UNIVERSITI PUTRA MALAYSIA

DIRECT BLOCK BACKWARD DIFFERENTIATION FORMULAS FOR SOLVING SECOND ORDER STIFF BOUNDARY VALUE PROBLEM

CHEW SOON YUEN

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DIRECT BLOCK BACKWARD DIFFERENTIATION FORMULAS FOR SOLVING SECOND ORDER STIFF BOUNDARY VALUE PROBLEM

By

CHEW SOON YUEN

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Master of Science

June 2016

DEDICATION

To

My Friends

and

My Lovely Parents

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Master of Science

DIRECT BLOCK BACKWARD DIFFERENTIATION FORMULAS FOR SOLVING SECOND ORDER STIFF BOUNDARY VALUE PROBLEM

By

CHEW SOON YUEN

June 2016

Chairman: Associate Professor Zarina Bibi Ibrahim, PhD

Faculty: Science

In this thesis, the direct method of Block Backward Differentiation Formula (BBDF) for solving two point boundary value problems (BVPs) directly was studied. The shooting technique will be implemented using constant step size. In order to overcome the numerical instabilities due to round off or truncation errors that occur in solving BVPs, the BBDF method will be adapted with multiple shooting techniques. Newton-Raphson method is also considered as a procedure for solving the second order BVPs.

Existing strategy for solving BVPs, is by reducing them to a system of first order ordinary differential equations (ODEs). This approach is well established but obviously it will enlarge the problem into a system of first order equations. However, the BBDF method in this thesis solves BVPs directly without reducing them to their first order differential equations. Besides that, the BBDF method can produce two approximate solutions at two points in each step. Furthermore, the BBDF method allows the differentiation coefficients to be stored and thus reduces the computational cost.

Another main focus in this thesis is to solve stiff BVPs, where more computational efforts are required to evaluate the Jacobian and solving the linear systems. Furthermore, stiff BVPs are difficult to solve due to the restriction on the step size of many numerical methods, except those with A-stability properties. Therefore, the BBDF method in this thesis will be used to solve stiff BVPs directly.

The source codes are written in C language and executed using MATLAB. Some numerical examples are given to illustrate the efficiency of the method.

Numerical results showed that the BBDF method manages to give acceptable results in terms of maximum error, number of iterations and execution time.

In conclusion, the proposed BBDF method in this thesis is suitable for solving directly the second order BVPs.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Master Sains

BLOK FORMULASI BEZA KE BELAKANG LANGSUNG UNTUK MENYELESAIKAN MASALAH NILAI SEMPADAN KAKU PERINGKAT KEDUA

Oleh

CHEW SOON YUEN

Jun 2016

Pengerusi: Profesor Madya Zarina Bibi Ibrahim, PhD

Fakulti: Sains

Di dalam tesis ini, kaedah Blok Formulasi Beza Ke Belakang (BFBB) untuk menyelesaikan masalah nilai sempadan dua titik (MNS) secara langsung telah dikaji. Teknik penembakan akan dilaksanakan dengan menggunakan saiz langkah malar. Dalam usaha untuk mengatasi ketidakstabilan berangka disebabkan oleh ralat pembundaran atau ralat pangkasan yang berlaku dalam menyelesaikan MNS, kaedah BFBB akan disesuaikan dengan teknik penembakan berganda. Kaedah Newton Raphson juga telah dipertimbangkan sebagai suatu prosedur untuk menyelesaikan MNS peringkat kedua.

Strategi yang sedia ada untuk menyelesaikan MNS, adalah dengan menurunkan masalah kepada sistem persamaan pembezaan biasa (PPB) peringkat pertama. Cara ini adalah mantap tetapi jelasnya ia akan membesarkan masalah kepada sistem persamaan peringkat pertama. Walau bagaimanapun, kaedah BFBB di dalam tesis ini menyelesaikan MNS secara langsung tanpa menurunkan MNS kepada persamaan pembezaan peringkat pertama. Selain itu, kaedah BFBB boleh menghasilkan dua penyelesaian hampiran pada dua titik dalam setiap langkah. Tambahan pula, kaedah BFBB membolehkan pekali pembezaan disimpan dan dengan ini mengurangkan kos pengiraan.

Fokus utama lain dalam tesis ini adalah untuk menyelesaikan MNS kaku, di mana lebih banyak usaha pengiraan dikehendaki untuk menilai Jakobian dan menyelesaikan sistem linear. Tambahan pula, MNS kaku sukar untuk diselesaikan kerana sekatan ke atas saiz langkah untuk kebanyakan kaedah berangka, kecuali kaedah yang mempunyai ciri-ciri kestabilan-A. Oleh itu, kaedah BFBB di dalam tesis ini akan digunakan untuk menyelesaikan MNS kaku secara langsung.

Kod sumber ditulis dalam bahasa C dan dilaksanakan dengan menggunakan MATLAB. Beberapa contoh berangka diberikan untuk menggambarkan kecekapan kaedah. Hasil berangka menunjukkan bahawa kaedah BFBB berjaya memberikan hasil yang boleh diterima dari segi ralat maksimum, bilangan lelaran dan masa pelaksanaan.

Kesimpulannya, kaedah BFBB yang diusulkan di dalam tesis ini adalah sesuai bagi menyelesaikan MNS peringkat kedua secara langsung.

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I certify that a Thesis Examination Committee has met on 15 June 2016 to conduct the final examination of Chew Soon Yuen on his thesis entitled "Direct Block Backward Differentiation Formulas for Solving Second Order Stiff Boundary Value Problem" in accordance with the Universities and University Colleges Act 1971 and the Constitution of the Universiti Putra Malaysia [P.U.(A) 106] 15 March 1998. The Committee recommends that the student be awarded the Master of Science.

Members of the Thesis Examination Committee were as follows:

Mohamad Rushdan b Md Said, PhD
Associate Professor
Faculty of Science
Universiti Putra Malaysia
(Chairman)

Leong Wah June, PhD
Associate Professor
Faculty of Science
Universiti Putra Malaysia
(Internal Examiner)

Mohammad Khatim Hasan, PhD
Associate Professor
Faculty of Information Science and Technology
University Kebangsaan Malaysia
(External Examiner)

ZULKARNAIN ZAINAL, PhD
Professor and Deputy Dean
School of Graduate Studies
Universiti Putra Malaysia

Date: 26 July 2016

This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Master of Science. The members of Supervisory Committee were as follows:

Zarina Bibi Ibrahim, PhD
Associate Professor
Faculty of Science
Universiti Putra Malaysia
(Chairman)

Norazak Bin Senu, PhD
Associate Professor
Faculty of Science
Universiti Putra Malaysia
(Member)

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School of Graduate Studies
Universiti Putra Malaysia

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Declaration by Graduate Student

I hereby confirm that:

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Declaration by Members of Supervisory Committee

This is to confirm that:

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Signature: ______________________________
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LIST OF ABBREVIATIONS

BVPs	Boundary Value Problems
BBDF	Block Backward Differentiation Formula
IVPs	Initial Value Problems
ODEs	Ordinary Differential Equations
RK4	Runge-Kutta method of order four via simple shooting technique.
1PSN4	One point direct method of order four via simple shooting technique adapted with Newton-Raphson method.
2PSN4	Two point direct method of order four via simple shooting technique adapted with Newton-Raphson method.
SBBDF	Block Backward Differentiation Formulas via simple shooting technique adapted with Newton-Raphson method.
SBDF	Classical one-point three-step Backward Differentiation Formula via simple shooting technique.
MBBDF	Block Backward Differentiation Formulas via multiple shooting technique adapted with Newton-Raphson method.
MBDF	Classical one-point three-step Backward Differentiation Formula via multiple shooting technique.
TFD	Fourth order tridiagonal finite difference method.
FTFD	Fitted fourth order tridiagonal finite difference method.
SONM	Seventh order numerical method.

CHAPTER 1

INTRODUCTION

1.1 Introduction

Boundary value problems (BVPs) arise in many areas of science and engineering including the modelling of chemical reactions, vibration problems and space technology. Problems in civil engineering such as deflection of a beam can be also formulated in terms of BVPs. For example, let us consider a simply supported beam with modulus of elasticity E, moment of inertia I, a uniform load w, length of the beam L and end tension T. If $y(x)$ denotes the deflection at each point x in the beam, then $y(x)$ satisfies the differential equation

$$y'' - \frac{T}{EI}y = \frac{wx(x-L)}{2EI}$$

with boundary condition $y(0) = y(L) = 0$. See Burden (1993) for details. Because of their importance, many algorithms have been proposed to find the solution of the BVPs.

There exist a large number of numerical methods to compute the solutions of BVPs. Among those methods, there are two main approaches for solving BVPs: indirect method and direct method. In indirect method, the higher order BVPs are reduced to an equivalent system of first order differential equations and then solved with numerical method which is computationally expensive. On the other hand, the two point direct multistep method in this thesis will be utilized to solve BVPs directly without reducing them to first order differential equations. Besides that, the two point direct method produce two approximate solutions at two points in each integrate step. Furthermore, the simple shooting method and multiple shooting techniques will be implemented to this direct method for solving BVPs.

1.2 Objective of the Thesis

The aim of this thesis is to propose a method for solving second order BVPs. The research objectives are:

1. to implement simple shooting technique in two point Block Backward Differentiation Formula (BBDF) developed by Ibrahim (2006) for solving second order BVPs directly using constant step size.

2. to implement multiple shooting technique in two point BBDF for solving second order BVPs directly using constant step size.

3. to develop the algorithm of two point BBDF for solving second order BVPs directly using constant step size.

4. to discuss and analyse the numerical solutions obtained by BBDF and BDF.

1.3 Scope of Study

This research is carried out to study a second order BVPs of the form as follows:

$$y'' = f(x, y, y'), \quad a \leq x \leq b \tag{1.1}$$

with Dirichlet boundary condition

$$y(a) = \alpha, \quad y(b) = \beta \tag{1.2}$$

where a, b, α, β are constants. The scopes of this research are:

i. To study the approach for solving second order BVPs using two point direct BBDF method with two type of shooting technique, $i.e.$ simple shooting technique and multiple shooting techniques.

ii. To present numerical results of BBDF codes using constant step size for solving second order BVPs.

1.4 Outline of the Thesis

This thesis is primarily divided into five chapters. In Chapter I, the objective of the thesis and scope of study have been stated.

A brief introduction to BVPs is given at the beginning of Chapter II. Existence theory and the definition of stiffness were given. This chapter also includes a review of earlier research on BVPs.

Chapter III presents a review on the derivation of the two point BBDF. Algorithm to solve BVPs using BBDF via simple shooting technique adapted with Newton Raphson method was developed. The numerical results and comparison with existed method were included.

Chapter IV explains the adaption of multiple shooting technique and an algorithm to solve BVPs directly using BBDF via multiple shooting technique adapted with Newton-Raphson method was developed. Numerical results and discussion were included.

The conclusions and suggestion for the future research on solving BVPs using BBDF were provided in Chapter V.

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