UNIVERSITI PUTRA MALAYSIA

DIRECT BLOCK BACKWARD DIFFERENTIATION FORMULAS FOR SOLVING SECOND ORDER STIFF BOUNDARY VALUE PROBLEM

CHEW SOON YUEN

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DIRECT BLOCK BACKWARD DIFFERENTIATION FORMULAS FOR SOLVING SECOND ORDER STIFF BOUNDARY VALUE PROBLEM

By

CHEW SOON YUEN

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Master of Science

June 2016
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DEDICATION

To

My Friends

and

My Lovely Parents
Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Master of Science

DIRECT BLOCK BACKWARD DIFFERENTIATION FORMULAS FOR SOLVING SECOND ORDER STIFF BOUNDARY VALUE PROBLEM

By

CHEW SOON YUEN

June 2016

Chairman: Associate Professor Zarina Bibi Ibrahim, PhD

Faculty: Science

In this thesis, the direct method of Block Backward Differentiation Formula (BBDF) for solving two point boundary value problems (BVPs) directly was studied. The shooting technique will be implemented using constant step size. In order to overcome the numerical instabilities due to round off or truncation errors that occur in solving BVPs, the BBDF method will be adapted with multiple shooting techniques. Newton-Raphson method is also considered as a procedure for solving the second order BVPs.

Existing strategy for solving BVPs, is by reducing them to a system of first order ordinary differential equations (ODEs). This approach is well established but obviously it will enlarge the problem into a system of first order equations. However, the BBDF method in this thesis solves BVPs directly without reducing them to their first order differential equations. Besides that, the BBDF method can produce two approximate solutions at two points in each step. Furthermore, the BBDF method allows the differentiation coefficients to be stored and thus reduces the computational cost.

Another main focus in this thesis is to solve stiff BVPs, where more computational efforts are required to evaluate the Jacobian and solving the linear systems. Furthermore, stiff BVPs are difficult to solve due to the restriction on the step size of many numerical methods, except those with A-stability properties. Therefore, the BBDF method in this thesis will be used to solve stiff BVPs directly.

The source codes are written in C language and executed using MATLAB. Some numerical examples are given to illustrate the efficiency of the method.
Numerical results showed that the BBDF method manages to give acceptable results in terms of maximum error, number of iterations and execution time.

In conclusion, the proposed BBDF method in this thesis is suitable for solving directly the second order BVPs.
Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Master Sains

BLOK FORMULASI BEZA KE BELAKANG LANGSUNG UNTUK MENYELESAIKAN MASALAH NILAI SEMPADAN KAKU PERINGKAT KEDUA

Oleh

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Kesimpulannya, kaedah BFBB yang diusulkan di dalam tesis ini adalah sesuai bagi menyelesaikan MNS peringkat kedua secara langsung.
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I also wish to express my thanks to my friends for their continuous support and help. Special thanks to my dearest parents for their continuous encouragement.
I certify that a Thesis Examination Committee has met on 15 June 2016 to conduct the final examination of Chew Soon Yuen on his thesis entitled "Direct Block Backward Differentiation Formulas for Solving Second Order Stiff Boundary Value Problem" in accordance with the Universities and University Colleges Act 1971 and the Constitution of the Universiti Putra Malaysia [P.U.(A) 106] 15 March 1998. The Committee recommends that the student be awarded the Master of Science.

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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABSTRACT</td>
<td>i</td>
</tr>
<tr>
<td>ABSTRAK</td>
<td>iii</td>
</tr>
<tr>
<td>ACKNOWLEDGEMENT</td>
<td>v</td>
</tr>
<tr>
<td>APPROVAL</td>
<td>vi</td>
</tr>
<tr>
<td>DECLARATION</td>
<td>viii</td>
</tr>
<tr>
<td>LIST OF TABLES</td>
<td>xii</td>
</tr>
<tr>
<td>LIST OF FIGURES</td>
<td>xiv</td>
</tr>
<tr>
<td>LIST OF ABBREVIATIONS</td>
<td>xv</td>
</tr>
</tbody>
</table>

## CHAPTERS

1. **INTRODUCTION**
   1.1 Introduction
   1.2 Objective of the Thesis
   1.3 Scope of Study
   1.4 Outline of the Thesis

2. **LITERATURE REVIEW**
   2.1 Introduction
   2.2 Boundary Value Problems
   2.3 Existence Theory
   2.4 Linear Multistep Method
   2.5 Newton-Raphson Method
   2.6 Stiff Systems of Ordinary Differential Equations
   2.7 Literature Review

3. **SOLVING BOUNDARY VALUE PROBLEM BY BLOCK BACKWARD DIFFERENTIATION FORMULAS USING SIMPLE SHOOTING TECHNIQUE**
   3.1 Introduction
   3.2 Review on Derivation of BBDF Method
   3.3 Derivation of the Predictors
   3.4 Implementation of BBDF Method Using Newton Iteration
   3.5 Simple Shooting Technique
   3.6 Code of SBBDF Algorithm
   3.7 Problem Tested
   3.8 Numerical Results
   3.9 Discussion

4. **SOLVING BOUNDARY VALUE PROBLEM BY BLOCK BACKWARD DIFFERENTIATION FORMULAS USING MULTIPLE SHOOTING TECHNIQUE**
   4.1 Introduction
   4.2 Multiple Shooting Technique
   4.3 Code of MBBDF Algorithm
# LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1</td>
<td>Numerical result for Problem 1</td>
</tr>
<tr>
<td>3.2</td>
<td>Numerical result for Problem 2</td>
</tr>
<tr>
<td>3.3</td>
<td>Numerical result for Problem 3</td>
</tr>
<tr>
<td>3.4</td>
<td>Numerical result for Problem 4</td>
</tr>
<tr>
<td>3.5</td>
<td>Numerical result for Problem 5</td>
</tr>
<tr>
<td>3.6</td>
<td>Numerical result for Problem 6</td>
</tr>
<tr>
<td>4.1</td>
<td>Numerical result for Problem 1</td>
</tr>
<tr>
<td>4.2</td>
<td>Numerical result for Problem 2</td>
</tr>
<tr>
<td>4.3</td>
<td>Numerical result for Problem 3</td>
</tr>
<tr>
<td>4.4</td>
<td>Numerical result for Problem 4</td>
</tr>
<tr>
<td>4.5</td>
<td>Numerical result for Problem 5</td>
</tr>
<tr>
<td>4.6</td>
<td>Numerical result for Problem 6</td>
</tr>
<tr>
<td>4.7</td>
<td>Comparison of the numerical result for Problem 7 with $\varepsilon = 10^{-2}$ at $h = 10^{-2}$</td>
</tr>
<tr>
<td>4.8</td>
<td>Comparison of the approximated errors for Problem 7 with $\varepsilon = 10^{-2}$ at $h = 10^{-2}$</td>
</tr>
<tr>
<td>4.9</td>
<td>Comparison of the numerical result for Problem 7 with $\varepsilon = 10^{-3}$ at $h = 10^{-2}$</td>
</tr>
<tr>
<td>4.10</td>
<td>Comparison of the approximated errors for Problem 7 with $\varepsilon = 10^{-3}$ at $h = 10^{-2}$</td>
</tr>
<tr>
<td>4.11</td>
<td>Comparison of the numerical result for Problem 7 with $\varepsilon = 10^{-4}$ at $h = 10^{-3}$</td>
</tr>
<tr>
<td>4.12</td>
<td>Comparison of the approximated errors for Problem 7 with $\varepsilon = 10^{-4}$ at $h = 10^{-3}$</td>
</tr>
<tr>
<td>4.13</td>
<td>Approximated solution of Problem 8 for $\lambda = 0.5$</td>
</tr>
<tr>
<td>4.14</td>
<td>Approximated error for Problem 8 with $\lambda = 0.5$</td>
</tr>
</tbody>
</table>
4.15 Approximated solution of Problem 8 for $\lambda = 1.0$ 45
4.16 Approximated error for Problem 8 with $\lambda = 1.0$ 46
4.17 Comparison between MBBDF and MBDF for solving Problem 7 with $\lambda = 0.5$ and $\lambda = 1.0$ 46
## LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1</td>
<td>Graph of $\log_{10}(\text{MAXE})$ plotted against $\log_{10}(\text{Time})$</td>
<td>33</td>
</tr>
<tr>
<td>4.2</td>
<td>Graph of $\log_{10}(\text{MAXE})$ plotted against $\log_{10}(h)$</td>
<td>33</td>
</tr>
<tr>
<td>4.3</td>
<td>Graph of $\log_{10}(\text{MAXE})$ plotted against $\log_{10}(\text{Time})$</td>
<td>34</td>
</tr>
<tr>
<td>4.4</td>
<td>Graph of $\log_{10}(\text{MAXE})$ plotted against $\log_{10}(h)$</td>
<td>35</td>
</tr>
<tr>
<td>4.5</td>
<td>Graph of $\log_{10}(\text{MAXE})$ plotted against $\log_{10}(\text{Time})$</td>
<td>36</td>
</tr>
<tr>
<td>4.6</td>
<td>Graph of $\log_{10}(\text{MAXE})$ plotted against $\log_{10}(h)$</td>
<td>36</td>
</tr>
<tr>
<td>4.7</td>
<td>Graph of $\log_{10}(\text{MAXE})$ plotted against $\log_{10}(\text{Time})$</td>
<td>37</td>
</tr>
<tr>
<td>4.8</td>
<td>Graph of $\log_{10}(\text{MAXE})$ plotted against $\log_{10}(h)$</td>
<td>38</td>
</tr>
<tr>
<td>4.9</td>
<td>Graph of $\log_{10}(\text{MAXE})$ plotted against $\log_{10}(\text{Time})$</td>
<td>39</td>
</tr>
<tr>
<td>4.10</td>
<td>Graph of $\log_{10}(\text{MAXE})$ plotted against $\log_{10}(h)$</td>
<td>39</td>
</tr>
</tbody>
</table>
### LIST OF ABBREVIATIONS

<table>
<thead>
<tr>
<th>Acronym</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>BVPs</td>
<td>Boundary Value Problems</td>
</tr>
<tr>
<td>BBDF</td>
<td>Block Backward Differentiation Formula</td>
</tr>
<tr>
<td>IVPs</td>
<td>Initial Value Problems</td>
</tr>
<tr>
<td>ODEs</td>
<td>Ordinary Differential Equations</td>
</tr>
<tr>
<td>RK4</td>
<td>Runge-Kutta method of order four via simple shooting technique.</td>
</tr>
<tr>
<td>1PSN4</td>
<td>One point direct method of order four via simple shooting technique adapted with Newton-Raphson method.</td>
</tr>
<tr>
<td>2PSN4</td>
<td>Two point direct method of order four via simple shooting technique adapted with Newton-Raphson method.</td>
</tr>
<tr>
<td>SBBDF</td>
<td>Block Backward Differentiation Formulas via simple shooting technique adapted with Newton-Raphson method.</td>
</tr>
<tr>
<td>SBDF</td>
<td>Classical one-point three-step Backward Differentiation Formula via simple shooting technique.</td>
</tr>
<tr>
<td>MBBDF</td>
<td>Block Backward Differentiation Formulas via multiple shooting technique adapted with Newton-Raphson method.</td>
</tr>
<tr>
<td>MBDF</td>
<td>Classical one-point three-step Backward Differentiation Formula via multiple shooting technique.</td>
</tr>
<tr>
<td>TFD</td>
<td>Fourth order tridiagonal finite difference method.</td>
</tr>
<tr>
<td>FTFD</td>
<td>Fitted fourth order tridiagonal finite difference method.</td>
</tr>
<tr>
<td>SONM</td>
<td>Seventh order numerical method.</td>
</tr>
</tbody>
</table>
CHAPTER 1

INTRODUCTION

1.1 Introduction

Boundary value problems (BVPs) arise in many areas of science and engineering including the modelling of chemical reactions, vibration problems and space technology. Problems in civil engineering such as deflection of a beam can be also formulated in terms of BVPs. For example, let us consider a simply supported beam with modulus of elasticity $E$, moment of inertia $I$, a uniform load $w$, length of the beam $L$ and end tension $T$. If $y(x)$ denotes the deflection at each point $x$ in the beam, then $y(x)$ satisfies the differential equation

$$y'' - \frac{T}{EI}y = \frac{wx(x - L)}{2EI}$$

with boundary condition $y(0) = y(L) = 0$. See Burden (1993) for details. Because of their importance, many algorithms have been proposed to find the solution of the BVPs.

There exist a large number of numerical methods to compute the solutions of BVPs. Among those methods, there are two main approaches for solving BVPs: indirect method and direct method. In indirect method, the higher order BVPs are reduced to an equivalent system of first order differential equations and then solved with numerical method which is computationally expensive. On the other hand, the two point direct multistep method in this thesis will be utilized to solve BVPs directly without reducing them to first order differential equations. Besides that, the two point direct method produce two approximate solutions at two points in each integrate step. Furthermore, the simple shooting method and multiple shooting techniques will be implemented to this direct method for solving BVPs.

1.2 Objective of the Thesis

The aim of this thesis is to propose a method for solving second order BVPs. The research objectives are:

1. to implement simple shooting technique in two point Block Backward Differentiation Formula (BBDF) developed by Ibrahim (2006) for solving second order BVPs directly using constant step size.
2. to implement multiple shooting technique in two point BBDF for solving second order BVPs directly using constant step size.

3. to develop the algorithm of two point BBDF for solving second order BVPs directly using constant step size.

4. to discuss and analyse the numerical solutions obtained by BBDF and BDF.

1.3 Scope of Study

This research is carried out to study a second order BVPs of the form as follows:

\[ y'' = f(x, y, y'), \quad a \leq x \leq b \]  

(1.1)

with Dirichlet boundary condition

\[ y(a) = \alpha, \quad y(b) = \beta \]  

(1.2)

where \( a, b, \alpha, \beta \) are constants. The scopes of this research are:

i. To study the approach for solving second order BVPs using two point direct BBDF method with two type of shooting technique, i.e. simple shooting technique and multiple shooting techniques.

ii. To present numerical results of BBDF codes using constant step size for solving second order BVPs.

1.4 Outline of the Thesis

This thesis is primarily divided into five chapters. In Chapter I, the objective of the thesis and scope of study have been stated.

A brief introduction to BVPs is given at the beginning of Chapter II. Existence theory and the definition of stiffness were given. This chapter also includes a review of earlier research on BVPs.

Chapter III presents a review on the derivation of the two point BBDF. Algorithm to solve BVPs using BBDF via simple shooting technique adapted with Newton Raphson method was developed. The numerical results and comparison with existed method were included.
Chapter IV explains the adaption of multiple shooting technique and an algorithm to solve BVPs directly using BBDF via multiple shooting technique adapted with Newton-Raphson method was developed. Numerical results and discussion were included.

The conclusions and suggestion for the future research on solving BVPs using BBDF were provided in Chapter V.
BIBLIOGRAPHY


52


