UNIVERSITI PUTRA MALAYSIA

DERIVATIONS AND CENTROIDS OF FINITE DIMENSIONAL DIALGEBRAS

MOHAMED ABUBAKAR HAGI MOHAMED

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DERIVATIONS AND CENTROIDS OF FINITE DIMENSIONAL DIALGEBRAS

By

MOHAMED ABUBAKAR HAGI MOHAMED

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Doctor of Philosophy

July 2016

DEDICATIONS

To spirits of my
dear Mum, Faduma Mohamed Ali, my dear Dad, Abubakar Hagi Mohamed and my
dear wife, Shamsa Ali Yabarow
whom I can feel every where, every time.

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Doctor of Philosophy

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Chairman : **Isamiddin S. Rakhimov, PhD**
Faculty : **Science**

The thesis deals with the derivations, generalized derivations and centroids of associative and diassociative algebras. In all the cases, we give an algorithm to find the derivation algebras. The algorithm is applied to compute the derivations of low dimensional diassociative algebras. Having found the derivations, we determine those diassociative algebras that are characteristically nilpotent. Some basic properties of derivations in terms of left and right multiplication operators are also given for each case.

We introduce a generalization of derivation of diassociative algebras and study its properties. The generalization of derivation depends on parameters (α, β, γ); we specify all possible values of the parameters. All of the generalized derivations of low-dimensional complex diassociative algebras are given.

We also introduce the concept of centroid for associative and diassociative algebras and study some of their properties. An algorithm to find centroids of algebras is given as well. The algorithm is then applied to determine the centroids of low-dimensional algebras.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

TERBITAN DAN SENTROID BAGI DIMENSI TERHINGGA DWIALJABAR

Oleh

MOHAMED ABUBAKAR HAGI MOHAMED

Julai 2016

Pengerusi : **Isamiddin S. Rakhimov, PhD**
Fakulti : **Sains**

Tesis ini berurusan dengan terbitan, terbitan teritlak dan sentroid bagi aljabar bersekutu dan dwibersekutu. Dalam semua kes, kami memberikan satu algoritma untuk mencari terbitan aljabar. Algoritma ini digunakan untuk mengira terbitan bagi dimensi rendah aljabar dwibersekutu. Setelah mendapati terbitan tersebut, kita menentukan aljabar dwibersekutu tersebut yang mana berciri nilpoten. Sesetengah sifat asas terbitan dalam sebutan pengoperasi-pengoperasi pendaraban kiri dan kanan juga diberi bagi setiap kes.

Kami memperkenalkan pengitlakan terbitan aljabar dwibersekutu dan mengkaji sifat-sifatnya. Pengitlakan terbitan ini bergantung pada parameter (α, β, γ); kita menentukan semua nilai yang mungkin bagi parameter. Semua terbitan teritlak berdimensi rendah bagi aljabar dwibersekutu kompleks diberikan.

Kami juga memperkenalkan konsep sentroid bagi aljabar bersekutu dan dwibersekutu dan mengkaji beberapa sifatnya. Algoritma untuk mencari sentroid aljabar tersebut juga diberikan. Algoritma ini kemudiannya digunakan untuk menentukan sentroid bagi aljabar berdimensi rendah.

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This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Doctor of Philosophy. The members of the Supervisory Committee were as follows:

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Universiti Putra Malaysia
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Universiti Putra Malaysia
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Universiti Putra Malaysia
(Member)

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Malaysian Institute of Industrial Technology
Universiti Kuala Lumpur, Malaysia
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LIST OF ABBREVIATIONS

As_p^q	q^{th} isomorphism class of associative algebras in dimension p.
$Dias_p^q$	q^{th} isomorphism class of diassociative algebras in dimension p.
$Dend_n^q$	q^{th} isomorphism class of dendriform algebras in dimension n.
$\dashv$	Left multiplication
$\vdash$	Right multiplication
D^{op}	Opposite dialgebra
$Der(D)$	The algebra of all derivations of the algebra D
$\mathbb{K}[x,y]$	Polynomial algebra
$Ann_R(D)$	Right annihilator of the diassociative algebra D
$Ann_L(D)$	Left annihilator of the diassociative algebra D
IC	Isomorphism classes of algebras
Derivation	The basic derivations
Dim	Dimensions of the algebra of derivations
L_a	Left multiplication operator by an element a
R_a	Right multiplication operator by an element a
$\Lambda(A)$	Centroid of associative algebra A
$\Gamma(D)$	Centroids of diassociative algebra D
$C(D)$	Central derivation of D
$\diamond$	Dimonoid of D
$\cong$	Isomorphism
$Z^1(A,M)$	First cocycles
$B^1(A,M)$	First coboundaries
$Z_D(H)$	centralizer of H in D
$\blacksquare$	End of the proof

CHAPTER 1

INTRODUCTION

1.1 Introduction

A derivation is a function on an algebra which generalizes certain features of the derivation operator. Specifically, given an algebra A over a ring or field $\mathbb{K}$, a $\mathbb{K}$-derivation is a $\mathbb{K}$-linear map d from A to itself that satisfies Leibniz's law: $d(xy) = (dx)y + x(dy)$. More generally, a $\mathbb{K}$-linear map D from A into an A-module M, satisfying the Leibniz law is also called a derivation. The collection of all $\mathbb{K}$-derivation of A to itself is denoted by $Der_{\mathbb{K}}(A)$. The collection of all K-derivation of A into an A-module M is denoted by $Der_{\mathbb{K}}(A,M)$. Plainly, the derivations of A to M are exactly the elements of $Z^1(A,M)$ 1-cocycles, while the so-called inner derivations are 1-coboundaries denoted by $B^1(A,M)$. Thus the derivations of A to M form an A-module with operations defined point wise, and the set of inner derivations is a submodule of derivations. The quotient of these modules is the first cohomology group $H^1(A,M)$. Derivations occur in many different contexts in diverse areas of mathematics. If the algebra A is noncommutative then the commutator with respect to an element of the algebra A defines a linear endomorphism of A to itself, which is a derivation over $\mathbb{K}$. Furthermore, the $\mathbb{K}$-module $Der(A)$ forms a Lie algebra with respect to Lie bracket defined by the commutator: $[d_1,d_2] = d_1 \circ d_2 - d_2 \circ d_1$.

1.1.1 Basic concepts

In this section, we introduce some basic concepts and notations that are used throughout this thesis. Most of the Lie algebra concepts can be found in any standard book on Lie algebras. For Leibniz algebra, the notations and concepts are referred by papers such as Loday and Pirashvili (1993), Albeverio et al. (2008), Albeverio et al. (2006), Rakhimov and Bekbaev (2010) and Ladra et al. (2011).

Definition 1.1 *Let $\mathbb{K}$ be a field, V be a vector space over $\mathbb{K}$ with a binary operation $f : V \times V \longrightarrow V$. If the binary operation is bilinear, i.e*

$$f(\alpha_1 x + \alpha_2 y, z) = \alpha_1 f(x,z) + \alpha_2 f(y,z),$$

$$f(z, \alpha_1 x + \alpha_2 y) = \alpha_1 f(z,x) + \alpha_2 f(z,y),$$

where $x,y,z \in V$ and $\alpha_1, \alpha_2 \in \mathbb{K}$. Then V is said to be an algebra over $\mathbb{K}$.

Let us introduce some classes of algebras which are closely related to the class of algebras in the thesis. The definitions and examples are mostly well-known and we bring them for completeness.

Definition 1.2 *An associative algebra A is a vector space over a field* $\mathbb{K}$ *equipped with bilinear map* $f : A \times A :\longrightarrow A$ *satisfying the associative law:*

$$f(f(x,y),z) = f(x, f(y,z)) \qquad \textit{for all} \qquad x,y,z \in A.$$

Further the notation $x \cdot y$ (even just xy) will be used for $f(x,y)$. Note that field, polynomial algebras, set of linear transformations on a vector space (quadratic matrices over a fixed field) are simple examples of associative algebras, where statements and hypotheses can be verified.

Definition 1.3 *Let A be an associative algebra. A linear transformation* $d : A \to A$ *is said to be derivation if*

$$d(x \cdot y) = d(x) \cdot y + x \cdot d(y)$$

for all $x, y \in A$.

Definition 1.4 *A Lie algebra L is a vector space over a field* $\mathbb{K}$ *equipped with a bilinear map*

$$[\cdot,\cdot] : L \times L \to L$$

satisfying the following conditions:

$$[x,x] = 0, \quad \textit{for all} \quad x \in L \tag{1.1}$$

$$\Big[[x,y],z\Big] + \Big[[y,z],x\Big] + \Big[[z,x],y\Big] = 0, \quad \textit{for all} \quad x,y,z \in L. \tag{1.2}$$

Remark 1.1

1. *The identity* (1.2) *is called the Jacobi identity.*
2. *Relation* (1.1) *implies the anticommutativity of the multiplication of L:*

$$[x,y] = -[y,x] \quad \textit{for all} \quad x,y \in L$$

In fact, we have

$$0 = [x+y, x+y] = [x,y] + [y,x].$$

Conversely, if the characteristic of the field $\mathbb{K}$ *is different from* 2, *anticommutativity of the bracket implies* $[x,x] = 0$.

Example 1.1 *Any vector space V has a Lie bracket defined by* $[x,y] = 0$ *for all* $x, y \in V$. *This is abelian Lie algebra structure on V. In particular, the field* $\mathbb{K}$ *may be regarded as a one-dimensional abelian Lie algebra.*

Example 1.2 *Let $\mathbb{K} = \mathbb{R}$. The vector product $(x,y) \mapsto x \wedge y$ defines the structure of a Lie algebra on $\mathbb{R}^3$. Explicitly, if $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ then $x \wedge y = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1)$*

Definition 1.5 *A Leibniz algebra L is a vector space over a field $\mathbb{K}$ equipped with a bilinear map*

$$[\cdot,\cdot] : L \times L \to L$$

satisfying the Leibniz identity

$$\Big[x,[y,z]\Big] = \Big[[x,y],z\Big] - \Big[[x,z],y\Big] \quad \text{for all} \quad x,y,z \in L. \tag{1.3}$$

A Leibniz algebra is a Lie algebra if the condition

$$[x,x] = 0 \quad \text{for all} \quad x \in L$$

is fulfilled.

Example 1.3 *Let A be an associative algebra and P be a projector in $End(A)$, that is $P^2 = P$. Suppose that $P(aP(b)) = P(P(a)b)$. Denote Px by $\bar{x}$. Define*

$$[x,y] = \bar{x}y - y\bar{x}.$$

This provides A with a Leibniz algebra structure. Indeed,
$[[x,y],z] - [x,[y,z]] + [y,[x,z]]$

$$\begin{aligned}
&= \overline{[x,y]}z - z\overline{[x,z]} - \bar{x}[y,z] + [y,z]\bar{x} + \bar{y}[x,z] - [x,z]\bar{y} \\
&= \overline{(\bar{x}y - y\bar{x})}z - z\overline{(\bar{x}y - y\bar{x})} - \bar{x}(\bar{y}z - z\bar{y}) + (\bar{y}z - z\bar{y})\bar{x} + \bar{y}(\bar{x}z - z\bar{x}) - (\bar{x}z - z\bar{x})\bar{y} \\
&= (\bar{x}\bar{y} - \bar{y}\bar{x})z - z(\bar{x}\bar{y} - \bar{y}\bar{x}) - \bar{x}\bar{y}z + \bar{x}z\bar{y} + \bar{y}z\bar{x} - z\bar{y}\bar{x} + z\bar{y}\bar{x}z - \bar{y}z\bar{x} - \bar{x}z\bar{y} + z\bar{x}\bar{y} \\
&= (\bar{x}\bar{y} - \bar{y}\bar{x})z - z(\bar{x}\bar{y} - \bar{y}\bar{x}) - \bar{x}\bar{y}z - z\bar{y}\bar{x} + \bar{y}\bar{x}z + z\bar{x}\bar{y} \\
&= 0.
\end{aligned}$$

In the above example we take P as an identity operator then one has a Lie algebra.

Example 1.4 *Let L be a Lie algebra and let M be a L-module with an action $M \times L \to L$, $(m,x) \mapsto mx$. Let $\psi : M \to L$ be a L-equivariant linear map, this is $\psi(mx) = [\psi(m), x]$, for all $m \in M$ and $x \in L$, then one can define a Leibniz structure on M as follows:*

$$[m,n]' := m\psi(n), \quad \text{for all} \quad m,n \in M.$$

Definition 1.6 *Let L and L_1 be two algebras over a field $\mathbb{K}$. A linear mapping ψ : $L \longrightarrow L_1$ is a homomorphism if*

$$\psi([x,y]_L) = [\psi(x), \psi(y)]_{L_1}, \quad \textit{for all} \quad x,y \in L. \tag{1.4}$$

The notion of diassociative algebras is a generalization of associative algebras, with two operations, which gives rise to Leibniz algebras instead of Lie algebras.

Definition 1.7 *Let D be a vector space over a field $\mathbb{K}$ equipped with two associative products $\vdash$ and $\dashv$: $D \times D \longrightarrow D$, If for any $x,y,z \in D$ satisfying the identities:*

$$\begin{aligned} x \dashv (y \dashv z) &= x \dashv (y \vdash z) \\ (x \vdash y) \dashv z &= x \vdash (y \dashv z) \\ (x \vdash y) \vdash z &= (x \dashv y) \vdash z, \end{aligned}$$

then $(D, \vdash, \dashv)$ is called an associative dialgebra or (the term diassociative algebra also is used).

The operations $\dashv$ and $\vdash$ are called left and right products, respectively. A bar-unit in D is an element e in D such that $x \dashv e = x = e \vdash x$ for all $x \in D$.
A bar-unit needs not to be unique. The subset of bar-units of D is called its halo. A unital dialgebra is a dialgebra with a specified bar-unit e.
Observe that if a dialgebra has a unit e, which satisfies $e \dashv x = x$ for any $x \in D$, then $\dashv = \vdash$ and D is an associative algebra with unit e.

Example 1.5 *If A is an associative algebra, then the formula $x \dashv y = xy = x \vdash y$ defines a structure of dialgebra on A.*

Example 1.6 *If (A,d) is a differential associative algebra with $d^2 = 0$, then the formulas $x \dashv y = xdy$ and $x \vdash y = dxy$ define a structure of dialgebra on A.*

Example 1.7 *Let $D = \mathbb{R}^n$. We define $\dashv$ and $\vdash$ on D as follows:*

$$(x \dashv y)_i = x_i \Big(\sum_{j=1}^{n} y_j \Big) \qquad for \quad 1 \leq i \leq n \tag{1.5}$$

and

$$(x \vdash y)_i = \Big(\sum_{j=1}^{n} x_j \Big) y_i \qquad for \quad 1 \leq i \leq n. \tag{1.6}$$

Then it is easy to check that $D = (\mathbb{R}^n, \dashv, \vdash)$ is an associative dialgebra.

Example 1.8 *Let V be a vector space and fix* $\phi \in V^*$ *(the algebraic dual), then one can define a dialgebra structure on V by setting* $x \dashv y = \phi(y)x$ *and* $x \vdash y = \phi(x)y$*, denoted by* V_ϕ*. If* $\phi \neq 0$*, then* V_ϕ *is a dialgebra with non-trivial bar-units.*

Example 1.9 *Let* $\mathbb{K}[x,y]$ *be the polynomial algebras over a field* $\mathbb{K}$ $(char\mathbb{K} = 0)$ *with two indeterminate* x, y*. We define the left product* $\dashv$ *and the right product* $\vdash$ *on* $\mathbb{K}[x,y]$ *as follows:*

$$f(x,y) \dashv g(x,y) = f(x,y)g(y,y)$$

and

$$f(x,y) \vdash g(x,y) = f(x,x)g(x,y).$$

Then $(\mathbb{K}[x,y], \dashv, \vdash)$ *is an associative dialgebra.*

Example 1.10 *Let* $(D, \dashv, \vdash)$ *be an associative dialgebra. Consider the module of* $n \times n$ *- matrices* $M_n(D) = M_n(\mathbb{K}) \otimes D$ *with products* $(\alpha \dashv \beta)_{ij} = \Sigma_k \alpha_{ik} \dashv \beta_{kj}$ *and* $(\alpha \vdash \beta)_{ij} = \Sigma_k \alpha_{ik} \vdash \beta_{kj}$*. Then* $(M_n(D), \dashv, \vdash)$ *is a diassociative algebra. Moreover, if* D_1 *and* D_2 *are diassociative algebras over a field* $\mathbb{K}$ *then their tensor product* $D_1 \bigotimes_{\mathbb{K}} D_2$ *is provided by a dialgebra structure defined as follows:*

$$(a \otimes a') \star (b \otimes b') = (a \star b) \otimes (a' \star b') \ \text{ for } \star = \vdash \text{ and } \dashv.$$

Definition 1.8 *A subset I of a dialgebra D is called a subalgebra of D, if I is the subspace of D and for any* $x, y \in I$*:*

$$x \dashv y,\ x \vdash y \in I.$$

Definition 1.9 *A two-sided ideal of associative dialgebra D is a subspace I such that* $x \star y$*,* $y \star x$ *are in I for all* $x \in D$*,* $y \in I$ *with* $\star = \vdash$ *and* $\dashv$*. Note that I is called the right and left ideal if* $y \vdash x$*,* $y \dashv x$ *are in I, and* $x \vdash y$*,* $x \dashv y$ *are in I, respectively, for all* $x \in D$*,* $y \in I$ *.*

An ideal I of associative dialgebra D is said to be nilpotent if it is nilpotent as a subalgebra of D.

It is observed that the sum $I_1 + I_2 = \{z \in D | z = x_1 + x_2,\ x_1 \in I_1 \text{ and } x_2 \in I_2\}$ of two nilpotent ideals I_1, I_2 of D is nilpotent. Therefore there exists unique maximal nilpotent ideal of D called nilradical. The nilradical plays an important role in the classification problem of algebras.

Definition 1.10 *Let* $(D_1, \dashv_1, \vdash_1)$*,* $(D_2, \dashv_2, \vdash_2)$ *be associative dialgebras over a field* $\mathbb{K}$*. Then a homomorphism from* D_1 *to* D_2 *is a* $\mathbb{K}$*- linear mapping* $\phi : D_1 \to D_2$ *such that*

$$\phi(x \dashv_1 y) = \phi(x) \dashv_2 \phi(y)$$

and

$$\phi(x \vdash_1 y) = \phi(x) \vdash_2 \phi(y)$$

for all $x, y \in D_1$.

The set of diassociative algebras in a fixed dimension n forms a category denoted by **Dias**.

Example 1.11 *Obviously, $I = \{0\}$ and D are two-sided ideals. As well as the kernel $Ker\varphi = \{x \in D_1 |\ \varphi(x) = 0\}$ of a homomorphism $\varphi : D_1 \longrightarrow D_2$ from diassociative algebra D_1 to D_2 is two-sided ideal in D_1 whereas the image $Im\varphi = \{y \in D_2 | \exists \quad x \in D_1 :\ \varphi(x) = y\}$ is just a subalgebra of D_2.*

Remark 1.2 *If $D_1 = D_2$ then a homomorphism is called endomorphism. In addition if ϕ is bijective then ϕ is called automorphism.*

Definition 1.11 *A derivation of associative dialgebra D is a linear transformation $d : D \to D$ satisfying*

$$d(x \dashv y) = d(x) \dashv y + x \dashv d(y)$$

and

$$d(x \vdash y) = d(x) \vdash y + x \vdash d(y)$$

for all $x, y \in D$.

The set of all derivations of associative dialgebra D is a subspace of $End_{\mathbb{K}}(D)$. This subspace equipped with the bracket $[d_1, d_2] = d_1 \circ d_2 - d_2 \circ d_1$ is a Lie algebra denoted by $Der(D)$.
In Lie algebras case the property to be characteristically nilpotent is an important property. For diassociative algebras case this property is defined similarly to that Lie algebras case as follows.

Definition 1.12 *An associative dialgebra D is called characteristically nilpotent if $Der(D)$ is nilpotent as an associative algebra.*

Associative dialgebra possess of two binary operations which are two right R_x, r_x and two left L_x, l_x multiplication operators defined as follows

$$R_x(y) := y \dashv x, \;\; r_x(y) := y \vdash x,$$

$$L_x(y) := x \dashv y, \;\; l_x(y) := x \vdash y.$$

Lemma 1.1 *(Rikhsiboev et al., 2014) The sets $R(D) = \{R_x | x \in D\}$, $L(D) = \{L_x | x \in D\}$, $r(D) = \{r_x | x \in D\}$, $l(D) = \{l_x | x \in D\}$ are subspaces of $End(D)$.*

Note that the following combinations of the right and left multiplication operators are also derivations of the diassociative algebra D:

$$L_x R_y + L_x L_y \text{ and } l_x r_y + l_x l_y.$$

Lemma 1.2 *(Rikhsiboev et al., 2014) For the right and left multiplication operators of diassociative algebras the following identities hold:*

$$R_x R_y = R_{r_x(y)}, \;\; R_x r_y = r_{R_x(y)}, \;\; r_x R_y = r_x r_y,$$

$$L_x L_y = L_x l_y, \;\; l_x L_y = L_{l_x(y)}, \;\; l_x l_y = l_{L_x(y)}.$$

The following proposition shows that diassociative algebra plays the analogous role as:

Proposition 1.1 *(Loday, J.L., and Frabetti, A., and Chapoton, F. and Goichot, F., 2001)*If D is a dialgebra and we define the bracket

$$[\cdot,\cdot] : D \times D \to D$$

by

$$[x,y] := x \dashv y - y \vdash x, \qquad \text{for all} \qquad x,y \in D.$$

Then $(D,[\cdot,\cdot])$ is a Leibniz algebra.

We recall the definitions of a few more classes of algebras also introduced by Loday that are closely related to the above mentioned two classes of algebras in order to complete Loday's categorical diagram. One of these classes is called the class of dendriform algebras.

Definition 1.13 *Dendriform algebra E is an algebra equipped with two binary operations*

$$\succ : E \times E \longrightarrow E, \prec : E \times E \longrightarrow E,$$

which $\forall x,y,z \in E$ satisfy the following axioms:

$$\begin{aligned} (x \prec y) \prec z &= x \prec (y \prec z) + x \prec (y \succ z), \\ (x \succ y) \prec z &= x \succ (y \prec z), \\ (x \prec y) \succ z + (x \succ y) \succ z &= x \succ (y \succ z). \end{aligned}$$

Example 1.12 *Let E be any algebra of operator valued-functions on the real line, closed under integral $\int_0^x dy$. One may wish to consider, for example, smooth $n \times n$ matrix-valued functions. Then E is a dendriform algebra for the operations:*

$$(A \prec B)(x) := A(x) \cdot \int_0^x B(y)dy \quad \text{and} \quad (A \succ B)(x) := \int_0^x A(y)dy \cdot B(x)$$

with $A,B \in E$.

Example 1.13 *(Matrices over dendriform algebras). Since in the axioms of a dendriform algebra the variables* a, b, c *stay in this order in all the monomials, the tensor product of two dendriform algebras is naturally a dendriform algebra. Similarly, let* $M_n(E)$ *be the module of* $n \times n$*- matrices with entries in the dendriform algebra* E*. Then the formulas*

$$(\alpha \prec \beta)_{ij} = \sum_k \alpha_{ik} \prec \beta_{kj} \quad \text{and} \quad (\alpha \succ \beta)_{ij} = \sum_k \alpha_{ik} \succ \beta_{kj}$$

make $M_n(E)$ *into a dendriform algebra.*

For Dendriform algebra the concept of homomorphism is defined similarly to that of diassociative algebras. Thus the class of Dendriform algebras forms a category denoted by **Dend**.

Proposition 1.2 *(Loday, J.L., and Frabetti, A., and Chapoton, F. and Goichot, F., 2001) For a dendriform algebra E, the product defined by*

$$x * y = x \prec y + x \succ y,$$

is associative .

Another class of algebras introduced by Loday is a class called Zinbiel algebras. The definition of the Zinbiel algebra is as follows.

Definition 1.14 *(?) Zinbiel algebra R is an algebra with a binary operation :* $R \times R \longrightarrow R$ *satisfying the condition :*

$$(x \cdot y) \cdot z = x \cdot (y \cdot z) + x \cdot (z \cdot y),$$

for $\forall x, y, z \in R$*.*

Homomorphism of Zinbiel algebras is a linear transformation preserving operations. The category of Zinbeil algebras as denoted by **Zinb**.

Proposition 1.3 *(Loday, J.L., and Frabetti, A., and Chapoton, F. and Goichot, F., 2001) Let R be a Zinbiel algebra and put*

$$x \prec y := x \cdot y, x \succ y := y \cdot x, \forall x, y \in R.$$

Then $(R, \prec, \succ)$*is a dendriform algebra. conversely, a commutative dendriform algebra E(i.e.dendriform algebra for which* $xy = yx$ *) is a Zinbiel algebra.*

Loday has showed that the classical relationship between Lie and associative algebras can be translated into an analogous relationship between Zinbiel and associative commutative algebras, with bracket $[x,y] = xy - yx$.
All of these classes of algebras can be assembled into a diagram called Loday diagram as in Fig. 1.1.

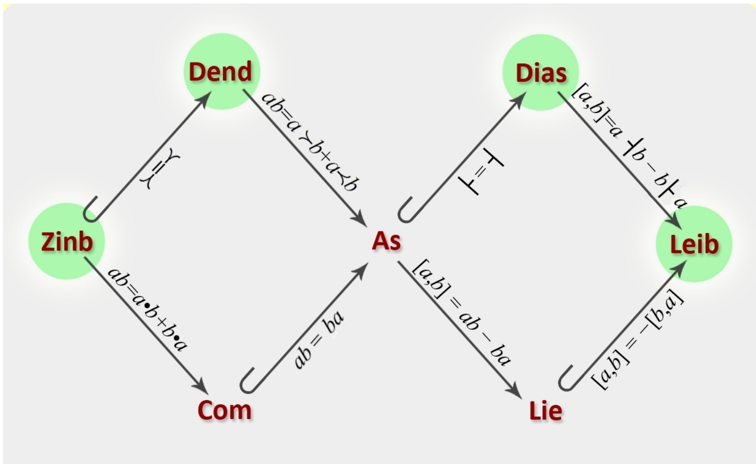

Figure 1.1: Loday Diagram

1.2 Motivation

The main motivation of the research is the application of the concept of derivation in analysis, differential equations, differential geometry, differential algebra, mechanics, physics and other many areas of science. In algebra the derivations are important invariant in studying the cohomological problems. The study of derivations of diassociative algebra provides solution to some important structural and cohomological problems. Extensions of the results on derivations to diassociative algebras are desirable. Particularly, there is an important class of diassociative algebras called characteristically nilpotent which forms one of the irreducible components of variety of diassociative algebras. One of the important invariant of algebras is the dimension of the derivation algebra. The dimension of the derivation algebra and the dimension of the automorphism group are complements to each other. These two invariants are useful in geometric classification of diassociative algebras.

The study of the generalized derivation can be use in the geometrical classification of diassociative algebras, notably in the study of the degenerations of the diassociative algebras.

The centroid are a key ingredient in the classification of associative and diassociative algebras. The centroid also plays an important role in understanding forms of an algebra: All scalar extensions of a simple algebra remain simple if and only if its centroid just consists of the scalars in the base field. In particular, for finite-dimensional simple associative algebras, the centroid is critical in investigating Brauer groups and division algebras. Another area where the centroid occurs naturally is in the study of derivations of an algebra.

1.3 Objectives of the research

The main objectives of this research are:

- To describe the algorithm for derivations of associative algebras, dialgebras and also compute their derivations on low dimensional cases.
- To give the list of isomorphism classes of three-dimensional diassociative algebras together with their properties.
- To introduce the concept of generalized derivations on diassociative algebras and give descriptions of the generalized derivations in low dimensional cases.
- To introduce the concept of centroids of associative, dialgebras and studied their properties.

1.4 Thesis Organization

The thesis contains five chapters. Now, we briefly mention the layout of the thesis as follows.

- Chapter one is a review of some important results on diassociative and dendriform algebras. We introduce the basic definitions of associative and diassociative algebras which will be used throughout the thesis.
- Chapter two is devoted to derivations of associative algebras. A simple algorithm to find the derivations is given and it is applied for low-dimensional cases.
- Chapter three focuses on studying the derivations of two, three and four-dimensional dialgebras. In tables the characteristically nilpotency of the above cases are indicated.
- In Chapter four, we give a description of (α, β, γ)-derivations of low dimensional dialgebras and determines if different types of subspaces of algebras.
- In Chapter five, we examine centroids of two, three and four-dimensional algebras and dialgebras are respectively, to determine which centroid is small.
- Chapter six makes some conclusions on results of the thesis and suggest a few problems for future research in these areas.

1.5 Literature Review

In 1993, Loday introduced the notion of Leibniz algebras, a generalization of Lie algebra, where the skew-symmetricity of the bracket is dropped and the Jacobi identity is replaced with the Leibniz identity (the identity has been called Leibniz identity by Loday due to its similarity to Leibniz rule, this is the reason for the class to be called by the name of Leibniz). Loday also showed that the link between Lie and associative algebras can be extended to analogous link between Leibniz algebras and a so-called associative dialgebras which is a generalization of associative algebras possessing two composition laws. Loday showed that if $D = (V, \dashv, \vdash)$ is a diassociative algebra structure on a vector space V then the Leibniz algebra structure on V is defined by the bracket $[x, y] := x \dashv y - y \vdash x$. Conversely, the universal enveloping algebra of Leibniz algebra has the structure of diassociative algebra. Loday has given some examples to motivate the study of these classes of algebras. On the structure of algebras from these classes not so much is known. Researchers (Loday, Pirashvili and others) mainly focused on their (co)homological problems.

An associative dialgebra is a vector space with two bilinear binary associative operations $\dashv, \vdash$, satisfying certain conditions (Loday, J.L., and Frabetti, A., and Chapoton, F. and Goichot, F., 2001). Associative algebras are particular case of the associative dialgebras when the two operations coincide.

The classification problem of algebras is one of the important problems of modern algebra. The problem has been successfully solved for semisimple parts of many classes of finite-dimensional algebras. However, the complete classification of solvable and nilpotent parts is still unsolved. Particularly, for Lie, Leibniz and associative algebras the solution to the problem in low-dimensional cases has been given with some conditions.

One of the approaches is to use the conditions on structure constants due to axioms of the class of algebras considered. As for associative dialgebras none of the above mentioned parts are studied.

The classification of associative algebras is an old and often recurring problem. The first investigation into it was perhaps done by Peirce (1881). Many interesting results related to the problem have appeared since then. Further works in this field can be found in Hazlett (1916) (nilpotent algebras of dimension ≤ 4 over $\mathbb{C}$), Mazzola (1979)- associative unitary algebras of dimension five over algebraic closed fields of characteristic not two, Mazzola (1980)- nilpotent commutative associative algebras of dimension ≤ 5, over algebraic closed fields of characteristic not two, three and recently, Poonen (2008)- nilpotent commutative associative algebras of dimension ≤ 5, over algebraically closed fields and De Graaf (2010) classify nilpotent associative algebras of dimensions ≤ 3 over any field, and four-dimensional commutative nilpotent associative algebras over finite fields and over $\mathbb{R}$.

A new era in the development of the theory of finite-dimensional associative algebras begun due to works of Wedderburn (1907), who obtained the fundamental results of this theory: description of the structure of semisimple algebras over a field, a theorem

on the lifting of the quotient by the radical, the theorem on the commutativity of finite division rings, and others.
Recent development of the theory of associative algebras was in the 80s of the last century, when some open, unsolved problems of this field were ultimately solved.

The next theorems are basis of the structural theory of associative algebras (see Kirichenko (2007)).

Theorem 1.1 (Wedderburn, 1907) Any finite-dimensional semisimple associative algebra A is uniquely decomposed into a direct sum of a number of simple algebra:

$$A = B_1 \oplus B_2 \oplus \cdots \oplus B_k.$$

Recall that an algebra is simple if it has nontrivial two-sided ideals.

Theorem 1.2 *Any finite-dimensional simple associative algebra A is isomorphic to the algebra of matrices $M_n(D)$ over a division ring D, the number n and the division ring D are uniquely determined by the algebra A.*

These theorems give a complete description of semisimple algebras. At the same time on the structure of nonsemisimple algebras, not much is known, even for an algebraically closed fields.

Complex associative algebras in dimension up to 5 were first classified by B. Pierce back in 1870, initially in the form of manuscripts, which appeared later in Pierce (1881). There are classifications of unital 3, 4 and 5-dimensional associative algebra by Gabriel (1975) and Mazzola (1979), respectively.

The Rota-Baxter algebra was first introduced by Baxter (1960) while doing probability study, which was then used and popularized by Rota (1969) and his co-workers. Loday (1993) introduced dendriform algebra notion related to dialgebra structure. The central point from our point of view is the intimate relation between the Rota-Baxter algebras and dendriform algebras. In 2002, Ebrahimi attempted to explore the relationship between Rota-Baxter operators and Loday-type algebras, i.e. dendriform di- and tri-algebras (see Ebrahimi-Fard (2002)). It has already been proved that associative algebras equipped with a Rota-Baxter operator of arbitrary weight always give rise to dendriform structures. The relationship between Rota-Baxter algebras and dendriform dialgebras and the study of the adjoint functors for the category of Rota-Baxter algebras and categories of dendriform di and trialgebras were considered in the work of Ebrahimi and Guo (2005 and 2007). The reformulation of the free dendriform algebra over the generator via a parenthesis setting was done by Leroux (2006). He also gave a brief survey on planar binary trees.

Dialgebra cohomology with coefficients was studied by Frabetti (1997, 2001) whereas Majumdar and Mukherjee (2002) considered the deformations of dialgebras. Dialgebras may appear in different contexts. For example dialgebra can be related to triple product as in (Pozhidaev, 2008). Lin and Zhang (2010) were the first to defined a new associative dialgebra over a polynomial algebra $F[x,y]$ with two indeterminates x and y. Bokut et al. (2010) made use of the Grobner-Shirshov basis for a dialgebra .

The idea of left-symmetric dialgebras was introduced by Felipe (2011). Bremner (2012) has explored some recent developments in the theory of associative and nonassociative dialgebras. The problems of finding special identities for dialgebras was studied by Kolesnikov and Voronin (2013). Zhang et al. (2014) has introduced the concepts of a totally compatible Lie dialgebra. The first paper concerning the Construction of dialgebras through bimodules over algebras is given by Salazar-Díaz et al. (2016).

There were several approaches to the study of generalized derivations on Lie algebras. For example Hartwig et al. (2006) defined the generalized derivation as a linear operator A satisfying the property $A[x,y] = [Ax, \tau y] + [\sigma x, Ay]$, where τ, σ are fixed elements of $EndL$. Such derivations have been called (σ, τ)-derivations of L. In Bresar (1991), he considered the following generalization: a linear transformation A of a Lie algebra L is said to be a generalized derivation if there exists $B \in DerL$ such that for all $x, y \in L$, the condition $A[x,y] = [Ax,y] + [x,By]$ holds. Leger and Luks (2000) studied a more general version defining the generalized derivation as $A \in EndL$ such that there exist $B, C \in EndL$ possessing the property $C[x,y] = [Ax,y] + [x,By]$. In Novotnỳ and Hrivnák (2008), the authors introduced a new version of a generalization of Lie algebra derivations and use them in algebraic and geometric classification problems in low-dimensional cases. Our interest is to study the generalized derivations of finite dimensional diassociative algebras. The algebra of derivations and generalized derivations are very useful in algebraic and geometric classification problems of algebras.

It is natural in our study to consider centroid of algebras since it is closely related to derivation. The definition of centroid for non associative algebra can be found in Jacobson (1979). In Jacobson (1979), it is also stated that the centroid of a simple Lie algebras is a field. Centroid plays a vital role in the classification of finite dimensional extended affine Lie algebras over arbitrary field of characteristic 0 (Neher, 2004). There have been considerable efforts on the centroid of some classes of algebras.

Melville (1992) investigated the centroids of some classes of nilpotent Lie algebras and their infinite dimensional analogues. Melville (1993) also, showed that for some Cartan algebras over an algebraically closed field of characteristic zero, the centroid of the cartan algebras is just the scalar. Benkart and Neher (2006) developed general results on centroids of Lie algebras and applied them to determine the centroid of extended affine Lie algebras, loop-like and Kac-Moody Lie algebras, and Lie algebras graded by finite root systems. McCrimmon (1999) used eigenvalue lemma to characterise the centroids of the basic simple Jordan algebras, triples and pairs. Richardson (2008) studied the centroids of quadratic Jordan superalgebra wherein it was shown that such superalgebra have no odd centroid.

The centroids of n-Lie algebras and centroid structures of n-Lie algebras have been studied in Bai et al. (2009). Ni (2014) has classified the centroids of Zinbiel algebras up to dimension $\leqslant 4$ and studied properties of its cenroids.

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