Toussaint, G., editor, *Computational Morphology*, pages 153–175, North-Holland.

Sapidis, N. and Perucchio, R. (1989). Advanced techniques for automatic finite element meshing from solid models. *Computer Aided Design*, 21(4):248–253.

Schroeder, W. and Shephard, M. (1988). Geometry-based fully automatic mesh generation and the Delaunay triangulation. *International Journal for Numerical Methods in Engineering*, 24:2503–2515.

Schulz, A. (2006). The Existence of a Pseudo-triangulation in a given Geometric Graph. In *proceedings of the 22nd European Workshop on Computational Geometry*, pages 17–20.

Sen, R. K. (1989). An approximate algorithm for minimum edge deletion bipartite subgraph problem. In *Proceedings of The 20th Southeastern Conference on Combinatorics, Graph Theory, and Computing*, volume 74, pages 38–54, Boca Raton, Florida, USA.

Sen, R. K. and Chaudhuri, P. (1989). A parallel approximate algorithm for minimum edge deletion bipartite subgraph problem. *Journal of Combinatorics, Information & System Sciences*, 14:111–123.

Silveira, R. I. (2009). *Optimization of polyhedral terrains*. PhD thesis, Utrecht University.

Srinivasan, V., Nackman, L. R., Tang, J. M., and Meshkat, S. N. (1992). Automatic mesh generation using the symmetric axis transformation of polygonal domains. In Allyn and Bacon, editors, *Proceedings of the IEEE*, volume 80, pages 1485–1501.

Wang, T. (1992). A C^2-quintic spline interpolation scheme on triangulation. *Computer Aided Geometric Design*, 9:379–386.

Wang, Y. and Aggarwal, J. (1986). Surface reconstruction and representation of 3-d scenes. *Pattern Recognition*, 19:197–207.

Zienkiewicz, O. C. and Taylor, R. L. (1989). *The Finite Element Method*, volume I. McGraw-Hill, London.