UNIVERSITI PUTRA MALAYSIA

HYBRID AND LINEAR MULTISTEP METHODS FOR SOLVING OSCILLATORY SECOND-ORDER DIFFERENTIAL EQUATIONS

SUFIA ZULFA BINTI AHMAD

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HYBRID AND LINEAR MULTISTEP METHODS FOR SOLVING OSCILLATORY SECOND-ORDER DIFFERENTIAL EQUATIONS

By

SUFIA ZULFA BINTI AHMAD

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Doctor of Philosophy

June 2018

DEDICATIONS

This humble work is dedicated to my beloved family,

friends, and future researches.

And above all,

To the almighty Allah.

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfillment of the requirement for the degree of Doctor of Philosophy

HYBRID AND LINEAR MULTISTEP METHODS FOR SOLVING OSCILLATORY SECOND-ORDER DIFFERENTIAL EQUATIONS

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June 2018

Chairman : Professor Fudziah Bt Ismail, PhD
Faculty : Science

This thesis is focused mainly on developing methods for solving special second order ordinary differential equations (ODEs) and delay differential equations (DDEs) with oscillatory solutions. The first part of this thesis is on the derivation of semi-implicit hybrid methods using the technique of trigonometrically-fitted for solving oscillatory ordinary as well as delay differential equations. The implementation of trigonometrically fitting technique is supposed to enhance the efficiency of the methods. Numerical results are illustrated using efficiency curve where the common logarithm of the maximum global error versus the CPU time is taken. Results indicated that the new method work efficiently for solving both ODEs and DDEs. The stability of the methods are presented.

In the second part of the thesis, phase-fitting technique is applied to the existing hybrid methods for solving oscillatory ODEs. The modification causes the nullifying of phase-lag of the methods. Numerical results illustrated that the new phase-fitted method is efficient compared to the existing fitted and non-fitted methods.

The derivation of vanishing phase-lag and amplification fitted semi-implicit hybrid method are shown in the third part of the thesis. The general formula of hybrid method is modified with additional coefficients which depend on the value of the fitted frequency. The theory of zero dissipation and zero dispersion techniques are investigated. Numerical solutions show that the new method is a promising tool for integrating oscillatory problems.

The fourth part of the thesis focuses on the derivation of block explicit hybrid methods. The new methods generate two points at every step length. The trigonometrically-fitting technique is adapted to the block methods to enhance the efficiency of the methods. Numerical result demonstrated that the new method performed better in accuracy and require lesser computational time compared to the methods in comparison.

Finally, two-step linear multistep methods with extra derivatives are derived using collocation technique. The method is developed using sequence of Chebyshev polynomials as the basis function. The new methods are then trigonometrically-fitted to improve the efficiency in solving oscillatory problems. The results signify that the methods are promising tools for the integration of oscillatory problems.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia
sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

KAEDAH HIBRID DAN MULTILANGKAH LINEAR BAGI PENYELESAIAN PERSAMAAN BEMBEZAAN PERINGKAT KEDUA YANG BERAYUN

Oleh

SUFIA ZULFA BINTI AHMAD

Jun 2018

Pengerusi : Professor Fudziah Bt Ismail, PhD
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Tumpuan utama tesis ini adalah untuk menerbitkan kaedah bagi menyelesaikan persamaan oembezaan biasa (PBB) dan persamaan peringkat pembezaan tunda (PPT) peringkat kedua yang berayun. Bahagian pertama tesis ini adalah mengenai kaedah separa-tersirat hibrid (STH) yang dibina menggunakan teknik suai secara trigonometri (SST) untuk menyelesaikan permasalahan berayun PBB dan PPT. Pelaksanaan teknik SST ini bertujuan untuk meningkatkan kecekapan sesuatu kaedah. Keputusan berangka digambarkan menggunakan lengkung kecekapan di mana logaritma biasa dari ralat global maksimum berbanding masa CPU diambil. Keputusan menunjukkan kaedah baharu ini dapat menyelesaikan masalah PBB and PPT dengan cekap. Kestabilan kaedah ini turut dipersembahkan.

Pada bahagian kedua tesis, teknik serakan secara fasa (SSF) digunakan ke atas kaedah hibrid yang sedia ada bagi penyelesaian PBB untuk masalah berayun. Pengubahsuaian ini mencetuskan kaedah serakan secara fasa sifar. Keputusan berangka menunjuk kaedah baharu lebih cekap dibandingkan dengan kaedah-kaedah sedia ada yang secara fasa dan tidak secara fasa.

Pembinaan kaedah berdasarkan teknik lesapan beserta serakan sifar secara fasa (LSSF) ditunjukkan dalam bahagian ketiga tesis ini. Persamaan umum kaedah hibrid ini diubahsuai menggunakan pekali-pekali tambahan yang nilainya bergantung kepada nilai frekuensi. Teori lesapan secara sifar dan serakan secara sifar ini dikaji. Kaedah baharu ini diuji untuk masalah yang berselang besar untuk membuktikan kecekapan kaedah baharu ini dalam penyelesaian

masalah berayun. Keputusan berangka menunjukkan kaedah baharu menjanjikan kaedah yang bersesuaian bagi menyelesaikan masalah berayun.

Bahagian keempat tesis ini merujuk kepada pembinaan kaedah tiga langkah blok tak-tersirat hibrid (BTTH). Kaedah baharu ini menghasilkan dua penyelesaian bagi setiap langkah yang dibina. Teknik SST digunakan ke atas kaedah-kaedah blok ini untuk meningkatkan kecekapan kaedah tersebut. Keputusan berangka menunjukkan kaedah baharu ini menjimatkan masa dan lebih cekap berbanding kaedah-kaedah yang dibandingkan.

Akhir sekali, kaedah linear multilangkah dengan derivatif tambahan diperoleh menggunakan teknik kolokasi. Kaedah ini dibina menggunakan jujukan polinomial Chebyshev sebagai fungsi asas. SST digunakan ke atas kaedah baharu ini untuk meningkatkan kecekapan bagi menyelesaikan masalah berayun. Keputusan menunjukkan bahawa kaedah hibrid adalah kaedah yang dijanjikan bersesuaian bagi mengamirkan masalah jenis berayun.

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The Most Merciful First and foremost

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I certify that a Thesis Examination Committee has met on 11 June 2018 to conduct the final examination of Sufia Zulfa binti Ahmad on her thesis entitled "Hybrid and Linear Multistep Methods for Solving Oscillatory Second-Order Differential Equations" in accordance with the Universities and University Colleges Act 1971 and the Constitution of the Universiti Putra Malaysia [P.U.(A) 106] 15 March 1998. The Committee recommends that the student be awarded the Doctor of Philosophy.

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LIST OF ABBREVIATIONS

BEHM	Block explicit hybrid method
BEHM3(4)	Three-stage fourth-order block explicit hybrid method
BEHM4(5)	Four-stage fifth-order block explicit hybrid method
DDE	Delay differential equation
DIRKN	Diagonally implicit Runge-Kutta Nyström
DIRKN(HS)	Diagonally implicit three-stage fourth-order Runge-Kutta Nyström method derived in Sommeijer (1987)
DIRKN3(4)	Diagonally implicit three-stage fourth-order Runge-Kutta Nyström method derived in Senu et al.(2010)
DIRKN4(4)	Diagonally implicit four-stage fourth-order Runge-Kutta Nyström method derived in Senu et al.(2011)
EHM3(4)	Explicit three-stage fourth-order hybrid method derived by Franco (2006)
EHM4(5)	Explicit four-stage fifth-order hybrid method derived by Franco (2006)
EHM5(6)	Explicit five-stage sixth-order hybrid method derived by Franco (2006)
EHM5(6)-(J)	Explicit five-stage sixth-order hybrid method with dispersion order 10 and dissipation order seven derived by Jikantoro et al.(2015)
IRKN4(5)	Four-stage fifth-order improved Runge-Kutta Nyström method developed by Faranak et al.(2012)
IVP	Initial Value Problem
LMM	Linear multistep method
LMMC(3)	A linear multistep method with collocation method of order three developed in Chapter 7.
LMMC(4)	A linear multistep method with collocation method of order four developed in Chapter 7.

MPAFRKN4(4)	A phase and amplification fitted four-stage fourth-order Runge-Kutta Nyström method by Papadopoulos et al. (2010)
NDDI	Newton divided difference interpolation
NPF-EHM4(5)	Phase-fitted four-stage fifth-order explicit hybrid method developed in Chapter 4
NPF-EHM5(6)	Phase-fitted five-stage sixth-order explicit hybrid method developed in Chapter 4
NSIHM3(4)	Three-stage fourth-order Semi implicit hybrid method developed in Chapter 2
NSIHM4(5)	Four-stage fifth-order Semi implicit hybrid method developed in Chapter 3
ODE	Ordinary differential equation
PFRKN4(4)	Explicit four-stage fourth-order phase-fitted Runge-Kutta Nyström method by Papadopoulos et al. (2009)
RK	Runge-Kutta method
RK7(6)	Explicit seven-stage sixth-order Runge-Kutta method by Butcher (2008)
RKN	Runge-Kutta Nyström method
RKN3(4)	Explicit three-stage fourth-order Runge-Kutta Nyström method by Hairer et al. (2010)
RKN4(5)	Explicit four-stage fifth-order Runge-Kutta Nyström method by Hairer et al. (2010)
SIHM	Semi implicit hybrid method
SIHM3(4)	Semi-implicit three-stage fourth-order hybrid method developed in Ahmad et al. (2013)
SIHM4(5)	Four-stage fifth-order Semi implicit hybrid method in Ahmad et al. (2013)
SIHM4(5)(8,7)	A four-stage fifth-order Semi implicit hybrid method with dispersion of order eight and dissipation of order seven derived in Chapter 5

TF-BEHM3(4)	Trigonometrically-fitted three-stage fourth-order block explicit hybrid method developed in Chapter 6
TF-BEHM4(5)	Trigonometrically-fitted four-stage fifth-order block explicit hybrid method developed in Chapter 6
TF-LMMC(3)	Trigonometrically-fitted linear multistep method with collocation method of order three developed in Chapter 7
TF-LMMC(4)	Trigonometrically-fitted linear multistep method with collocation method of order four developed in Chapter 7
TF-NSIHM3(4)	Trigonometrically-fitted three-stage fourth-order Semi implicit hybrid method developed in Chapter 2
TF-NSIHM4(5)	Trigonometrically-fitted four-stage fifth-order Semi implicit hybrid method developed in Chapter 3
VPA-SIHM4(5)	Vanishing phase-lag and amplification error four-stage fifth-order Semi implicit hybrid method developed in Chapter 5

CHAPTER 1

INTRODUCTION

1.1 Differential Equations

Differential equations are known as mathematical formulas or equations that relate some functions and its derivatives. The relationship of functions and derivatives can simply be described as the physical quantities and their rate of change. Differential equations have become essential studies that include in variety of disciplines especially in the field of engineering, mathematics, astronomic and sciences. For example engineers design spacecraft by considering the radiation environment, duration of the research program, and computation on how much shielding is needed for the spacecraft to survive till the end of its mission. Therefore, a more realistic model involving differential equation which calculate and estimate the orbit path and radiation belt rates which can be performed using a simple graphical integration by estimating areas under the curves. There are several types of differential equations such as ordinary, partial, delay, fuzzy, and non-linear differential equations. In this thesis, we are going to focus on solving special second order ordinary differential equations and delay differential equations with oscillatory solutions.

1.2 Two – step Hybrid Methods

The general formula of s-stage two-step explicit hybrid method for numerical integration of initial value problems (IVPs) as proposed in Franco (2006) is in the form of

$$Y_i = (1 + c_i)y_n - c_i y_{n-1} + h^2 \sum_{j=1}^{s} a_{ij} f\left(x_n + c_j h, Y_j\right), \tag{1.1}$$

$$y_{n+1} = 2y_n - y_{n-1} + h^2 \sum_{i=1}^{s} b_i\, f(x_n + c_i h, Y_i) \tag{1.2}$$

where $i = 1, \dots, s,$ for $i > j$.The coefficients of b_i, c_i , and a_{ij}can be represented in Butcher tableau as in Table 1.1.

Table 1.1: The s-stage of explicit hybrid method

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array} = \begin{array}{c|cccccc} -1 & 0 & & & & \\ 0 & 0 & 0 & & & \\ c_3 & a_{3,1} & a_{3,2} & 0 & & \\ \vdots & \vdots & \vdots & \ddots & \ddots & \\ c_s & a_{s,1} & a_{s,2} & \dots & a_{s,s-1} & 0 \\ \hline & b_1 & b_2 & \dots & b_{s-1} & b_s \end{array}$$

Semi-implicit hybrid method (SIHM) can be written in the form of

$$Y_i = (1+c_i)y_n - c_i y_{n-1} + h^2 \sum_{j=1}^{s} a_{ij} f\left(x_n + c_j h, Y_j\right), \tag{1.3}$$

$$y_{n+1} = 2y_n - y_{n-1} + h^2 \sum_{i=1}^{s} b_i \, f(x_n + c_i h, Y_i), \tag{1.4}$$

where $i = 1, \dots, s$, and $i \geq j$. The first two nodes are defined as $c_1 = -1$, and $c_2 = 0$ given the equation (1.3) and (1.4) as

$$Y_1 = y_{n-1}, Y_2 = y_n, \tag{1.5}$$

$$Y_i = (1+c_i)y_n - c_i y_{n-1} + h^2 \sum_{j=1}^{i} a_{ij} f\left(x_n + c_j h, Y_j\right), \tag{1.6}$$

$$y_{n+1} = 2y_n - y_{n-1} + h^2 \left(b_1 f_{n-1} + b_2 f_n + \sum_{i=3}^{s} b_i f(x_n + c_i h, Y_i) \right), \tag{1.7}$$

where $i = 3, \dots, s, i \geq j$, and functions $f_{n-1} = f(x_{n-1}, y_{n-1})$ and $f_n = f(x_n, y_n)$. The s-stages SIHM can be written in Butcher tableau as in Table 1.2.

Table 1.2: The s-stage of semi-implicit hybrid methods

$$\begin{array}{c|cccccc} -1 & 0 & & & & \\ 0 & 0 & 0 & & & \\ c_3 & a_{3,1} & a_{3,2} & a_{3,3} & & \\ \vdots & \vdots & \vdots & \ddots & \ddots & \\ c_s & a_{s,1} & a_{s,2} & \dots & a_{s,s-1} & a_{s,s} \\ \hline & b_1 & b_2 & \dots & b_{s-1} & b_s \end{array}$$

1.2.1 Local Truncation Error and Order Condition of Hybrid Method

The study of algebraic order condition of explicit hybrid method was done by Coleman (2003). The methods are characterized by two s-dimensional vectors, $\boldsymbol{b}$ and $\boldsymbol{c}$, with elements b_i and c_i, respectively, and $s \times s$ matrix $\mathbf{A}$ with elements

a_{ij}. In vector notation, for an autonomous system of equation $y" = f(y)$, (1.3) and (1.4) can be written in the form of

$$y_{n+1} = 2y_n - y_{n-1} + h^2(\boldsymbol{b}^T \otimes I)f(\boldsymbol{Y}),$$

$$\boldsymbol{Y} = (\boldsymbol{e} + \boldsymbol{c})\otimes y_n - \boldsymbol{c}\otimes y_{n-1} + h^2(A\otimes I)f(\boldsymbol{Y}), \tag{1.8}$$

where $\boldsymbol{e} = (1, \dots, 1)^T$. The order conditions for two-step hybrid methods are derived by considering them as one-step methods of the form

$$u_n = u_{n-1} + h\phi(u_{n-1}, h), \tag{1.9}$$

where u_n is an appropriately defined numerical solution vector, and some starting procedure is used to generate u_0. This approach is prompted by the work of Hairer and Warner (2012) for a class of two-step RK methods for differential equations of first order. The first equation in (1.8) can be written as a pair of equations by defining $F_n \coloneqq (y_{n+1} - y_n)/h$ so that

$$y_{n=}y_{n-1} + hF_{n-1}, \quad F_n = F_{n-1} + h(\boldsymbol{b}^T \otimes I)f(\boldsymbol{Y}).$$

These equations can be written as (1.9) with

$$u_n = \binom{y_n}{F_n}, \quad \phi(u_{n-1}, h) = \begin{pmatrix} F_{n-1} \\ (\boldsymbol{b}^T \otimes I)f(\boldsymbol{Y}) \end{pmatrix},$$

where $\boldsymbol{Y}$ is defined by

$$\boldsymbol{Y} = (\boldsymbol{e} + \boldsymbol{c})\otimes y_n - \boldsymbol{c}\otimes y_{n-1} + h^2(A\otimes I)f(\boldsymbol{Y})$$

$$= \boldsymbol{e}\otimes y_{n-1} + h(\boldsymbol{e} + \boldsymbol{c})\otimes F_{n-1} + h^2(A\otimes I)f(\boldsymbol{Y}). \tag{1.10}$$

The vector u_n is an approximation for $z_n = z(x_n, h)$, where z is the exact-value function defined by

$$z(x, h) = \begin{pmatrix} y(x) \\ \dfrac{y(x+h) - y(x)}{h} \end{pmatrix}$$

The local truncation error of the method at x_n is

$$d_n = z_n - z_{n-1} - h\phi(z_{n-1}, h) \tag{1.11}$$

for

$$\phi(z_{n-1}, h) = \begin{pmatrix} \dfrac{y(x_n) - y(x_{n-1})}{h} \\ (\boldsymbol{b}^T \otimes I)f(\boldsymbol{Y}) \end{pmatrix} \tag{1.12}$$

where $\boldsymbol{Y}$ is now defined implicitly as (1.10). The order conditions for semi-implicit hybrid method up to order seven which is derived in Coleman (2003) are listed in Table 1.3.

Table 1.3: Order condition

Tree t	$\rho(t)$	Order condition
t_{21}	2	$\sum_{i=1}^{s} b_i = 1$
t_{31}	3	$\sum_{i=1}^{s} b_i c_i = 0$
t_{41}	4	$\sum_{i=1}^{s} b_i {c_i}^2 = \frac{1}{6}$
t_{42}		$\sum_{i=1}^{s} b_i a_{ij} = \frac{1}{12}$
t_{51}	5	$\sum_{i=1}^{s} b_i {c_i}^3 = 0$
t_{52}		$\sum_{i=1}^{s} b_i c_i a_{ij} = \frac{1}{12}$
t_{53}		$\sum_{i=1}^{s} b_i a_{ij} c_j = 0$
t_{61}	6	$\sum_{i=1}^{s} b_i {c_i}^4 = \frac{1}{15}$
t_{62}		$\sum_{i=1}^{s} b_i {c_i}^2 a_{ij} = \frac{1}{30}$
t_{63}		$\sum_{i=1}^{s} b_i c_i a_{ij} c_j = -\frac{1}{60}$
t_{64}		$\sum_{i=1}^{s} b_i a_{ij} a_{ik} = \frac{7}{120}$
t_{65}		$\sum_{i=1}^{s} b_i a_{ij} {c_j}^2 = \frac{1}{180}$
t_{66}		$\sum_{i=1}^{s} b_i a_{ij} a_{jk} = \frac{1}{360}$
t_{71}	7	$\sum_{i=1}^{s} b_i {c_i}^5 = 0$
t_{72}		$\sum_{i=1}^{s} b_i {c_i}^3 a_{ij} = \frac{1}{30}$

t_{73}	$\sum_{i=1}^{s} b_i {c_i}^2 a_{ij} c_j = 0$
t_{74}	$\sum_{i=1}^{s} b_i c_i a_{ij} a_{ik} = \frac{1}{30}$
t_{75}	$\sum_{i=1}^{s} b_i\, c_i a_{ij} {c_j}^2 = \frac{1}{72}$
t_{76}	$\sum_{i=1}^{s} b_i\, c_i a_{ij} a_{jk} = -\frac{1}{720}$
t_{77}	$\sum_{i=1}^{s} b_i a_{ij} a_{ik} c_k = -\frac{1}{120}$
t_{78}	$\sum_{i=1}^{s} b_i a_{ij} {c_j}^3 = 0$
t_{79}	$\sum_{i=1}^{s} b_i\, a_{ij} c_j a_{jk} = \frac{1}{360}$
$t_{7,10}$	$\sum_{i=1}^{s} b_i a_{ij} a_{jk} c_k = 0$

where value of $i \geq j \geq k$. The simplifying condition for hybrid method is

$$\sum_{i}^{s} a_{ij} = \frac{({c_i}^2 + c_i)}{2}.$$

The error constant for the pth – order method can be defined as

$$C_{p+1} = \left\| \left(e_{p+1}(t_1) \right), \ldots, e_{p+1}(t_k) \right\|_2 \tag{1.13}$$

where k is the number of trees of order $p + 2(p(t_i) = p + 2)$ and $e_{p+1}(t_i)$ is the local truncation error which was defined in Coleman (2003).

1.2.2 Analysis of the Periodicity, Absolute Stability, Dispersion and Dissipation Errors

Phase analysis can be divided into two parts: homogeneous which the phase error are accumulated as s increases and inhomogeneous which phase error is constant in time. Franco (2006) proposed that phase analysis is investigate using the second order homogeneous linear test model, $y''(x) = -\lambda^2 y(x)$.

We apply the test equation $y''(x) = (i\lambda)^2 y(x) = -\lambda^2 y(x)$, for $\lambda > 0$ by replacing $f(x, y) = -\lambda^2 y(x)$ into equation (1.3) and (1.4) and we obtain

$$Y_i = (1+c_i)y_n - c_i y_{n-1} - h^2 \sum_{j=1}^{s} a_{ij}\, \lambda^2 y(x), i = 1, \dots, s, \tag{1.14}$$

$$y_{n+1} = 2y_n - y_{n-1} - h^2 \sum_{i=1}^{s} b_i\, \lambda^2 y(x). \tag{1.15}$$

By substituting $H = h\lambda$, equations (1.14) and (1.15) can be expressed as

$$Y_i = (1+c_i)y_n - c_i y_{n-1} - H^2 \sum_{j=1}^{s} a_{ij}\, y(x), \qquad i = 1, \dots, s, \tag{1.16}$$

$$y_{n+1} = 2y_n - y_{n-1} - H^2 \sum_{i=1}^{s} b_i\, y(x), \tag{1.17}$$

and for s-stage, equation (1.16) will give

$$\begin{aligned} Y_1 &= (1+c_1)y_n - c_1 y_{n-1} - H^2(a_{11}Y_1 + a_{12}Y_2 + \dots + a_{1s}Y_s) \\ Y_2 &= (1+c_2)y_n - c_2 y_{n-1} - H^2(a_{21}Y_1 + a_{22}Y_2 + \dots + a_{2s}Y_s) \\ &\vdots \\ Y_s &= (1+c_s)y_n - c_s y_{n-1} - H^2(a_{s1}Y_1 + a_{s2}Y_2 + \dots + a_{ss}Y_s) \end{aligned} \tag{1.18}$$

Then, (1.18) and (1.17) can be written in vector form respectively as below:

$$\boldsymbol{Y} = (\boldsymbol{e} + \boldsymbol{c})y_n - \boldsymbol{c}y_{n-1} - H^2 \boldsymbol{A}\boldsymbol{Y}, \tag{1.19}$$

$$y_{n+1} = 2y_n - y_{n-1} - H^2 \boldsymbol{b}^{\boldsymbol{T}}\boldsymbol{Y}, \tag{1.20}$$

where

$$\boldsymbol{Y} = \begin{pmatrix} Y_1 \\ \vdots \\ Y_s \end{pmatrix}, \boldsymbol{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_s \end{pmatrix}, \boldsymbol{e} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \boldsymbol{A} = \begin{pmatrix} a_{11} & \dots & a_{1s} \\ \vdots & \ddots & \vdots \\ a_{s1} & \dots & a_{ss} \end{pmatrix}, \text{ and } \boldsymbol{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_s \end{pmatrix}.$$

By rearranging (1.19) we obtain

$$\boldsymbol{Y} = (\boldsymbol{I} + H^2\boldsymbol{A})^{-1}(\boldsymbol{e} + \boldsymbol{c})y_n - (\boldsymbol{I} + H^2\boldsymbol{A})^{-1} c y_{n-1} \tag{1.21}$$

where $(\boldsymbol{I} + H^2\boldsymbol{A})^{-1} \neq 0$.

Substituting (1.21) into (1.20), the following equation is obtained

$$y_{n+1} = \left(2 - H^2\boldsymbol{b}^{\boldsymbol{T}}(\boldsymbol{I} + H^2\boldsymbol{A})^{-1}(\boldsymbol{e} + \boldsymbol{c})\right)y_n - (1 - H^2\boldsymbol{b}^{\boldsymbol{T}}(\boldsymbol{I} + H^2\boldsymbol{A})^{-1}c)y_{n-1}. \tag{1.22}$$

We simplify equation (1.22) and obtain the following recursion relation which represents the stability polynomial of hybrid method as

$$\mathrm{P}(\xi, H) = \xi^2 - S(H^2)\xi + P(H^2) = 0 \tag{1.23}$$

where $$S(H^2) = 2 - H^2\boldsymbol{b}^{\boldsymbol{T}}(\boldsymbol{I} + H^2\boldsymbol{A})^{-1}(\boldsymbol{e} + \boldsymbol{c}) \quad (1.24)$$

$$P(H^2) = 1 - H^2\boldsymbol{b}^{\boldsymbol{T}}(\boldsymbol{I} + H^2\boldsymbol{A})^{-1}\boldsymbol{c}. \quad (1.25)$$

The numerical solution defined by the difference equation (1.23) should be periodic where the necessary conditions are

$$P(H^2) \equiv 1, \quad \text{and} \quad |S(H^2)| < 2, \quad \forall H \epsilon (0, H_p) \quad (1.26)$$

and interval $(0, H_p)$ is known as the periodicity interval of the method. The method is said to be zero dissipative $(d(H) = 0)$ if it satisfied conditions in (1.26). Otherwise, the method is said to have finite order of dissipation, the integration process is stable if the coefficients of polynomial in (1.26) satisfy the conditions

$$P(H^2) < 1, \quad \text{and} \quad |S(H^2)| < 1 + P(H^2), \quad \forall H \epsilon (0, H_s) \quad (1.27)$$

and interval $(0, H_s)$ is known as the interval of absolute stability of the method.

The analysis of phase-lag was firstly introduced by Bursa and Nigro (1980). The phase analysis can be divided into inhomogeneous and homogeneous test models. The inhomogeneous is describe when the phase error is constant in time, while for homogeneous the phase error are accumulated as n increases. As proposed by Franco (2006), the phase analysis is investigated using the second order homogeneous linear test model, $y''(x) = -\lambda^2 y(x)$. The steps to define phase analysis of hybrid method are the same as equations (1.14) to (1.25). Given that the exact solution for the homogeneous test equation

$$y'' = (i\lambda)^2 y(x), \quad (1.28)$$

which is

$$y(x_n) = 2|\varpi|\cos(X + nH). \quad (1.29)$$

The numerical solution of (1.4) is in the form of

$$y_n = 2|c||\rho|^n \cos(\omega + n\varphi). \quad (1.30)$$

This leads to the following definition by Van der Houwen and Sommeijer (1989).

Definition 1 (Apply the hybrid method (1.1) and (1.2) to (1.28)) Define the phase-lag $\phi(H) = H - \varphi$. If $\phi(H) = O(H^{q+1})$, then the hybrid method is said to be dispersive of order q. While, the quantity $d(H) = 1 - |\rho|$ is called as amplification error and if $d(H) = O(H^{r+1})$, then the hybrid method is said to have dissipation order r.

The error $\phi(H)$ and $d(H)$ are accumulated in the numerical process and therefore a cause of inaccuracy which leads to many integration steps to be performed. Hence, in this study we will focus on increasing the order of dispersion q (defined by $\phi(H) = O(H^{q+1})$) and the order of dissipation r (defined by $d(H) = O(H^{r+1})$).

Dispersion (phase-lag) is described as the angle between the exact and the approximated solution, while dissipation (amplification error) is the distance from a standard cyclic solution which can be seen in example in Figure 1.1.

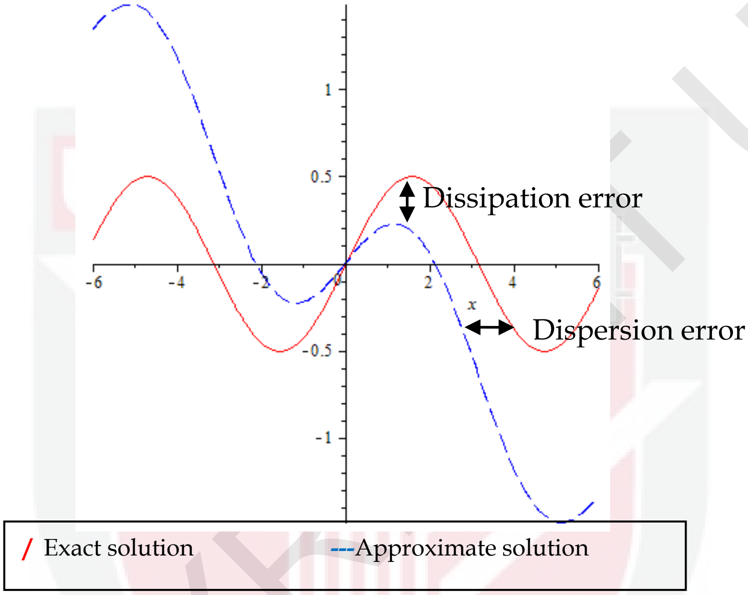

Figure 1.1: Example of the position of Dissipation and Dispersion errors

From Definition 1, it follows the nomenclature given by Van der Houwen and Sommeijer (1987)

$$\phi(H) = H - \cos^{-1}\left(\frac{S(H^2)}{2\sqrt{P(H^2)}}\right), \tag{1.31}$$

$$d(H) = 1 - \sqrt{P(H^2)}, \tag{1.32}$$

which defined as the dispersion error (phase-lag) and dissipation error (amplification error) respectively.

1.3 Problem Statement

We consider the solution of special second order ordinary differential equations (ODEs) and delay differential equations (DDEs) with oscillatory solutions directly using various hybrid methods. Usually, methods with higher possible order of dispersion and dissipation errors are derived using mathematical software such as Maples. Hence, various fitted techniques are applied to the numerical methods which produce methods with zero-dispersion and zero-dissipative properties. Consequently, these approaches should provide a significant improvement in accuracy of the solutions and produce less expensive methods.

We concern about developing methods with smaller error and require lesser time to execute. The conventional approach is by using block method which execute few solutions of the problems for each function evaluation. However, the existing block methods in the literature are usually not accurate for solving oscillating problems. We proposed two-point block explicit hybrid method which two solutions of the problems are evaluated simultaneously at each step. The approach required half of the execution time compare to the existing explicit hybrid method. We aim for the accuracy to be preserved.

It is possible to solve special second order ODEs for oscillatory solutions with linear multistep method. Due to the simplicity approach of the existing interpolation and collocation methods, we proposed linear multistep method with extra derivative by using collocation technique. Hence, application of the extra derivatives in the formulae as well as the fitted technique should provide a significant improvement in the accuracy.

1.4 The Objective of the Thesis

The main focus of this thesis is to develop numerical methods that are suitable and competent to solve special second order ODEs and DDEs with oscillating solutions. Various fitting techniques are taken into consideration when developing the new methods.

The objectives of this thesis are to develop:

1. Trigonometrically-fitted semi-implicit hybrid methods that are suitable for solving both ODEs and DDEs.
2. New phase-fitted explicit hybrid methods for solving ODEs.
3. Vanishing phase-lag and amplification error semi-implicit hybrid method for solving ODEs.
4. Block explicit hybrid methods for integrating ODEs.
5. Trigonometrically-fitted linear multi-step method with collocation technique using Chebyshev polynomial as the basis function for solving ODEs.

1.5 Outline of the Thesis

In this section, we provide a brief description of the thesis. Chapter 1 begins with the introduction of the order conditions and the theoretical analysis of dispersion and dissipation relations of semi-implicit hybrid methods. Chapter 2 provides the reviews of some of the previous works on the numerical solutions for ODEs and DDEs.

In Chapter 3, the derivation of three-stage fourth-order trigonometrically-fitted semi implicit hybrid method is presented. The method is used to solve both ODEs and DDEs for oscillatory problems. Extending the idea, the derivation of four-stage fifth-order trigonometrically-fitted semi implicit hybrid method for solving oscillatory ODEs and DDEs is presented in Chapter 4.

In Chapter 5, the analysis of phase-lag of order infinity is presented. New phase-fitted explicit hybrid methods of order five and six are developed based on the existing hybrid methods in the literature. The applications of the methods in comparison with existing fitted and non-fitted methods for solving oscillatory ODEs are shown.

Chapter 6 discussed the technique of nullifying the dispersion and dissipation error of a method. Semi implicit hybrid method of four-stage fifth-order with vanishing phase-lag and amplification error for solving highly oscillatory problems is derived in this chapter.

Chapter 7 begins with the construction of order conditions for block explicit hybrid method by using Taylor series expansion and its derivatives. We derive four and five order block explicit hybrid methods. Then the methods are trigonometrically-fitted. Numerical results illustrated the efficiency of the new method with existing methods in comparison.

In Chapter 8, we derive the linear multistep methods with extra derivative by using collocation technique. The method is developed using the sequence of Chebyshev polynomials as the basis function. Trigonometrically-fitting technique is adapted to the new method. The applications of these new linear multistep methods for solving oscillatory ODEs are shown.

Finally, conclusion of the thesis is given in Chapter 9 and future work is also recommended.

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