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ITERATIVE METHODS FOR SOLVING NONLINEAR EQUATIONS WITH MULTIPLE ZEROS

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By

NUR ALIF AKID JAMALUDIN

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Master of Science

May 2018

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DEDICATIONS

To:
my lovely parent,
Rokiah Mat Desa
&
Jamaludin Abd Hadi
and
my youngest brother,
Muhammad Alif Akmal Jamaludin

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Master of Science

ITERATIVE METHODS FOR SOLVING NONLINEAR EQUATIONS WITH MULTIPLE ZEROS

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NUR ALIF AKID JAMALUDIN

May 2018

Chairman : **Nik Mohd Asri Nik Long, PhD**
Faculty : **Science**

This thesis discusses the problem of finding the multiple zeros of nonlinear equations. Six two-step methods without memory are developed. Five of them posses third order convergence and an optimal fourth order of convergence. The optimal order of convergence is determined by applying the Kung-Traub conjecture. These method were constructed by modifying the Victory and Neta's method, Osada's method, Halley's method and Chebyshev's method. All these methods are free from second derivative function. Numerical computation shows that the newly modified methods performed better in term of error. The multiplicity of roots for the test functions have been known beforehand. Basin of attraction described that our methods have bigger choice of initial guess.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Master Sains

KAEDAH LELARAN UNTUK MENYELESAIKAN PERSAMAAN TAK LINEAR DENGAN PUNCA BERULANG

Oleh

NUR ALIF AKID JAMALUDIN

Mei 2018

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Tesis ini membincangkan masalah mencari punca berulang bagi persamaan tak linear. Enam kaedah lelaran dua langkah tanpa memori dibangunkan. Lima daripadanya mempunyai penumpuan darjah ketiga dan satu lagi penumpuan optimal darjah keempat. Penumpuan darjah optimal ditentukan oleh konjektur Kung-Traub. Kaedah ini dibina dengan ubah suai kaedah Victory dan Neta, kaedah Osada, kaedah Halley dan kaedah Chebyshev. Kaedah-kaedah ini adalah bebas daripada perbezaan funsi peringkat kedua. Pengiraan berangka menunjukan kaedah baharu yang di ubah suai adalah lebih baik dari segi ralat. Bilangan punca berulang untuk setiap fungsi ujian diketahui sebelumnya. Bekas tarikan menerangkan bahawa kaedah kami mempunyai pilihan nilai awal yang lebih besar.

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I certify that a Thesis Examination Committee has met on 4 May 2018 to conduct the final examination of Nur Alif Akid bin Jamaludin on his thesis entitled "Iterative Methods for Solving Nonlinear Equations with Multiple Zeros" in accordance with the Universities and University Colleges Act 1971 and the Constitution of the Universiti Putra Malaysia [P.U.(A) 106] 15 March 1998. The Committee recommends that the student be awarded the Master of Science.

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LIST OF ABBREVIATIONS

x_0	Initial approximation
x^*	Root or zeros
m	Multiplicity of roots
ρ	Order of convergence
τ	Number of evaluation of functions
ϕ	Iteration function
EI	Efficiency index
COC	Computational order of convergence
ACOC	Approximated computational order of convergence
PM1	Modified of Victory and Neta's method for 3^{rd} order convergence
PM2	Modified of Osada's method for 3^{rd} order convergence
PM3	Modified of Halley's method 1 for 3^{rd} order convergence
PM4	Modified of Halley's method 2 for 3^{rd} order convergence
PM5	Modified of Chebyshev's method for 3^{rd} order convergence
OM	Modified of Osada's method for optimal 4^{rd} order convergence

CHAPTER 1

INTRODUCTION

Solving root-finding problem of nonlinear equation is an important research work in the theory and practice, not only in applied mathematics, but also in many branches of engineering science, physics, computer sciences and others. Let $f(x)$ be real single-valued function. If $f(x^*) = 0$, then x^* is known as zero of $f(x)$ or root of the equation

$$f(x) = 0. \tag{1.1}$$

Assume that $f(x)$ has certain number of continuous derivative in the neighbourhood of zeros, x^*. Root of equation can be found analytically in some special cases. Finding roots problem are commonly solved by an approximation to the zero, x^* by introduce some iterative methods.

Let $x_k, x_{k-1}, ..., x_{k-n}$ be $n+1$ approximate to x^*. Let x_{k+1} be determined uniquely by the information obtained at points $x_k, x_{k-1}, ..., x_{k-n}$. Let the function that map $x_k, x_{k-1}, ..., x_{k-n}$ into x_{k+1} be denoted as ϕ. Thus

$$x_{k+1} = \phi(x_k, x_{k-1}, ..., x_{k-n}), \tag{1.2}$$

where ϕ is called as an iteration function.

1.1 Classification of iterative methods

Traub (1982) classified the iterative methods by the information requires for approximation to roots, x^*.

1. One-point iterative methods without memory

   Let x_{k+1} be determined by the only new information at x_k, which no previous information is reused.

$$x_{k+1} = \phi(x_k) \tag{1.3}$$

where ϕ is iteration function. One-point iterative method are mostly used for root-finding problem. The well-known method for one-point iterative method is Newton's method.

2. One-point iterative methods with memory

   Let x_{k+1} be defined a new approximate information at x_k and the information at $x_{k-1},...,x_{k-n}$ are reused. Thus

$$x_{k+1} = \phi(x_k;x_{k-1},...,x_{k-n}), \tag{1.4}$$

where semicolon in (1.4) is used to separate from the new data are used from the point at which previous information are reused. The most popular iterative method for one-point method with memory is secant method.

3. Multipoint iterative method without memory

   Let x_{k+1} be determined by the new information at $x_k, w_i(x_k),...,w_n(x_k)$, $k \geqslant 1$, only one new information from previous iteration are used and no old information is reused,

$$x_{k+1} = \phi(x_k, w_i(x_k),...,w_n(x_k)) \tag{1.5}$$

where ϕ is called a multipoint iteration function without memory. There are extensive applications on multipoint iterative method without memory in approximation at x^*, for example Ostrowski (1966), Schröder and Stewart (1998) and Homeier (2009).

4. Multipoint iterative method with memory

   Let z_j represents the $i+1$ quantities $x_j, w_1(x_j),...,w_k(x_j),(k \geqslant 1)$. Thus

$$x_{k+1} = \phi(z_k;z_{k-1},...,z_{k-n}), \tag{1.6}$$

where ϕ is said to be multipoint iteration function with memory.

1.2 Initial Approximation

For any iterative methods for solving nonlinear equation, $f(x) = 0$ must require the knowledge of initial approximation, x_0 in finding the zeros, x^*. Thus, the initial guess, x_0 needs to be chosen properly and close to the sought of zeros for having good convergence of any iterative methods either for one-point iterative methods as well multipoint iterative methods. The advantage of the choice of initial approximation is become more important if we applied on high-order iterative methods due to sensitivity of perturbation.

1.3 Efficiency Index

Let τ be the number of function evaluations per iteration and ρ be an order of methods. Kung and Traub (1974) defined the information efficiency or coefficient of efficiency of an iterative method by the ratio

$$IE = \frac{\rho}{\tau}. \tag{1.7}$$

Other definition was introduced by Ostrowski (1966), which is called the efficiency index, written as

$$EI = \rho^{\frac{1}{\tau}} = \sqrt[\tau]{\rho}. \tag{1.8}$$

Kung and Traub (1974) gave a more realistic estimation of the computational efficiency by defining the computational cost as $\tau = \sum \tau^{(j)}$, where $\tau^{(j)}$ is the computational cost of f and its derivatives $f^{(j)}(f^{(0)} \equiv f)$. The computational cost τ is usually expressed by the number of evaluations of the function and its derivatives. The main intention of establishing a new method is to obtain a method with the best possible efficiency index. It is fascinating to attain as high as possible convergence order with fixed number of function evaluation per iteration. For the case of multipoint method without memory this interest is closely related to the optimal order of convergence considered in the Kung-Traub conjecture (Kung and Traub, 1974).
Kung-Traub's conjecture: Multipoint iterative methods without memory, demanding $n+1$ function evaluations per iteration, have order of convergence at most 2^n.
According to Kung-Traub conjecture the optimal efficiency index is equal to

$$EI_n^{(optimal)} = 2^{\left(\frac{n}{n+1}\right)}.$$

1.4 Computational Order of Convergence

The convergence rate is defined by the order of convergence. Computional Order of Convergence (COC) is use to check the order of convergence of an iterative method during its practical implementation and estimate how much it differs from the theoretical order.
Weerakoon and Fernando (2000) introduced the formula for calculation of COC, as

$$COC \approx \frac{\ln |(x_{k+1} - x^*) / (x_k - x^*)|}{\ln |(x_k - x^*) / (x_{k-1} - x^*)|}. \tag{1.9}$$

The COC has been used in many papers to test numerically the order of convergence of new methods whose order has been previously studied theoretically, such as Bi et al. (2009), Ferrara et al. (2015) and Sharifi et al. (2016). Another approach that avoid the use of unknown zero x^* was studied by Grau-Sánchez et al. (2010) by introducing a more realistic relationship, approximated computational order convergence (ACOC) defined by

$$ACOC \approx \frac{\ln |(x_{k+1} - x_k) / (x_k - x_{k-1})|}{\ln |(x_k - x_{k-1}) / (x_{k-1} - x_{k-2})|}. \tag{1.10}$$

1.5 One-point methods

1.5.1 One-point iterative methods for simple root

The most popular one-point iterative method for solving nonlinear equation is Newton's method or also known as Newton-Raphson's method (Petkovic et al., 2012) written as

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}; (k = 1, 2, 3, ...), \tag{1.11}$$

which is quadratically convergence. The new approximation of x_{k+1} to the root of x^* is produced by the tangent line of $f(x_k)$ at point x_k. We assume

$$f'(x_k) = \frac{f(x_{k+1}) - f(x_k)}{x_{k+1} - x_k}, \tag{1.12}$$

when $f(x_{k+1}) = 0$, yield method (1.11).

For the small values of h the approximation

$$f'(x) \approx \frac{f(x+h)-f(x)}{h}, \tag{1.13}$$

holds. By choosing two consecutive points approximation x_{k-1} and x_k the approximation to the first derivative in (1.13) becomes

$$f'(x_k) \approx \frac{f(x_k)-f(x_{k-1})}{x_k-x_{k-1}}, \tag{1.14}$$

substitutes (1.14) into (1.11) yields

$$x_{k+1} = x_k - \frac{x_k - x_{k-1}}{f(x_k)-f(x_{k-1})} f(x_k), \tag{1.15}$$

which is second order convergence method. This method also known as secant method (Petkovic et al., 2012). Method (1.15) is the example of iterative method which is free from any derivative functions.
The well known Halley's method (Petkovic et al., 2012) is the example of third order iterative method for simple root, given by

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \frac{1}{1-\frac{f(x_k)f''(x_k)}{2f'(x_k)^2}}. \tag{1.16}$$

From method (1.16) the famous Chebyshev's method (Petkovic et al., 2012) is given by

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}\Big(1+\frac{f(x_k)f''(x_k)}{2f'(x_k)^2}\Big), \tag{1.17}$$

which cubically convergence.

1.5.2 Iterative method by using Traub's relation

Let x^* be a root of multiplicity m of function f to generalize the basic sequence of root solver. The following difference-differential recurrence relation are derived by Traub (1982) which is given by

$$\phi_{\rho+1}(x) = \phi_\rho(x) - \frac{m}{\rho}\frac{f(x)}{f'(x)}\phi'_\rho(x), \ldots \tag{1.18}$$

where $\phi_\rho(x)$ determined as an iterative method of order ρ.

Theorem 1.1 *(Petkovic et al., 2012) Let $\phi_\rho(x)$ be an iteration function which defines the method $x_{k+1} = \phi_\rho(x_k)(k = 0, 1, \ldots)$ of order ρ for finding a simple or multiple zeros of the given sufficiently differentiable function f. Then the iterative method*

$$\phi_{\rho+1}(x_k) = \phi_\rho(x_k) - \frac{m}{\rho}\frac{f(x_k)}{f'(x_k)}\phi'_\rho(x)\ (\rho \geq)2, (k = 0, 1, 2, \ldots) \tag{1.19}$$

arise from (1.18), has the order of convergence $\rho + 1$

The example of method that apply Traub's relation is third-order Halley-like method (Petkovic et al., 2012) for multiple roots, $H(x)$ as in equation (2.5) . By finding the derivative of $H'(x)$ in (2.5) and equation (1.18), we have the following formulae

$$H_4(x) = H(x) - \frac{m}{3}\frac{f(x_k)}{f'(x_k)}H'(x). \tag{1.20}$$

For simplicity, let $u = \dfrac{f(x)}{f'(x)}$ and $C_j = \dfrac{f^j(x)}{j!f'(x)}$ $(j = 2, 3, \ldots)$. Hence the fourth-order method (1.20) becomes

$$H_4(x) = x - \frac{mu(7 + 6m - m^2 - m^2 - 12muC_2 + 12m^2u^2(C_2^2 - C_3))}{3(m + 1 - 2muC_2)^2} \tag{1.21}$$

The construction of higher-order method using H_4 can be proceed but these iterative method will becomes slightly large and more complex form.

1.6 Multipoint methods

1.6.1 Composite multipoint method

This type of construction methods is defined in Traub (1982). The first example of composition method is Newton-Halley's method (Petkovic et al., 2012) which written as

$$\begin{cases} y_k &= x_k - \dfrac{f(x_k)}{f'(x_k)}, \\ x_{k+1} &= y_k - \dfrac{f(x_k)}{f'(x_k) - \dfrac{f(y_k)f''(y_k)}{2f'(y_k)}}, \end{cases} \tag{1.22}$$

The efficiency index (1.8) of Newton's method, (1.11), Halley's method, (1.16) and the Newton-Halley's method, (1.22) are

$$EI_N = \sqrt[2]{2} = 1.414, \tag{1.23}$$

$$EI_H = \sqrt[3]{3} = 1.442 \tag{1.24}$$

and

$$EI_{NH} = \sqrt[5]{6} = 1.431, \tag{1.25}$$

respectively. From the value of efficiency index, shows that the composite method (1.22) is not example of succeseful two-point method, because for construction of composite iterative method, it necessary for method to have decrease the number of function evaluation, inducing the increase of computational efficiency.
The other example of composite method is the combination of Newton (1.11) and secant methods (1.15), yields Newton-secant's method (Petkovic et al., 2012)

$$x_{k+1} = x_k - \frac{\frac{f(x_k)}{f'(x_k)} f(x_k)}{f(x_k) - f(x_k - \frac{f(x_k)}{f'(x_k)})}, \tag{1.26}$$

which is cubically convergence and its require three function evaluations per step. Thus, the efficiency index of Newton-secant's method (1.26) is $EI_{NS} = \sqrt[3]{3} = 1.442$, which is greater than efficiency index of Newton-Halley's method (1.22). Note that Newton-secant's methods (1.26) are not an optimal methods in the sense of Kung-Traub conjecture, which assume that for three function evaluations should provide the optimal order four (see Section 1.3). The application of composite method for multiple roots is applied by Chun et al. (2009).

1.6.2 Optimal two-point method of Jarratt's type

Construction of optimal two-point method is using Jarratt's scheme in Jarratt (1966). Let define the functions

$$w_1(x) = \frac{f(x)}{f'(x)},\ w_2(x) = \frac{f(x)}{f'\big(x+\beta w_1(x)\big)}.$$

Traub (1982) showed that the iterative formulae

$$x_{k+1} = x_k - a_1 w_1(x_k) - a_2 w_2(x_k), \tag{1.27}$$

which have cubic convergence and required one evaluation of $f(x)$ and two of $f'(x)$ per iteration. In order to obtain fourth order convergence, Jarratt (1966) presented the similar class of iterative methods of the form

$$x_{k+1} = x_k - \phi_1(x_k) - \phi_2(x_k), \tag{1.28}$$

where

$$\phi_1(x) = a_1 w_1(x) - a_2 w_2(x),$$
$$\phi_2(x) = \frac{f(x)}{b_1 f'(x) + b2 f'(x+\beta w_1(x))}.$$

Jarratt (1966) successfully increased the order of convergence (1.27) from three to four without additional function evaluation. This type of construction of optimal two-point method for simple root are extensively apply in Kou et al. (2007), Chun and Ham (2008), Basu (2008) and Sharma et al. (2009).

1.6.3 Jarratt-like method for optimal two-point method

Let consider a two-point method in the form

$$
\begin{cases} y_k &= x_k - \frac{2}{3}\frac{f(x_k)}{f'(x_k)}, \\ x_{k+1} &= x_k - Q(t_k)\frac{f(x_k)}{f'(x_k)}, \end{cases} \tag{1.29}
$$

where $Q(t_k)$ is weight function to be determined the fourth order convergence of method (1.29). The weight function $Q(t_k)$ be approximated by its Taylor's polynomial of third degree at the point $t = 1$, given by

$$
Q(t) \approx Q(1) + Q'(1)(t-1) + \frac{Q''(1)}{2}(t-1)^2 + \frac{Q'''(1)}{6}(t-1)^3;\ t = \frac{f'(y)}{f'(x)}. \tag{1.30}
$$

Let $e_k = x_{k+1} - x^*$, $C_j = \frac{f^j(x)}{j!f'(x)}$ $(j = 2,3,...)$ and substitute (1.30) into (1.29) yields

$$
\begin{aligned} e_{k+1} &= (1-Q(1))e_k + C_2\big(Q(1) + \frac{4}{3}Q'(1)\big)e_k^2 - \frac{2}{9}\Bigg(9C_2^2Q(1) - 9C_3Q(1) \\ &+ 24C_2^2Q'(1) - 12C_3Q'(1) + 4C_2^2Q''(1)\Bigg) + \frac{1}{81}\Bigg(324C_2^3Q(1) \\ &- 567C_2C_3Q(1) + 243C_4Q(1) + 1404C_2^3Q'(1) - 1512C_2C_3Q'(1) \\ &+ 312C_4Q'(1) + 504C_2^3Q''(1) - 288C_2C_3Q''(1) + 32C_2^3Q'''(1)\Bigg)e_k^4 + O(e_k^5). \end{aligned} \tag{1.31}
$$

In order the method of (1.29) to be fourth order convergence, then the coefficient in (1.31) of e_k, e_k^2 and e_k^3 must be vanished. These condition are satisfied when the weight function $Q(t)$ in (1.29) have the following properties :

$$
Q(1) = 1,\ Q'(1) = -\frac{3}{4},\ Q''(1) = \frac{9}{4},\ |Q'''(1) < \infty|
$$

Thus, the error term (1.29) becomes

$$
e_{k+1} = \Big(-C_2C_3 + \frac{1}{9}C_4 + C_2^3\big(5 + \frac{32}{81}Q'''(1)\big)\Big)e_k^4 + O(e_k^5). \tag{1.32}
$$

1.7 Objectives

The main objectives of the research are :

- to modified a two-point iterative methods of nonlinear equation for multiple zeros which is free from second derivative functions.
- to attain the third and fourth order of convergence iterative method for computing multiple zeros.
- to obtain numerical result of modified methods by using test functions.
- to investigate the modified methods in term of availability in choosing initial guess by using basin of attraction.
- to compare the developed methods with the other existing iterative methods in term of numerical performance and basin of attraction with same order of convergence.

1.8 Scopes of the research

We are dealing with solving the nonlinear equation of multiple roots. The approximation of the roots are based on the iterative method without memory, which is only used some new informations and no old information been reused per iteration.

We apply the proposed methods and other existing methods with several test functions with the known of their multiplicity roots. The calculation of absolute error have been used to observe the convergence behaviour of those methods. We introduce basin of attraction for those method to observe the accessibility of initial approximation.

REFERENCES

Basu, D. (2008). From third to fourth order variant of newtons method for simple roots. *Applied Mathematics and Computation*, 202(2):886–892.

Behl, R., Cordero, A., Motsa, S. S., Torregrosa, J. R., and Kanwar, V. (2016). An optimal fourth-order family of methods for multiple roots and its dynamics. *Numerical Algorithms*, 71(4):775–796.

Bi, W., Ren, H., and Wu, Q. (2009). Three-step iterative methods with eighth-order convergence for solving nonlinear equations. *Journal of Computational and Applied Mathematics*, 225(1):105–112.

Bodewig, E. (1946). Sur la méthode laguerre pour lapproximation des racines de certaines équations algébriques et sur la critique dhermite. *Indag. Math*, 8:570–580.

Chun, C., Bae, H. J., and Neta, B. (2009). New families of nonlinear third-order solvers for finding multiple roots. *Computers and Mathematics with Applications*, 57(9):1574–1582.

Chun, C. and Ham, Y. (2008). Some second-derivative-free variants of super-halley method with fourth-order convergence. *Applied Mathematics and Computation*, 195(2):537–541.

Dong, C. (1982). A basic theorem of constructing an iterative formula of the higher order for computing multiple roots of an equation. *Mathematics Numerical Sinica*, 11:445–450.

Dong, C. (1987). A family of multipoint iterative functions for finding multiple roots of equations. *International Journal of Computer Mathematics*, 21(3-4):363–367.

Ferrara, M., Sharifi, S., and Salimi, M. (2015). Computing multiple zeros by using a parameter in Newton-Secant method. *SeMA Journal*, pages 1–9.

Geum, Y. H., Kim, Y. I., and Neta, B. (2015). A class of two-point sixth-order multiple-zero finders of modified double-newton type and their dynamics. *Applied Mathematics and Computation*, 270:387–400.

Geum, Y. H., Kim, Y. I., and Neta, B. (2018). Constructing a family of optimal eighth-order modified newton-type multiple-zero finders along with the dynamics behind their purely imaginary extraneous fixed points. *Journal of Computational and Applied Mathematics*, 333:131–156.

Grau-Sánchez, M., Noguera, M., and Gutiérrez, J. M. (2010). On some computational orders of convergence. *Applied Mathematics Letters*, 23(4):472–478.

Hansen, E. and Patrick, M. (1976). A family of root finding methods. *Numerische Mathematik*, 27(3):257–269.

Hazrat, R. (2010). *Mathematica: a problem-centered approach*. Springer.

Homeier, H. H. H. (2009). On Newton-type methods for multiple roots with cubic convergence. *Journal of Computational and Applied Mathematics*, 231(1):249–254.

Jarratt, P. (1966). Some fourth order multipoint iterative methods for solving equations. *Mathematics of Computation*, 20(95):434–437.

Jarratt, P. (1969). Some efficient fourth order multipoint methods for solving equations. *BIT Numerical Mathematics*, 9(2):119–124.

Kim, Y. I. and Geum, Y. H. (2013). A two-parameter family of fourth-order iterative methods with optimal convergence for multiple zeros. *Journal of Applied Mathematics*, 2013.

Kou, J., Li, Y., and Wang, X. (2007). Fourth-order iterative methods free from second derivative. *Applied mathematics and computation*, 184(2):880–885.

Kung, H. T. and Traub, J. F. (1974). Optimal order of one-point and multipoint iteration. *Journal of the ACM (JACM)*, 21(4):643–651.

Li, S., Cheng, L. Z., and Neta, B. (2010). Some fourth-order nonlinear solvers with closed formulae for multiple roots. *Computers and Mathematics with Applications*, 59(1):126–135.

Li, S., Li, H., and Cheng, L. (2009a). Some second-derivative-free variants of halleys method for multiple roots. *Applied Mathematics and Computation*, 215(6):2192–2198.

Li, S., Liao, X., and Cheng, L. (2009b). A new fourth-order iterative method for finding multiple roots of nonlinear equations. *Applied Mathematics and Computation*, 215(3):1288–1292.

Liu, B. and Zhou, X. (2013). A new family of fourth-order methods for multiple roots of nonlinear equations. *Nonlinear Analysis: Modelling and Control*, 18(2):143–152.

Loney, N. W. (2016). *Applied mathematical methods for chemical engineers*. CRC Press, London.

Neta, B. (2008). New third order nonlinear solvers for multiple roots. *Applied Mathematics and Computation*, 202(1):162–170.

Osada, N. (1994). An optimal multiple root-finding method of order three. *Journal of Computational and Applied Mathematics*, 51(1):131–133.

Ostrowski, A. M. (1966). *Solution of equations and systems of equations*, volume 9. Academic Press, New York.

Petkovic, M., Neta, B., Petkovic, L., and Dzunic, J. (2012). *Multipoint methods for solving nonlinear equations*. Academic Press, London.

Potra, F. A. and Pták, V. (1984). *Nondiscrete induction and iterative processes*, volume 103. Pitman Advanced Publishing Program.

Schröder, E. and Stewart, G. W. (1998). On infinitely many algorithms for solving equations. *Translation of paper by E. Schröder*, UMIACS-TR-92-121.

Sharifi, S., Ferrara, M., Nik Long, N. M. A., and Salimi, M. (2015). Modified Potra-Pták method to determine the multiple zeros of nonlinear equations. *arXiv preprint arXiv:1510.00319*.

Sharifi, S., Salimi, M., Siegmund, S., and Lotfi, T. (2016). A new class of optimal four-point methods with convergence order 16 for solving nonlinear equations. *Mathematics and Computers in Simulation*, 119:69–90.

Sharma, J. R., Guha, R. K., and Sharma, R. (2009). Some variants of hansen–patrick method with third and fourth order convergence. *Applied Mathematics and computation*, 214(1):171–177.

Sharma, J. R. and Sharma, R. (2010). Modified jarratt method for computing multiple roots. *Applied Mathematics and Computation*, 217(2):878–881.

Traub, J. F. (1982). *Iterative methods for the solution of equations*. American Mathematical Soc.

Victory Jr, H. D. and Neta, B. (1983). A higher order method for multiple zeros of nonlinear functions. *International Journal of Computer Mathematics*, 12(3-4):329–335.

Weerakoon, S. and Fernando, T. G. I. (2000). A variant of Newton's method with accelerated third-order convergence. *Applied Mathematics Letters*, 13(8):87–93.

Zhou, X., Chen, X., and Song, Y. (2011). Constructing higher-order methods for obtaining the multiple roots of nonlinear equations. *Journal of Computational and Applied Mathematics*, 235(14):4199–4206.

Zhou, X., Chen, X., and Song, Y. (2013). Families of third and fourth order methods for multiple roots of nonlinear equations. *Applied Mathematics and Computation*, 219(11):6030–6038.