UNIVERSITI PUTRA MALAYSIA

TRIGONOMETRICALLY-FITTED EXPLICIT RUNGE-KUTTA-NYSTR¨O M METHODS FOR SOLVING SPECIAL SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS WITH PERIODIC SOLUTIONS

MUSA AHMED DEMBA

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By

MUSA AHMED DEMBA

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Master of Science

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DEDICATIONS

Mum: Fatima Nuhu (Ayiya)
Dad: Alhaji Ahmed Demba
Brother: Alhaji Mohammed Demba (Majidadi)

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Master of Science

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Chairman : **Associate Professor Norazak Senu, PhD**
Faculty : **Science**

In this study, a trigonometrically-fitted explicit Runge-Kutta-Nystrom (RKN) methods are proposed for the integration of initial-value problems (IVPs) of special second-order ordinary differential equations (ODEs) with periodic behavior. The derivation of fourth and fifth-order trigonometrically-fitted explicit RKN methods using constant step length and an embedded trigonometrically-fitted explicit 4(3) and 5(4) pairs of RKN methods for variable step length have been developed.

The numerical results obtained show that the new trigonometrically-fitted explicit RKN methods developed for constant and variable step length are more accurate and efficient than several existing methods in the literature.

Meanwhile, a symplectic trigonometrically-fitted explicit RKN methods for solving Hamiltonian system with periodic solutions were derived. However, it is well known that the local error of a non-symplectic method is smaller than that of the symplectic method, the error produce during the integration process is slower for the symplectic method. Thus, for a large interval of integration of Hamiltonian systems the symplectic method will be more efficient than the non-symplectic method. The numerical results obtained show that the symplectic methods incorporated with trigonometric fitting technique are more efficient than the non-symplectic methods when solving IVPs with periodic character.

In conclusion, a trigonometrically-fitted explicit RKN methods were derived for solving special second-order ODEs with periodic solutions. The local truncation error (LTE)

of each method derived was computed, the absolute stability interval of the methods derived were discussed. Numerical experiment performed show the accuracy and efficiency in terms of function evaluation per step of the new methods in comparison with other existing methods.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Master Sains

KAEDAH RUNGE-KUTTA-NYSTROM TAK TERSIRAT SUAI SECARA TRIGONOMETRI UNTUK MENYELESAIKAN PERSAMAAN PEMBEZAAN BIASA PERINGKAT DUA BERKALA

Oleh

MUSA AHMED DEMBA

Jun 2016

Pengerusi :**Prof. Madya Norazak Senu, PhD**
Fakulti : **Sains**

Dalam kajian ini, kaedah Runge-Kutta-Nystrom (RKN) tak tersirat suai secara trigonometri dicadangkan untuk mengkamir masalah nilai awal (MNA) bagi persamaan pembezaan biasa (PPB) berperingkat dua dengan kelakuan berkala. Penerbitan kaedah RKN tak tersirat suai secara trigonometri peringkat empat dan lima untuk saiz langkah tetap dan kaedah benaman tak tersirat suai secara trigonometri untuk pasangan 4(3) dan 5(4) RKN untuk saiz langkah boleh ubah telah diterbitkan.

Keputusan berangka diperolehi menunjukkan kaedah baharu suai secara trigonometri RKN yang diterbitkan untuk saiz langkah tetap dan boleh ubah adalah lebih jitu dan cekap berbanding kaedah sedia ada dalam literatur.

Sementara itu, kaedah RKN simplektik tak tersirat suai secara trigonometri untuk menyelesaikan sistem Hamiltonian dengan penyelesaian berkala diterbitkan. Walau bagaimanapun, telah diketahui bahawa ralat setempat bagi kaedah tak simplektik adalah lebih kecil berbanding kaedah simplektik, ralat dihasilkan semasa proses pengamiran adalah perlahan bagi kaedah simplektik. Oleh itu, bagi selang kamiran yang besar untuk sistem Hamiltonian kaedah simplektik adalah lebih cekap berbanding kaedah tak simplektik. Keputusan berangka diperolehi bagi kaedah simplektik bersama dengan teknik suai secara trigonometri adalah lebih cekap berbanding kaedah tak simplektik apabila menyelesaikan MNA dengan kelakuan berkala.

Kesimpulannya, kaedah RKN tak tersirat suai secara trigonometri diterbitkan untuk menyelesaikan PPB peringkat dua dengan penyelesaian berkala. Ralat pangkasan tempatan (RPT) bagi setiap kaedah diterbitkan telah dikira, analisa kestabilan mutlak bagi

kaedah dibincangkan. Perlaksanaan eksperimen berangka menunjukkan kejituan dan kecekapan bagi kaedah baharu berbanding kaedah sedia ada yang lain.

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This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Master of Science. The members of the Supervisory Committee were as follows:

Norazak Senu, PhD
Associate Professor
Faculty of Science
Universiti Putra Malaysia
(Chairperson)

Fudziah Ismail, PhD
Professor
Faculty of Science
Universiti Putra Malaysia
(Member)

BUJANG KIM HUAT, PhD
Professor and Dean
School of Graduate Studies
Universiti Putra Malaysia

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Signature: ______________________________
Name of Chairman of Supervisory Committee
Associate Professor Norazak Senu, PhD

Signature: ______________________________
Name of Member of Supervisory Committee
Professor Fudziah Ismail, PhD

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LIST OF ABBREVIATIONS

PDEs	Partial Differential Equations
ODEs	Ordinary Differential Equations
IVPs	Initial Value Problems
LTE	Local Truncation Error of y
LTEder	Local Truncation Error of y'
MAXE	Maximum Error
RKN	Runge-Kutta-Nystr¨om method
RK	Runge-Kutta method
RKN4G	The fourth-order three-stage Runge-Kutta-Nystr¨om method obtained by Garcia et al. (2002).
ETFRKN4G	The fourth-order three-stage trigonometrically-fitted Runge-Kutta-Nystr¨om method derived in chapter 2.
RKN4-4	The fourth-order four-stage Runge-Kutta-Nystrom method given by Xinyuan Wu and Wang (2013).
PFERKN4P	The fourth-order four-stage RKN method with phase-lag order infinity obtained by Papadopoulos et al. (2009).
ETFRKN4 (4, 8, 5)	M Trigonometrically fitted fourth-order four-stage method derived in chapter 2.
RKN4 (4, 8, 5) M	The RKN method with dispersion order eight and dissipation order five obtained by Senu (2009).
RKN4 (4, 8, 5) S	The RKN method with dispersion order eight and dissipation order five obtained by Senu (2009).
RK4	The classical Runge-Kutta method given by Butcher (2008).
ETFRKN4(4)S	The new fourth order four-stage explicit trigonometrically-fitted RKN method derived in Chapter 2.
RK5F	The Fehlberg fifth order RK method given by Butcher (2008).
RK5D	The Dormand fifth order RK method given by Butcher (2008).
ETFRKN5H	The new four-stage fifth order explicit trigonometrically-fitted RKN method derived in chapter 2.
RKN5H	The fifth order four-stage RKN method obtained by Hairer and Wanner (1993).
ETFRKN5 (4) M	The new four-stage fifth order explicit trigonometrically-fitted RKN method derived in chapter 2.
ETFRKN5 (4) S	The new four-stage fifth order explicit trigonometrically-fitted RKN method derived in chapter 2.
SETFRKN4 (3, 6)	The new four-stage third order symplectic explicit trigonometricallyfitted RKN method derived in chapter 3.
SETFRKN3 (3, 6)	The new three-stage third order symplectic explicit trigonometricallyfitted RKN method derived in chapter 3.
SRKN4 (3, 6)	The four-stage third order symplectic explicit RKN method with phaselag order six derived by Mohamad (2013).
SRKN3 (3, 6)	The three-stage third order symplectic explicit RKN method with phase-lag order six derived by Mohamad (2013).

RKN3(3,6,¥)	The three-stage third order explicit RKN method with phase-lag order six and zero dissipative derived by Senu (2009).
EETFRKN 4(3)	The new 4(3) pair of embedded explicit trigonometrically-fitted RKN method derived in chapter 4.
ERKN4 (3) S	The 4(3) pair of embedded explicit RKN method obtained by Senu (2009).
ERKN4 (3)	The 4(3) pair of embedded explicit RKN method obtained by Van de Vyver (2005a).
ERKN6 (4)6	ER A 6(4) pair of embedded explicit RKN method obtained by Anastassi and Kosti (2015).
DOP4 (3)	The five stage Dormand embedded RK method given by Butcher (2008).
MERK 4(3)	The five stage Merson embedded RK method given by Butcher (2008).
EETFRKN 5(4) M	The new 5(4) pair of embedded explicit trigonometrically-fitted RKN pair derived in chapter 4.
ARKN 5(3)	A 5(3) pair of explicit ARKN method derived by Franco (2003).
DOP5 (4)	A 5(4) embedded RK method given by Butcher (2008).

CHAPTER 1

INTRODUCTION

1.1 The Initial Value Problem

The initial value problem for a system of k special second order ODEs is defined as:

$$y'' = f(x,y),\ y(x_0) = \delta \tag{1.1}$$

where $y(x) = [y_1(x), y_2(x), y_3(x), ..., y_k(x)]^T$, $f(x,y) = [f_1(x,y), f_2(x,y), f_3(x,y), ..., f_k(x,y)]^T, x \in [a,b]$ and $\delta = [\delta_1, \delta_2, \delta_3, ..., \delta_k]^T$ is the vector of initial conditions.

1.2 Existence and Uniqueness of Solution

Initial value problems describe a problem together with the behaviour of it's path taken at some initial points of the independent variable x. The question is how dependable are they in guessing the nature of the same path? Some of the characteristic of initial value problems that answer this question, as given by Butcher (2008), are existence of solution, uniqueness of the solution if it exists and the sensitivity of the solution to a small perturbation to the initial information. One of the well known conditions that guarantees these characteristics is the Lipschitz condition.

Definition 1.1 A function $f : R \times R^d \to R^d$ is said to satisfy Lipschitz condition in its second variable if there exist a constant L such that for any $x \in [a,b]$ and $y_1, y_2 \in R^d$,

$$\|f(x,y_1) - f(x,y_2)\| \leq L\|y_1 - y_2\|, \tag{1.2}$$

where L is called Lipschitz constant.

Theorem 1.1 :(Existence and Uniqueness)

Let $f(x,y(x))$ be defined and continuous $\forall$ points $(x,y(x))$ in a domain D defined by $x \in [a,b], y \in (-\infty,\infty)$, a and b are finite, and that $f(x,y(x))$ satisfies Lipschitz condition. Then for any given number ζ, $\exists$ a unique solution $y(x)$ of the IVP (1.1), where $\forall$ $(x,y(x)) \in D$, $y(x)$ is continuous and differentiable.Butcher (2008).

In this thesis, we assume that $f(x,y(x))$ of the IVP (1.2) satisfies Lipschitz condition so that a unique solution is guaranteed.

1.3 The Objectives of the Thesis

The Objective of this thesis is to construct an improved numerical methods based on explicit RKN method that can accurately and efficiently integrate second order IVPs of the form (1.2), in which it is known that its solutions are periodic in nature. The new methods are tested for both constant and variable step length. To achieve this, we propose the following methods:

- To derive explicit trigonometrically-fitted RKN (ETFRKN) methods for the solution of the IVPs (1.2).
- To derive symplectic explicit trigonometrically-fitted RKN (SETFRKN) methods for the solution of the IVPs (1.2).
- To derive embedded explicit trigonometrically-fitted RKN (EETFRKN) methods for the solution of the IVPs (1.2) in variable step size code.
- To compare the numerical results of the derived methods with other existing methods in the scientific literature.

1.4 Organization of the Thesis

In Chapter 1 of this thesis, introductory background on the development of numerical solution of ODEs is discussed. Initial value problems and theory of the existence of their solution is presented. Runge-Kutta-Nyström method is briefly discussed and its algebraic order conditions. Local truncation error as well as absolute stability analysis of RKN method is presented. Derivation of a trigonometrically-fitted explicit Runge-Kutta-Nyström method for solving periodic special second order ordinary differential equations is presented. In Chapter 2, literature review is given. In Chapter 3, we derive a three-stage fourth order, a four-stage fourth order and a four-stage fifth order trigonometrically-fitted explicit Runge-Kutta-Nyström methods. Algebraic order and local truncation error of each of these methods derived has been analyzed as well as the absolute stability interval of each of the method. Numerical results are presented and comparison of their performance is made with the RK and RKN methods given in the literature such as Butcher (2008), Garcia (2002), Hairer (1993) and Senu et al. (2009) for solving periodic second order ODEs.

In Chapter 4, we derive a three-stage third order symplectic trigonometrically-fitted explicit Runge-Kutta-Nyström method and a four-stage fourth order symplectic explicit trigonometrically-fitted Runge-Kutta-Nyström method. Algebraic order and local truncation error of each of the method derived has been analyzed as well as the absolute stability analysis of each of the method. numerical results are presented and the efficacy of the new methods in comparison with other existing symplectic and non-symplectic methods has been seen.

In Chapter 5, we derive a four-stage embedded trigonometrically-fitted explicit Runge-Kutta-Nyström methods. A four-stage $4(3)$ and a four-stage $5(4)$ pairs of an embedded trigonometrically-fitted explicit Runge-Kutta-Nyström method are constructed. Algebraic order and local truncation error of each of the method derived has been analyzed as well as the absolute stability interval of each of the method. Numerical experiment was performed and the efficiency of the new method over some well known method has been seen. Finally, in Chapter 6 we give the summary of the whole thesis, conclusion and future work.

1.5 Runge-Kutta-Nyström (RKN) Method

A Runge-Kutta-Nyström method is a Runge-Kutta method designed specifically for solving special second order ODEs of the form (1.2) by E.J.Nyström in 1925.

RKN methods are generally divided in to two type

- explicit methods
- implicit methods

A Runge–Kutta–Nyström method is said to be explicit if $a_{ij} = 0$ for $i \leq j$ and implicit elsewhere. In this study, our focus is on the explicit RKN type.

The general form of an explicit s-stage RKN method is given by:

$$y_{n+1} = y_n + hy'_n + h^2 \sum_{i=1}^{s} b_i f(x_n + c_i h, Y_i), \tag{1.3}$$

$$y'_{n+1} = y'_n + h \sum_{i=1}^{s} d_i f(x_n + cih, Y_i), \tag{1.4}$$

$$Y_i = y_n + c_i hy'_n + h^2 \sum_{j=1}^{i-1} a_{ij} f(x_n + cih, Y_j),. \tag{1.5}$$

or

$$y_{n+1} = y_n + hy'_n + h^2 \sum_{i=1}^{s} b_i k_i, \tag{1.6}$$

$$y'_{n+1} = y'_n + h \sum_{i=1}^{s} d_i k_i, \tag{1.7}$$

$$k_i = f(x_n + cih, y_n + c_i y'_n + h^2 \sum_{j=1}^{i-1} a_{ij} k_j),. \tag{1.8}$$

or in Butcher Tableau as :

where A is a matrix $[a_{ij}]_{s\times s}$, $c = [c_1, c_2, ..., c_s]^T$, $b = [b_1, b_2, ..., b_s]^T$ and $d = [d_1, d_2, ..., d_s]^T$. The parameters $a_{ij}, c_i, b_i,$ and d_i appearing in the method are assumed

$$\begin{array}{c|c} c & A \\ \hline & b^T \\ & d^T \end{array}$$

to be real numbers.

1.6 Algebraic Conditions for RKN Method

The order conditions for the RKN method can be obtained by expanding the local truncation error (LTE) directly. The RKN method (1.6)−(1.8) can be expressed as:

$$y_{n+1} = y_n + h\Psi(x_n, y_n, h), \tag{1.9}$$
$$y'_{n+1} = y'_n + h\Psi'(x_n, y_n, h), \tag{1.10}$$

where the functions $\Psi(x_n, y_n, h)$ and $\Psi'(x_n, y_n, h)$ are define as:

$$\Psi(x_n, y_n, h) = y'_n + h\sum_{i=1}^{s} b_i k_i, \tag{1.11}$$

$$\Psi'(x_n, y_n, h) = \sum_{i=1}^{s} d_i k_i. \tag{1.12}$$

where

$$k_i = f(x_n + c_i h, y_n + c_i y'_n + h^2 \sum_{j=1}^{i-1} a_{ij} k_j)$$

If Ω represent the Taylor series increment function, the LTE of the actual solution and that of its derivative can be obtained by substituting the actual solution $y(x)$ of the equation (1.1) into the RKN increment function as given below:

$$LTE_{n+1} = h[\Psi - \Omega],\ LTE'_{n+1} = [\Psi' - \Omega']$$

where

$$\begin{aligned} \Omega &= y'_n + \frac{1}{2}hy_n'' + \frac{1}{6}h^2 y'''_n + \frac{1}{24}h^3 y_n^{(iv)} + \frac{1}{120}h^4 y_n^{(v)} + \ldots + \frac{1}{p!}h^{p-1}y_n^{(p)}, \\ \Omega' &= y_n'' + \frac{1}{2}hy'''_n + \frac{1}{6}h^2 y_n^{(iv)} + \frac{1}{24}h^3 y_n^{(v)} + \frac{1}{120}h^4 y_n^{(vi)} + \ldots + \frac{1}{(p-1)!}h^{p-2}y_n^{(p)} \end{aligned}$$

Expressing the above equations in terms of elementary differentials. We first define some elementary differentials from the total derivative of the function defined in (1.2) as follows: The total derivative of $f(x,y) = F_1^{(2)}$ is given by

$$
\begin{aligned}
\frac{dF}{dx} &= \frac{\partial}{\partial x}f + \frac{\partial}{\partial y}f\frac{dy}{dx} \\
&= \frac{d}{dx}(y\prime\prime) \\
&= y''' \\
&= F_x \\
&= F_1{}^{(3)}
\end{aligned} \tag{1.13}
$$

Differentiating equation (1.13) with respect to x we have:

$$
\begin{aligned}
\frac{d}{dx}F_x &= \frac{d}{dx}f_x + \frac{d}{x}f_y y' \\
&= f_{xx} + 2f_{xy}y' + f_{yy}(y')^2 + f f_y \\
&= \frac{d}{dx}(y''') \\
&= y^{(iv)} \\
&= F_{xx} \\
&= F_1{}^{(4)}
\end{aligned} \tag{1.14}
$$

Similarly, differentiating equation (1.14) with respect to x we have:

$$
\begin{aligned}
\frac{d}{dx}F_{xx} &= \frac{d}{dx}f_{xx} + 2\frac{d}{dx}(f_{xy}y') + \frac{d}{dx}(f_{yy}(y')^2) + \frac{d}{dx}(f f_y) \\
&= f_{xxx} + f_{xxy}y' + 2f_{xyx}y' + 2f_{xyy}(y')^2 + 2f_{xy}f + f_{yyx}(y')^2 \\
&\quad + f_{yyy}(y')^3 + 2y' f_{yy} f + f f_{yx} + f f_{yy} y' + f_y f_x + (f_y)^2 y' \\
&= \frac{d}{dx}(y^{(iv)}) \\
&= y^{(v)} \\
&= F_{xxx} \\
&= F_1{}^{(5)}
\end{aligned} \tag{1.15}
$$

Following similar pattern, we can derive all the elementary differentials. Expressing Ω

and Ω' in terms of elementary differentials, we have:

$$\begin{aligned}\Omega &= y'_n + \frac{1}{2}hF_1^{(2)} + \frac{1}{6}h^2F_1^{(3)} + \frac{1}{24}h^3F_1^{(4)} + \frac{1}{120}h^4(F_1^{(5)} + F_2^{(5)}) + O(h^5),\\ \Omega' &= F_1^{(2)} + \frac{1}{2}hF_1^{(3)} + \frac{1}{6}h^2F_1^{(4)} + \frac{1}{24}h^3(F_1^{(5)} + F_2^{(5)}) + O(h^4).\end{aligned}$$

Using the above elementary differentials, the increment function Ψ and Ψ' can be express as:

$$\begin{aligned}\sum_{i=1}^{s} b_i k_i &= \sum_{i=1}^{s} b_i F_1{}^{(2)} + h\sum_{i=1}^{s} b_i c_i F_1{}^{(3)}\\ &\quad + \frac{1}{2}h^2(\sum_{i=1}^{s} b_i c_i^2 F_1{}^{(4)}) + O(h^3),\end{aligned} \tag{1.16}$$

$$\begin{aligned}\sum_{i=1}^{s} d_i k_i &= \sum_{i=1}^{s} d_i F_1{}^{(2)} + h\sum_{i=1}^{s} d_i c_i F_1{}^{(3)}\\ &\quad + \frac{1}{2}h^2(\sum_{i=1}^{s} d_i c_i^2 F_1{}^{(4)}) + O(h^3).\end{aligned} \tag{1.17}$$

The LTE of y and that of its derivatives can therefore be express as:

$$\begin{aligned}LTE_{n+1} &= h^2\Big[\Big(\sum_{i=1}^{s} b_i F_1{}^{(2)} + h\sum_{i=1}^{s} b_i c_i F_1{}^{(3)} + \frac{1}{2}h^2\sum_{i=1}^{s} b_i c_i^2 F_1{}^{(4)} + \ldots\Big)\\ &\quad - \Big(\frac{1}{2}F_1{}^{(2)} + \frac{1}{6}F_1{}^{(3)} + \frac{1}{24}F_1{}^{(4)} + \ldots\Big)\Big],\end{aligned} \tag{1.18}$$

$$\begin{aligned}LTE'_{n+1} &= h\Big[\Big(\sum_{i=1}^{s} d_i F_1{}^{(2)} + h\sum_{i=1}^{s} d_i c_i F_1{}^{(3)} + \frac{1}{2}h^2\sum_{i=1}^{s} d_i c_i^2 F_1{}^{(4)} + \ldots\Big)\\ &\quad - \Big(F_1{}^{(2)} + \frac{1}{2}F_1{}^{(3)} + \frac{1}{6}F_1{}^{(4)} + \ldots\Big)\Big].\end{aligned} \tag{1.19}$$

Simplifying equation (1.18) and (1.19), we have:

$$\begin{aligned}LTE_{n+1} &= h^2\Big[\Big(\sum_{i=1}^{s} b_i - \frac{1}{2}F_1{}^{(2)}\Big) + \Big(\sum_{i=1}^{s} b_i c_i - \frac{1}{6}\Big)hF_1{}^{(3)}\\ &+ \Big(\frac{1}{2}\sum_{i=1}^{s} b_i c_i^2 - \frac{1}{24}\Big)h^2F_1{}^{(4)} + \ldots\Big],\end{aligned} \tag{1.20}$$

$$
\begin{aligned}
LTE'_{n+1} &= h\Big[\Big(\sum_{i=1}^{s} d_i - 1\Big)F_1{}^{(2)} + \Big(\sum_{i=1}^{s} d_i c_i - \frac{1}{2}\Big)hF_1{}^{(3)} \\
&\quad + \Big(\frac{1}{2}\sum_{i=1}^{s} d_i c_i^2 - \frac{1}{6}\Big)h^2 F_1{}^{(4)} + \ldots\Big]. \qquad (1.21)
\end{aligned}
$$

Using equation (1.20) and (1.21), the order conditions for an s-stage RKN process up to order six is given below:

For y:

Order 2: $$\sum b_i = \frac{1}{2}, \qquad (1.22)$$

Order 3: $$\sum b_i c_i = \frac{1}{6}, \qquad (1.23)$$

Order 4: $$\frac{1}{2}\sum b_i c_i^2 = \frac{1}{24}, \qquad (1.24)$$

Order 5: $$\frac{1}{6}\sum b_i c_i^3 = \frac{1}{120}, \qquad (1.25)$$

$$\sum b_i a_{ij} c_j = \frac{1}{120}, \qquad (1.26)$$

Order 6: $$\frac{1}{24}\sum b_i c_i^4 = \frac{1}{720}, \qquad (1.27)$$

$$\frac{1}{4}\sum b_i c_i a_{ij} c_j = \frac{1}{720}, \qquad (1.28)$$

$$\frac{1}{2}\sum b_i a_{ij} c_j^2 = \frac{1}{720}, \qquad (1.29)$$

For y':

$$\textbf{Order 1:} \quad \sum d_i = 1, \tag{1.30}$$

$$\textbf{Order 2:} \quad \sum d_i c_i = \frac{1}{2}, \tag{1.31}$$

$$\textbf{Order 3:} \quad \frac{1}{2}\sum d_i c_i^2 = \frac{1}{6}, \tag{1.32}$$

$$\textbf{Order 4:} \quad \frac{1}{6}\sum d_i c_i^3 = \frac{1}{24}, \tag{1.33}$$

$$\sum d_i a_{ij} c_j = \frac{1}{24}, \tag{1.34}$$

$$\textbf{Order 5:} \quad \frac{1}{24}\sum d_i c_i^4 = \frac{1}{120}, \tag{1.35}$$

$$\frac{1}{4}\sum d_i c_i a_{ij} c_j = \frac{1}{120}, \tag{1.36}$$

$$\frac{1}{2}\sum d_i a_{ij} c_j^2 = \frac{1}{120}, \tag{1.37}$$

$$\textbf{Order 6:} \quad \frac{1}{120}\sum d_i c_i^5 = \frac{1}{720}, \tag{1.38}$$

$$\frac{1}{20}\sum d_i c_i^2 a_{ij} c_j = \frac{1}{720}, \tag{1.39}$$

$$\frac{1}{10}\sum d_i c_i a_{ij} c_j^2 = \frac{1}{720}, \tag{1.40}$$

$$\frac{1}{6}\sum d_i a_{ij} c_j^3 = \frac{1}{720}, \tag{1.41}$$

$$\sum d_i a_{ij} a_{jk} c_k = \frac{1}{720}. \tag{1.42}$$

All subscripts i, j, k run to s from 1. Most methods needs c_i to satisfy the condition given below:

$$\frac{1}{2}c_i^2 = \sum_{j=1}^{s} a_{ij}, \ (i = 1, 2, ..., s). \tag{1.43}$$

For a higher order RKN methods, a simplifying assumption given by Butcher (2008) is usually used to reduce the number of order conditions as given by the equation below:

$$b_i = d_i(1 - c_i), \ (i = 1, 2, ..., s). \tag{1.44}$$

1.7 Local Truncation Error

Dormand (1996) proposed that having achieved a particular order of accuracy, the best strategy for practical purposes is to minimize the error norms. The quantities of the norms of the local truncation error coefficients are:

$$\left\|\tau^{(p+1)}\right\|_2 = \sqrt{(\sum_{j=1}^{n_{p+1}} \tau_j^{(p+1)})^2} \text{ for } y_n, \tag{1.45}$$

and

$$\left\|\tau'^{(p+1)}\right\|_2 = \sqrt{(\sum_{j=1}^{n_{p+1}} \tau_j'^{(p+1)})^2} \text{ for } y'_n. \tag{1.46}$$

Given below is the error coefficients up to order six for RKN processes :

For y:

Order 2: $$\tau_1^{(2)} = \sum b_i - \frac{1}{2}, \tag{1.47}$$

Order 3: $$\tau_1^{(3)} = \sum b_i c_i - \frac{1}{6}, \tag{1.48}$$

Order 4: $$\tau_1^{(4)} = \frac{1}{2}\sum b_i c_i^2 - \frac{1}{24}, \tag{1.49}$$

Order 5: $$\tau_1^{(5)} = \frac{1}{6}\sum b_i c_i^3 - \frac{1}{120}, \tag{1.50}$$

$$\tau_2^{(5)} = \sum b_i a_{ij} c_j - \frac{1}{120}, \tag{1.51}$$

Order 6: $$\tau_1^{(6)} = \frac{1}{24}\sum b_i c_i^4 - \frac{1}{720}, \tag{1.52}$$

$$\tau_2^{(6)} = \frac{1}{4}\sum b_i c_i a_{ij} c_j - \frac{1}{720}, \tag{1.53}$$

$$\tau_3^{(6)} = \frac{1}{2}\sum b_i a_{ij} c_j^2 - \frac{1}{720}, \tag{1.54}$$

For y':

$$\textbf{Order 1:} \quad \tau_1'^{(1)} = \sum d_i - 1, \tag{1.55}$$

$$\textbf{Order 2:} \quad \tau_1'^{(2)} = \sum d_i c_i - \frac{1}{2}, \tag{1.56}$$

$$\textbf{Order 3:} \quad \tau_1'^{(3)} = \frac{1}{2}\sum d_i c_i^2 - \frac{1}{6}, \tag{1.57}$$

$$\textbf{Order 4:} \quad \tau_1'^{(4)} = \frac{1}{6}\sum d_i c_i^3 - \frac{1}{24}, \tag{1.58}$$

$$\tau_2'^{(4)} = \sum d_i a_{ij} c_j - \frac{1}{24}, \tag{1.59}$$

$$\textbf{Order 5:} \quad \tau_1'^{(5)} = \frac{1}{24}\sum d_i c_i^4 - \frac{1}{120}, \tag{1.60}$$

$$\tau_2'^{(5)} = \frac{1}{4}\sum d_i c_i a_{ij} c_j - \frac{1}{120}, \tag{1.61}$$

$$\tau_3'^{(5)} = \frac{1}{2}\sum d_i a_{ij} c_j^2 - \frac{1}{120}, \tag{1.62}$$

$$\textbf{Order 6:} \quad \tau_1'^{(6)} = \frac{1}{120}\sum d_i c_i^5 - \frac{1}{720}, \tag{1.63}$$

$$\tau_2'^{(6)} = \frac{1}{20}\sum d_i c_i^2 a_{ij} c_j - \frac{1}{720}, \tag{1.64}$$

$$\tau_3'^{(6)} = \frac{1}{10}\sum d_i c_i a_{ij} c_j^2 - \frac{1}{720}, \tag{1.65}$$

$$\tau_4'^{(6)} = \frac{1}{6}\sum d_i a_{ij} c_j^3 - \frac{1}{720}, \tag{1.66}$$

$$\tau_5'^{(6)} = \sum d_i a_{ij} a_{jk} c_k - \frac{1}{720}. \tag{1.67}$$

1.8 Absolute Stability Analysis

The analysis of the absolute stability of RKN method is based on the test equation

$$y'' = -w^2 y, \; w \in \mathbb{R}. \tag{1.68}$$

Substituting $f(x,y) = -w^2 y$, $f(x_n + c_i h, Y_i) = -w^2 Y_i$ and $f(x_n + c_i h, Y_j) = -w^2 Y_j$ in equations (1.3)–(1.4) and multiply equation (1.4) by h, we have:

$$y_{n+1} = y_n + h y_n' + h^2 \sum_{i=1}^{s} b_i(-w^2 Y_i), \tag{1.69}$$

$$h y_{n+1}' = h y_n' + h^2 \sum_{i=1}^{s} d_i(-w^2 Y_i), \tag{1.70}$$

$$Y_i = y_n + c_i h y_n' + h^2 \sum_{j=1}^{s} a_{ij}(-w^2 Y_j). \tag{1.71}$$

Now, define $M_{n+1} = \begin{pmatrix} y_{n+1} \\ hy'_{n+1} \end{pmatrix}$, $M_n = \begin{pmatrix} y_n \\ hy'_n \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and $H = -v^2$, where $v = wh$. Simplifying equation (1.69) and (1.70), we have

$$M_{n+1} = BM_n - v^2 \begin{pmatrix} b_1 & \cdots & b_s \\ d_1 & \cdots & d_s \end{pmatrix} \begin{pmatrix} Y_1 \\ \vdots \\ Y_s \end{pmatrix} \tag{1.72}$$

where, from equation (1.71), we have

$$\begin{pmatrix} Y_1 \\ \vdots \\ Y_s \end{pmatrix} = \left[I + v^2 \begin{pmatrix} a_{11} & \cdots & a_{1s} \\ \vdots & \ddots & \vdots \\ a_{s1} & \cdots & a_{ss} \end{pmatrix} \right]^{-1} \begin{pmatrix} 1 & c_1 \\ \vdots & \vdots \\ 1 & c_s \end{pmatrix} M_n. \tag{1.73}$$

Substituting equation (1.73) into equation (1.72), we have

$$M_{n+1} = \begin{pmatrix} 1 - v^2 b^T L^{-1} e & v(1 - v^2 b^T L^{-1} c) \\ -v d^T L^{-1} e & 1 - v^2 d^T L^{-1} c \end{pmatrix} M_n \tag{1.74}$$

where $L = I + v^2 A$, $A = \begin{pmatrix} a_{11} & \cdots & a_{1s} \\ \vdots & \ddots & \vdots \\ a_{s1} & \cdots & a_{ss} \end{pmatrix}$, $b = \left[b_1 \cdots b_s \right]^T$,
$d = [d_1 \cdots d_s]^T$, $e = [1 \cdots 1]^T$ and $c = [c_1 \cdots c_s]^T$.
The stability function associated with this method is given by

$$P(\lambda, v^2) = det[\lambda I - M]$$

where

$$\begin{aligned} M &= \begin{pmatrix} M_{11}(v^2) & M_{12}(v^2) \\ M_{21}(v^2) & M_{22}(v^2) \end{pmatrix} \\ &= \begin{pmatrix} 1 - v^2 b^T L^{-1} e & v(1 - v^2 b^T L^{-1} c) \\ -v d^T L^{-1} e & 1 - v^2 d^T L^{-1} c \end{pmatrix}. \end{aligned} \tag{1.75}$$

It is assumed that $M(v^2)$ has complex conjugate eigenvalues for sufficiently small values of v as stated by Van der Houwen and Sommeijer (1989). With this assumption, an oscillatory numerical solution is derived, whose behavior depends on the eigenvalues of $M(v^2)$, which is called the stability matrix. The characteristic equation of $M(v^2)$ is

$$\lambda^2 - trace(M(v^2))\lambda + det(M(v^2)) = 0 \tag{1.76}$$

In order to analyse the absolute stability of RKN method (1.3)–(1.5), we adopt the following definitions. Let $\lambda_{1,2}$ denotes the eigenvalues of $M(v^2)$.

Definition 1.2 An interval $(-v_b^2, 0)$ is called the interval of absolute stability of the method if, $\forall -v^2 \in (-v_b^2, 0)$, $|\lambda_{1,2}| < 1$, where $\lambda_{1,2}$ are the roots of the polynomial (1.76)

Definition 1.3 The interval $(-v_b^2, 0), v_b^2 > 0$, where $\forall -v^2 \in (-v_b^2, 0)$ such that the conditions

$$\mid trace(M(v^2)) \mid < det(M(v^2)) + 1 \text{ and } det(M(v^2)) < 1$$

are satisfied, is called the interval of absolute stability for RKN method.

From definitions (1.2) and (1.3), if $-v_b^2 = \infty$ then the method is said to be R-stable.

In computing the interval of absolute stability, we adopt the approach introduced by Paternoster and Cafaro (1998). The stability region is the region enclosed by the set of points for which $\mid \lambda_{1,2} \mid = 1$. Substituting $\lambda = e^{i\theta}$ into (1.76) for values of $\theta \in [0, 2\pi]$ and solve for H, then the boundary of the stability region can be mapped out by using MAPLE Package.

1.9 A Trigonometrically-Fitting Explicit Runge–Kutta–Nyström Method Technique

In this section, we will discuss a trigonometrically-fitted explicit RKN method technique.

Definition 1.4 A Runge-Kutta-Nyström method (1.3)–(1.5) is said to be trigonometrically-fitted if it integrates exactly the function e^{iwx} and e^{-iwx} or equivalently $\sin(wx)$ and $\cos(wx)$ with $w > 0$ the principal frequency of the problem when applied to the test equation $y'' = -w^2 y$; leading to a system of equations as derived as follows:

When an explicit Runge-Kutta-Nyström method (1.3) $-$ (1.5) is applied to the test equation $y'' = -w^2 y$, the method become:

$$y_{n+1} = y_n + hy'_n + h^2 \sum_{i=1}^{s} b_i f(x_n + c_i h, Y_i), \tag{1.77}$$

$$y'_{n+1} = y'_n + h \sum_{i=1}^{s} d_i f(x_n + cih, Y_i), \tag{1.78}$$

with

$$Y_1 = y_n + c_1 hy'_n, \tag{1.79}$$

$$Y_2 = y_n + c_2 hy'_n - h^2 a_{21} w^2 Y_1, \tag{1.80}$$

$$Y_3 = y_n + c_3 hy'_n + h^2(-a_{31} w^2 Y_1 - a_{32} w^2 Y_2), \tag{1.81}$$

$$Y_4 = y_n + c_4 hy'_n + h^2(-a_{41} w^2 Y_1 - a_{42} w^2 Y_2 - a_{43} w^2 Y_3), \tag{1.82}$$

$$\vdots$$

$$Y_i = y_n + c_i hy'_n + h^2 \sum_{j=1}^{s} a_{ij}(-w^2 Y_j), \tag{1.83}$$

which results in

$$y_{n+1} = y_n + hy'_n + h^2 \sum_{i=1}^{s} b_i(-w^2 Y_i), \tag{1.84}$$

and

$$y'_{n+1} = y'_n + h \sum_{i=1}^{s} d_i(-w^2 Y_i). \tag{1.85}$$

Now, let $y_n = e^{Iwx_n}$. Evaluating for y_{n+1}, y'_n and y'_{n+1} and substituting in the equations (1.79) – (1.85). Next, by using $e^{Iv} = \cos(v) + I\sin(v)$ and comparing the real and imaginary part, we obtain the following system of equations:

$$\cos(v) = 1 - v^2 \sum_{i=1}^{s} b_i(1 - v^2 \sum_{j=1}^{i-1} a_{ij} Y_j e^{-Iwx_n}), \tag{1.86}$$

$$\sin(v) = v - v^2 \sum_{i=1}^{s} b_i c_i v, \tag{1.87}$$

$$\sin(v) = v \sum_{i=1}^{s} d_i(1 - v^2 \sum_{j=1}^{i-1} a_{ij} Y_j e^{-Iwx_n}), \tag{1.88}$$

$$\cos(v) = 1 - v^2 \sum_{i=1}^{s} d_i c_i. \tag{1.89}$$

1.10 Scope of Study

The main aim of this research is to solve second order ordinary differential equations; whose first derivative does not appear explicitly of the form (1.2). In which it is known that their solutions are periodic in nature. In this work, our focus is to solve (1.2) by the use of a trigonometrically-fitted explicit RKN methods.

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