UNIVERSITI PUTRA MALAYSIA

EVASION DIFFERENTIAL GAME FROM MANY PURSUERS OF ONE EVADER WHOSE CONTROL SET IS A SECTOR

SHARIFAH ANISAH BINTI SYED MAFDZOT

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EVASION DIFFERENTIAL GAME FROM MANY PURSUERS OF ONE EVADER WHOSE CONTROL SET IS A SECTOR

By

SHARIFAH ANISAH BINTI SYED MAFDZOT

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Master of Science

February 2016

DEDICATIONS

This thesis is dedicated to:

My parents
Syed Mafdzot bin Syed Abdullah & Tuan Asselah binti T Dalam
for their endless love, support and encouragement.

My husband
Muhammad Marwan bin Bidin
for his great support, love, care and patience.

My brothers and sisters
Sharifah Huda Fauziah, Sharifah Norhayati, Sharifah Shahrulliza, Syed Mohd Amin, Sharifah Mahfudzah, & Syed Ahmad Hanif
who has been a great source of motivation and inspiration.

My friends
who believe in the richness of learning.

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Master of Science

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SHARIFAH ANISAH BINTI SYED MAFDZOT

February 2016

Chairman: Idham Arif bin Alias, PhD
Faculty: Science

From mathematical point of view, game involves a number of players, a set of strategies for each player, and analysis of the game outcome which conclude either victory or defeat for each player involved. A common type of game is often called the pursuit-evasion game. Pursuit-evasion game is one of the widely studied game theory where it involves players of two opposite sides, which are pursuer and evader. The pursuer's goal is to capture the evader while oppositely, the evader is to avoid being captured. Strategy to be constructed for players depend on the purpose of the game. It can be solved as an evasion problem for the evader. In this case, strategy for evader will be constructed and behavior of the pursuer is assumed to be any. On the other hand, the game could also be a pursuit problem and thus construction of the strategy is for the pursuer, with assumption that the evader can move freely.

In this thesis, we study an evasion differential game of many pursuers $x_1,\ldots,x_m$ against one evader y in the plane $\mathbb{R}^2$. Movements of the players are described by simple differential equations. Control functions of players are subjected to geometric constraints where maximum speed of each pursuer is equal to 1, and maximum speed of the evader is $\alpha > 1$. Control set of the evader is a sector $\mathbf{S}$ with radius α. We say that evasion is possible if $x_i(t) \neq y(t)$ for all $t \geq 0$ and $i = 1,\ldots,m$. In other words, the evasion problem is solved when it is proved that the position of the evader never coincides with the position of each pursuer at all time. To achieve the solution, conditions of evasion that guarantee the evasion from any initial positions of players are to be found.

We examine game with one pursuer by constructing the evader's strategy, checking the admissibility, and estimating distances between the evader and pursuer for the possibility of evasion. Then we show that evasion game is solvable for the case of k pursuers.

The motivation behind the study is to construct a new admissible strategy for the evasion to be possible in the evasion differential game of one evader versus many pursuers, which were studied in many works before.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Sarjana Sains

PERMAINAN PENGELAKAN PEMBEZAAN BEBERAPA PEMANGSA UNTUK SATU MANGSA YANG MANA KAWALAN SETNYA IALAH SATU SEKTOR

Oleh

SHARIFAH ANISAH BINTI SYED MAFDZOT

Februari 2016

Pengerusi: Idham Arif bin Alias, PhD
Fakulti: Sains

Dari sudut pandangan matematik, permainan melibatkan beberapa pemain, satu set strategi untuk setiap pemain, dan analisis hasil permainan yang menentukan samada menang atau kalah bagi setiap pemain yang terlibat. Jenis permainan umum sering dikenali sebagai permainan mangsa-pemangsa. Permainan mangsa-pemangsa adalah salah satu teori permainan yang dipelajari dengan meluas di mana ia melibatkan pemain-pemain dari dua pihak yang bertentangan, iaitu pemangsa dan mangsa. Strategi yang dirangka untuk pemain-pemain bergantung kepada matlamat permainan. Ianya boleh diselesaikan sebagai masalah pengelakan untuk mangsa. Dalam kes ini, strategi untuk mangsa akan dirangka dan kelakuan pemangsa dianggap sebagai apa sahaja. Sebaliknya, permainan boleh juga menjadi masalah penangkapan dan dengan itu perangkaan strategi adalah untuk pemangsa, dengan anggapan bahawa mangsa bergerak bebas.

Dalam tesis ini, kami mengkaji permainan pengelakan pembezaan bagi beberapa pemangsa $x_1,\ldots,x_m$ terhadap satu mangsa y dalam satah $\mathbb{R}^2$. Pergerakan pemain diterangkan oleh persamaan pembezaan mudah. Fungsi kawalan bagi pemain adalah tertakluk kepada kekangan geometrik di mana kelajuan maksimum setiap pemangsa adalah sama dengan 1, dan kelajuan maksima mangsa adalah $\alpha > 1$. Kawalan set mangsa adalah sektor $\mathbf{S}$ dengan jejari α. Pengelakan akan berlaku jika $x_i(t) \neq y(t)$ untuk semua $t \geq 0$ dan $i = 1,\ldots,m$. Dalam erti kata lain, masalah pengelakan selesai jika telah terbukti bahawa kedudukan mangsa tidak pernah bertepatan dengan kedudukan setiap pemangsa pada setiap masa. Untuk mendapat penyelesaian, syarat bagi pengelak yang menjamin pengelakan dari mana-mana kedudukan awal pemain perlu dicari.

Kami mengkaji permainan dengan satu pemangsa dengan membina strategi untuk mangsa, memeriksa kesahihan strategi, dan menganggarkan jarak antara mangsa dan

pemangsa untuk kemungkinan mangsa melepaskan diri. Kemudian kami membuktikan permainan mangsa dapat melepaskan diri bagi kes k pemangsa.

Motivasi di sebalik kajian ini adalah untuk membina strategi baru yang boleh diterima bagi pengelakan untuk berlaku dalam permainan pengelakan pembezaan bagi satu mangsa melawan beberapa pemangsa, di mana banyak kajian telah dibuat sebelumnya.

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I would also like to thank all of my friends who have supported me to strive towards my goal. Thank you.

I certify that a Thesis Examination Committee has met on 18 February 2016 to conduct the final examination of Sharifah Anisah binti Syed Mafdzot on his thesis entitled ”Evasion Differential Game from Many Pursuers of One Evader whose Control Set is a Sector” in accordance with the Universities and University Colleges Act 1971 and the Constitution of the Universiti Putra Malaysia [P.U.(A) 106] 15 March 1998. The Committee recommends that the student be awarded the Master of Science.

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This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Master of Science.

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Declaration by graduate student

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This is to confirm that:
- the research conducted and the writing of this thesis was under our supervision;
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Member of
Supervisory
Committee: ______________________________

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LIST OF ABBREVIATIONS

P	Pursuer
P_i	ith pursuer
E	Evader
RE	Real evader
FE	Fictitious evader
FEs	Fictitious evaders
$\dot{x}$	First derivative of x
$\dot{y}$	First derivative of y
$\dot{x}_i$	First derivative of x_i
u_i	Control parameter of the ith pursuer
v	Control parameter of the evader
w_i	Control parameter of the ith fictitious evader
$u_i(t)$	Control function of the ith pursuer
$v(t)$	Control function of the evader
$x_i(t)$	State of the ith pursuer
$y(t)$	State of the evader
V	Strategy of the evader
$\mathbb{R}^2$	Two-dimensional euclidean space/plane
$\mathbb{R}^n$	n-dimensional euclidean space
l_2	Hilbert space
$\mathbf{S}$	Sector
$H(0,\rho)$	Circle of radius ρ centered at origin
$conv\{x_{10}, x_{20}, \dots, x_{m0}\}$	Convex hull of points $x_{10}, \dots, x_{m0}$
$intconv\{x_{10}, x_{20}, \dots, x_{m0}\}$	Interior convex hull of points $x_{10}, \dots, x_{m0}$
α	Speed of the evader/radius
∂X	Boundary of the set X
$mes\,A$	Measure of the set A

LIST OF BASIC DEFINITIONS

Definition 1:
A set S is called a convex set if the line segment joining any pair of points of S lies entirely in S.

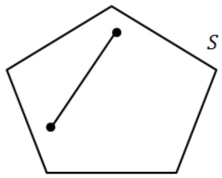

Figure 1: An example of a convex set

Definition 2:
A subset of Euclidean space is said to be closed if it contains all its limit points.

Definition 3:
A subset of Euclidean space is said to be bounded if it has all its points lie within some fixed distance of each other.

Definition 4:
A set S of real numbers is compact if and only if it is closed and bounded. Examples of closed and bounded subset of Euclidean space are a closed interval, a rectangle, or even a finite set of points. Thus, compact convex set is a closed bounded convex set.

Definition 5:
Let M be a set of points in the plane. The convex hull of M is the smallest convex polygon that contains all the points of M.

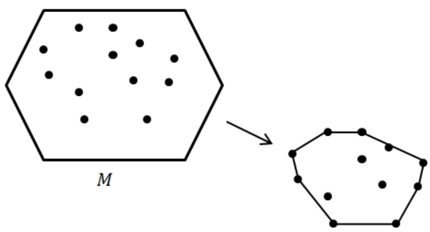

Figure 2: An example of a convex hull

CHAPTER 1

INTRODUCTION

1.1 Differential Games

1.1.1 A Brief Description

Game Theory is a formal study of decision-making where several players making choices that potentially affect the interests of the other players. In Differential Games Theory, movement of players are described by differential equations where state of players develops or depends continuously in time and therefore, $x(t)$ and the derivative of $x(t)$ or higher derivatives of $x(t)$ are variables that play parts in the calculation of the problem.

This area has been increasingly important in the application to many other fields such as military, economy, management, optimal control, engineering, biology and others. For example, in warfare technology, strategy is to be constructed for missiles to move towards any airplane regardless of the movement of the plane. On the other hand, in a different problem of differential game, strategy is to be formulated for a military plane to avoid any missiles launched against it.

In economy, the application of differential game theory has covered areas such as capital accumulation, industrial organization, oligopolies, marketing and environmental economics. An example is that of the similarity of the price-setting of oligopolies to the Prisoner's Dilemma. If an oligopoly situation exists, the companies are able to set prices if they choose to cooperate with each other. If they cooperate, both are able to set higher prices, leading to higher profits. However, if one company decides to defect by lowering its price, it will get higher sales, and, consequently, bigger profits than its competitor(s), who will receive lower profits. If both companies decide to defect, i.e. lower prices, a price war will ensue, in which case neither company will profit, since it will retain its market share and experience lower revenues at the same time. Lim (1999) defined oligopoly as a market with few large firms collectively controlling large market share and aware of interdependence of their profits and impacts of each firm's strategic decision on their profits and market shares. Game theory also finding its applications in computer science such as network security, game programming, sensor networks and internet usage. There are many other examples of the application of differential game theory in several fields.

Differential game was first introduced by Isaacs (1965) in which there are two players with opposing goals. One of them is called Pursuer and denoted as P, and the other is called Evader and denoted as E. The pursuer's goal is simply to capture the evader while the evader's goal is certainly, to avoid being captured.

1.1.2 Types of Differential Games

In general, there are two types of differential games which are as follows:

1. Pursuit differential games.

   In this game, the strategy for the pursuer will be constructed but there is no restriction on the movement of the evader.

   The game or the pursuit is completed when the positions of P and E coincide at a finite time τ, that is, $x(\tau) = y(\tau)$ for some time $\tau \geq 0$.

2. Evasion differential games.

   In this game, the strategy for the evader will be constructed but there is no restriction on the movement of the pursuer.

   The evader is to avoid being captured by the pursuer indefinitely, that is, $x(t) \neq y(t)$ for all $t \geq 0$.

   In other words, the position of the pursuer and the evader will not coincide at all time and we say, the evasion is possible.

1.1.3 Types of Constraints

Movements of players in differential games are described by some differential equations. We discuss some types of differential games.

i. Differential games with state/phase constraints.

   This is a constraint of the position (state) of the players at all time, where movement of players can only occur within some kind of areas in a given space. For example, in the Lion and Man game, both players can only move within a given circle. Here, the state constraint is a circle in $\mathbb{R}^2$.

ii. Differential games with integral constraint.

   This is a constraint which is exhausted by consumption, and it is in the form of integral. Examples are energy, fuel, food and finance which are exhausted over time as they are consumed. A common integral constraint for a player is energy which is bounded, and written as $\int_0^\infty |u(t)|^2 dt \leq \rho^2$ for some positive value ρ where $|u(t)|$ is the speed of the player.

iii. Differential games with geometric constraint.

   Constraint of the form $u(t) \in P \subset R^n$ is called geometric constraint where u is the control paramater and P is a subset of R^n. For example, in the case of simple motion, the constraint $|u| \leq 1$ means $u(t) \in P \subset \mathbb{R}^2$ where P is a circle of radius 1. In other words, the speed of the player is bounded by 1.

The thesis consider problems of simple motion evasion differential game of

- one evader versus one pursuer,
- one evader versus two pursuers,
- one evader versus finitely many pursuers.

All these problems occur in the plane $\mathbb{R}^2$ and the control functions of players are bounded by geometric constraint. In general, movements of players are described by the following differential equation:

$$\dot{x}_i = u_i, \;\; x_i(0) = x_{i0}, \;\; i = 1, \ldots, m,$$
$$\dot{y} = v, \;\; y(0) = y_0,$$

where u_i and v are the control parameters of P_i and E respectively, x_{i0} and y_0 are the initial positions of P_i and E respectively with $x_{i0} \neq y_0$.

The important feature in this game is that, the direction of trajectory of the evader can only be within a sector which will be defined. The control set of the evader is a sector. However for each pursuer, the control set is a circle and this means, pursuer can move in any direction.

1.2 Lion and Man Game

We introduce differential game theory by presenting a classical pursuit-evasion game called Lion and Man game. It was originally posed in 1925 by Rado to find out a strategy for a pursuer (lion) to catch the evader (man) in a given environment. When the lion catch the man, it means that the man and the lion coincide after a finite time. The goal of the lion is to get close to, and possibly catch the man for any trajectory of the man. In turn, the man tries to avoid being captured. The main question here is that, can the lion catch the man?

Here we are going to show that evasion is possible in differential game of the Lion and Man. In this problem, the man constructs strategy to ensure the possibility of not being captured by the lion indefinitely. We say that evasion is possible in the game.

1.2.1 Description of Differential Game

Both lion and man have same motion capabilities and move with maximum speeds equal to 1. The positions of the players are always being in a circle for any trajectory, and must not leave the given circle.

They have perfect information about each other's position, but have different goals. The lion wants to decrease his distance to zero distance for some finite time, while the man wants to avoid being captured by the lion. To show that the lion cannot catch the man, strategy for the man need to be formulated since this is evasion problem.

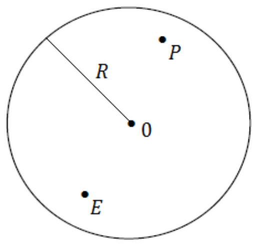

Figure 1.1: State of the players

In this game, we consider the lion as the Pursuer (P) and man as the Evader (E). The diagram in Figure 1.1 show states of the players which are always in a circle.

The movement of the players are governed by the following equations

$$\begin{aligned} P &: \quad \dot{x} = u, \quad |u| \le 1 \\ E &: \quad \dot{y} = v, \quad |v| \le 1 \end{aligned}$$

where u and v are the control parameters of the P and E respectively. We assume that the radius of the circle is R. In this evasion game, $|u| \le 1$ and $|v| \le 1$ are the geometric constraint on controls of the pursuer and evader respectively.

1.2.2 Strategy for the Evader

There are three cases possible for movement of the players in the lion and man game as pictured in the following figure:

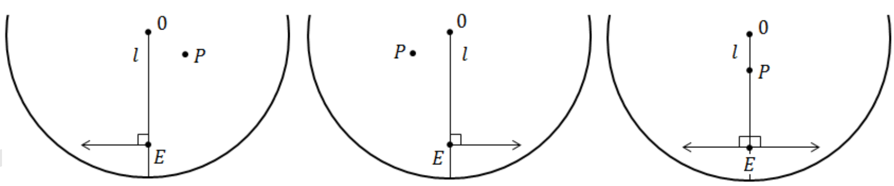

Figure 1.2: Movement of Lion and Man

In each case, movement of E depend on whether P is on the left or the right side of straight line OE where O is the origin of the circle. In case (i), E will move to the left perpendicularly to line OE since P is on the right side of OE. In case (ii), E move to the right. If P is on the line OE as in case (iii), E can move either to the left or right. The result of this differential game is the following theorem.

Theorem 1.1 In the game of Lion and Man, evasion is possible.

Proof:

1. Construction of the evader's (Man) strategy.

   First, we assume the evader to move with maximum speed 1. Second, the strategy of the evader in terms of movement is constructed this way (see Figure 1.3). Without loss of generality, we assume that E is inside the circle.

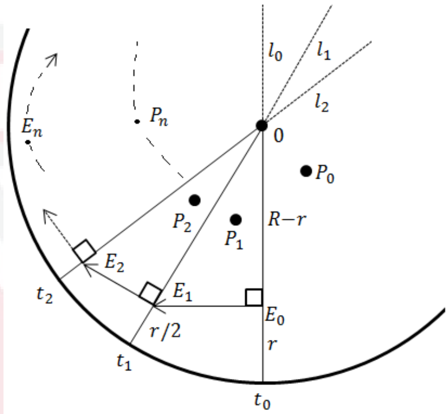

Figure 1.3: State of players in a circle

   From Figure 1.3, E_i and P_i, for $i \in \{0, 1, 2, \ldots, n, \ldots\}$ denote point of position of E and P respectively at time t_i. In addition, l_i denote the straight line connecting the origin O with E_i. Now, at each time t_i, the distance between E_i to circumference equals $\frac{r}{i+1}$. There are three possible cases of movement of E (see Figure 1.2). Based on Figure 1.3, the position $P(t_i)$ of the Pursuer P_i at time t_i is either on the right of l_i, or on the left of l_i, or on the line l_i. If P_i is on the right of l_i, then evader moves from E_i to the left of l_i perpendicularly. If P_i is on the left of l_i, then E_i moves to the right of l_i perpendicularly. Finally, if P_i is on the line of l_i, then E_i moves either to the left or to the right of l_i perpendicularly. Without loss of generality, we assume that P_i is always on the right of l_i or on the line of l_i, and therefore E choses to move the left of l_i perpendicularly.

   Specifically at time t_0, the distance between E_0 to circumference is r. If the position of P_0 is on the right or on the line of l_0, then E_0 will move to the left perpendicularly to l_0 until the point E_1, where the distance between E_1 to circumference is $\frac{r}{2}$. At E_1, if the position of P_1 is on the right or on the line of l_1, then E_1 will move to the left perpendicularly to l_1 until the point E_2, where the distance between E_2 to circumference is $\frac{r}{3}$. The strategy will continue in a similar manner.

2. Admissibility of the strategy

   We need to show two parts for the strategy to be admissible. First, according to the geometric constraint of the evader, evader always move with speed v which is less than or equal to 1, that is $|v| \leq 1$. For this evasion game, we assume E to move with maximum speed 1. Since $1 \leq 1$, the strategy is admissible in terms of speed.

   Second, the distance E_i with the circumference of the circle is always equal to $\frac{r}{i+1}$, $i \in \{0, 1, 2, \ldots\}$. Since $\frac{r}{i+1} > 0$ for each i, this means E never leave the circle. Therefore, the strategy is admissible in terms of the state of the evader at all time.

1.2.3 Evasion is Possible

The proof consists of two parts.

a. First we show that on each section E_iE_{i+1}, $i = 0, 1, 2, \ldots$ evasion is possible.

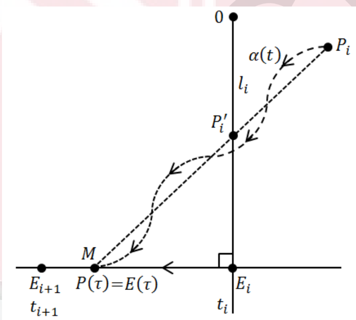

Figure 1.4: Section E_iE_{i+1}

Let $E_i = E(t_i)$, $P_i = P(t_i)$ and the position of P_i is on the right or on the line of l_i. We assume P_i is moving with the speed of $\alpha(t)$ where $0 \leq \alpha(t) \leq 1$. This is pictured in Figure 1.4.

Now, assume the contrary which is, pursuit is completed at some time τ at point M. Thus,

$$P(\tau) = E(\tau) = M.$$

On the other hand, E is moving with maximum speed 1 from E_i to E_{i+1}. Thus,

$$E_iM = 1.(\tau - t_i) = \tau - t_i.$$

Now, distance travelled by P from P_i to M is calculated as follows:

$$\widetilde{P_iM} = \int_{t_i}^{\tau} \alpha(t)\, dt \leq \int_{t_i}^{\tau} 1\, dt = \tau - t_i$$

where $\widetilde{P_iM}$ is length of the curve. Thus,

$$\tau - t_i = E_iM \geq \widetilde{P_iM} \geq P_iM \geq P_i'M > E_iM.$$

where P_iM and $P_i'M$ are segments of straight lines and $P_i'M$ is the hypotenus of the right triangle $MP_i'E_i$.

This is a contradiction and it shows that our assumption $P(\tau) = E(\tau)$, is wrong. Thus, on each section E_iE_{i+1} evasion is possible or the evader will not be captured by the pursuer.

b. Second, we estimate the total time spent by the evader in avoiding pursuer throughout the game.

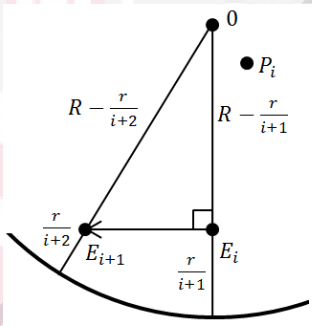

Figure 1.5: Triangle of state of players

The time spent by the evader to travel the section E_iE_{i+1} is equal to

$$T_i = \frac{E_iE_{i+1}}{1} = E_iE_{i+1} \text{ where (see Figure 1.5),}$$

$$E_iE_{i+1} = \sqrt{\left(R - \frac{r}{i+2}\right)^2 - \left(R - \frac{r}{i+1}\right)^2} \geq \frac{r}{i+2}.$$

Thus, for the sections $E_0E_1, E_1E_2, \ldots, E_nE_{n+1}$ we obtain

$$T_0 = E_0E_1 = \sqrt{\left(R - \frac{r}{2}\right)^2 - (R-r)^2},$$

$$T_1 = E_1E_2 = \sqrt{\left(R - \frac{r}{3}\right)^2 - \left(R - \frac{r}{2}\right)^2},$$

$$\vdots$$

$$T_n = E_nE_{n+1} = \sqrt{\left(R - \frac{r}{n+2}\right)^2 - \left(R - \frac{r}{n+1}\right)^2}. \qquad (1.1)$$

Now, $T_n \geq \frac{r}{n+2}$ as shown below:

$$\begin{aligned}
T_n^2 &= \left(R-\frac{r}{n+2}\right)^2-\left(R-\frac{r}{n+1}\right)^2 \\
&= \left(\left(R-\frac{r}{n+2}\right)-\left(R-\frac{r}{n+1}\right)\right)\left(\left(R-\frac{r}{n+2}\right)+\left(R-\frac{r}{n+1}\right)\right) \\
&= \left(\frac{r}{n+1}-\frac{r}{n+2}\right)\left(2R-\frac{r}{n+2}-\frac{r}{n+1}\right) \\
&\geq \left(\frac{r(n+2)-r(n+1)}{(n+1)(n+2)}\right)\left(\frac{2r(n+2)(n+1)-r(n+1)-r(n+2)}{(n+1)(n+2)}\right) \\
&= \left(\frac{r}{(n+1)(n+2)}\right)\left(\frac{2rn^2+6rn+4r-rn-r-rn-2r}{(n+1)(n+2)}\right) \\
&= \frac{r^2\left(2n^2+4n+1\right)}{(n+1)^2(n+2)^2} \\
&\geq \frac{r^2(n+1)^2}{(n+1)^2(n+2)^2} \\
&= \frac{r^2}{(n+2)^2}.
\end{aligned}$$

Hence,

$$T_n = \frac{E_nE_{n+1}}{1} \geq \frac{r}{n+2}.$$

Calculating the sum of times spent up to point E_n we have,

$$\begin{aligned}
\sum_{i=0}^{n} Ti &\geq \sum_{i=0}^{n} \frac{r}{i+2} \\
&= r\sum_{i=0}^{n} \frac{1}{i+2}
\end{aligned}$$

But for $n \to \infty$, $\sum_{i=0}^{n} \frac{1}{i+2} \to \infty$, that is the series $\sum_{i=0}^{\infty} \frac{1}{i+2}$ is divergent.

Therefore the total time spent by the evader is infinity which means he will not be captured at all. The proof is completed. □

1.3 Research Problem

We consider a simple motion evasion differential game of many pursuers $x_1, \ldots, x_m$ versus one evader y with geometric contraints as the control functions of players on a plane. The game is described by following differential equations

$$\begin{aligned}
&\dot{x}_i = u_i,\ x_i(0) = x_{i0},\ |u_i| \leq 1,\ i = 1, \ldots, m, \\
&\dot{y} = v,\ y(0) = y_0,\ v \in \mathbf{S},
\end{aligned} \tag{1.2}$$

$x_i, u_i, x_{i0}, y, v, y_0 \in \mathbb{R}^2$ where **S** is a sector.

We say that evasion is possible in this game if the position of the evader never coincides with that of any pursuer, that is $x_i(t) \neq y(t)$ for all $t \geq 0$, and $i = 1, \ldots, m$.

1.4 Research Objectives

The objectives of this thesis are as follows:

- to construct an evasion strategy which guarantees possibility of evasion along any strip of positive width in the game of one pursuer and one evader.
- to obtain sufficient conditions of evasion from any initial positions of pursuers.
- to construct an evasion strategy in the game of many pursuers.

1.5 Thesis Organization

The thesis is divided into eight chapters as follows:

Chapter 1 includes a brief introduction to differential games and present a classic example of differential games which is lion and man game.

Chapter 2 is a discussion about related past results through literature review. This chapter reviews the method for solving evasion differential games with geometric constraints as the control function of players.

Chapter 3 is an exploration of a simple pursuit differential game using Parallel Strategy. This is a fundamental knowledge in differential game theory.

Chapter 4 is a study of a differential game of evasion from one pursuer posted by Chernous'ko (1976) and a discussion of the strategy for the case of many pursuers constructed by Chernous'ko (1976). The present evasion problem is described by the same differential equation as Chernous'ko (1976) but with different control set for the evader. We introduce new approach and strategy for the evader which is compatible with the control set. As in any evasion differential game, we solve the problem by the following steps:

1. construct strategy for the evader
2. show the admissibility of the strategy
3. prove that evasion is possible.

Chapter 5 is a description of a simple motion evasion differential game for the case of one pursuer against one evader in the plane $\mathbb{R}^2$. We prove the main theorem by construction of the strategies, checking the admissibility of the strategies and estimating distances between the evader and pursuer at any time to show the possibility of the evasion.

Chapter 6 is a study of a simple motion evasion differential game for the case of two pursuers against one evader in the plane $\mathbb{R}^2$. In this chapter, the concept of fictitious evader is introduced. We prove the main theorem by construction of the strategies of evader and fictitious evader and estimating distances between the evader and pursuers at any time to show the possibility of the evasion.

Chapter 7 is the main component of the thesis of which we study an evasion differential game described by simple motion differential equations in the plane $\mathbb{R}^2$, for the case of many pursuers against one evader. It is shown that evasion is possible for the case of k pursuers, by construction of the evader's strategy which is admissible, and by estimating distance at any time between the evader and each pursuer.

Chapter 8 consist of conclusion and future work of this thesis.

REFERENCES

Alias, I. A., Raja Ramli, R. N., Ibragimov, G. I., and Narzullaev, A. (2015). Simple Motion Pursuit Differential Game of Many Pursuers and One Evader on Convex Compact Set. *International Journal of Pure and Applied Mathematics*, 102(4):733–745.

Alias, I. A., Salleh, Y., and Azamov, A. (2013). Evasion Differential Game of Two Evaders and One Pursuer with Integral Constraints. In *Proceedings of the International Conference on Mathematical Sciences and Statistics 2013 (AIP Conference Proceedings)*, volume 1557, pages 140–145.

Azamov, A. A. (2011). *Fundamentals of Theory of Discrete games*. Niso Poligraf, Tashkent, Uzbekistan.

Azimov, A. Y. (1974). A Linear Differential Evasion Game with Integral Constraints on the Controls. *USSR Computational Mathematics and Mathematical Physics*, 14(6):56–65.

Blagodatskikh, A. I. and Petrov, N. N. (2009). *Conflict Interaction between Groups of Controlled Objects*. Udmurt State University, Izhevsk.

Borowko, P., Rzymowski, W., and Stachura, A. (1988). Evasion from Many Pursuers in the Simple Motion Case. *Journal of Mathematical Analysis and Applications*, 135(1):75–80.

Chernous'ko, F. L. (1976). A Problem of Evasion from Many Pursuers. *Prikladnaya Matematika I Mekhanika*, 40(1):14–24.

Chernousko, F. L. and Melikyan, A. A. (1978). *Game Problems of Control and Search (in Russian)*. Nauka, Moscow, USSR.

Chernousko, F. L. and Zak, V. L. (1985). On Differential Games of Evasion from Many Pursuers. *Journal of Optimization Theory and Applications*, 46(4):461–470.

Chikrii, A. A. (1997). *Conflict-Controlled Processes*. Kluwer, Dordrecht.

Chikrii, A. A. and Prokopovich, P. V. (1992). Simple Pursuit of One Evader by A Group. *Cybernetics and Systems Analysis*, 28(3):438–444.

Chodun, W. (1989). Differential Games of Evasion with Many Pursuers. *Journal of Mathematical Analysis and Applications*, 142(2):370–389.

Friedman, A. (1971). *Differential Games*. John Wiley & Sons, New York.

Grigorenko, N. L. (1990). *Mathematical Methods of Control of Several Dynamic Processes*. Moscow State University, Moscow.

Gusiatnikov, P. B. and Mokhon'ko, E. Z. (1980). On l_∞ Escape in a Linear Many-Person Differential Game with Integral Constraints. *Journal of Applied Mathematics and Mechanics*, 44(4):436–440.

Hajek, O. (1975). *Pursuit Games*. Academic Press, New York.

Ibragimov, G. I. and Mat Hasim, R. (2010). An Evasion Differential Game in Hilbert space. *International Game Theory Review*, 12(3):239–251.

Ibragimov, G. I., Salimi, M., and Amini, M. (2012). Evasion from Many Pursuers in Simple Motion Differential Game with Integral Constraints. *European Journal of Operational Research*, 218(2):505–511.

Ibragimov, G. I. and Salleh, Y. (2012). Simple Motion Evasion Differential Game of Many Pursuers and One Evader with Integral Constraints on Control Functions of Players. *Journal of Applied Mathmatics*, 2012:1–10.

Isaacs, R. (1965). *Differential Games*. John Wiley & Sons, New York, USA.

Ivanov, R. P. (1980). Simple Pursuit-Evasion on a Compact Set. *Doklady Akademii Nauk SSSR*, 254(6):1318–1321.

Ivanov, R. P. and Ledyaev, Y. S. (1981). Optimality of Pursuit Time in a Simple Motion Differential Game of Many Objects. *Trudy Matematicheskogo Instituta imeni V. A. Steklova*, 158:87–97.

Krasovskii, N. N. (1970). *Game Problems of Contact of Motions (in Russian)*. Nauka, Moscow, USSR.

Krasovskii, N. N. and Subbotin, A. I. (1974). *Positional Differential Games (in Russian)*. Nauka, Moscow, USSR.

Kuchkarov, A. S., Risman, M. H., and Malik, A. H. (2012). Differential games with many pursuers when evader moves on the surface of a cylinder. *ANZIAM J.*, 53(E):E1–E20.

Leong, W. J. and Ibragimov, G. I. (2008). A Multiperson Pursuit Problem on A Closed Convex Set in Hilbert Space. *Far East Journal of Applied Mathematics*, 33(2):205–214.

Lim, J. (1999). Fun, Games & Economics:An Appraisal of Game Theory in Economics. *Undergraduate Journal of Economics*, pages 2–22.

Petrov, N. N. and Shchelchkov, K. A. (2012). To the Problem of Chernous'ko. *Vestnik Udmurtskogo Universiteta. Matematika. Mekhanika. Komp'yuternye Nauki*, (4):62–67.

Pontryagin, L. S. (1971). A Linear Differential Game of Evasion. *R Mat. Inst. Akad. Nauk SSSR*, 112(1):30–63.

Pontryagin, L. S. (1988). *Selected Works*. Nauka, Moscow.

Pshenichnii, B. N. (1976). Simple Pursuit by Several Objects. *Cybernetics and System Analysis*, 12(3):145–146.

Rzymowski, W. (1984). On the Game of n+1 Cars. *Journal of Mathematical Analysis and Applications*, 99(1):109–122.

Salimi, M., Ibragimov, G. I., Siegmund, S., and Sharifi, S. (2015). On a Fixed Duration Pursuit Differential Game with Geometric and Integral Constraints. *Dynamic Games and Applications*, pages 1–17.

Satimov, N. Y. and Rikhsiev, B. B. (2000). *Methods of Solving of Evasion Problems in Mathematical Control Theory*. Fan, Tashkent, Uzbekistan.

Zak, V. L. (1978). On A Problem of Evading Many Pursuers. *Journal of Applied Mathematics*, 43(3):456–465.