UNIVERSITI PUTRA MALAYSIA

NEWTON-KANTOROVICH METHOD FOR SOLVING ONE- AND TWO-DIMENSIONAL NONLINEAR VOLTERRA INTEGRAL EQUATIONS OF THE SECOND KIND

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By

HAMEED HUSAM HAMEED

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Doctor of Philosophy

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DEDICATIONS

To Her Dignity, Hadhrat Fatima Al-Zahraa (peace upon her).

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Doctor of Philosophy

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Chair: Assoc. Prof. Zainidin Eshkuvatov, PhD
Faculty: Science

The problems of nonlinear Volterra integral equations (VIEs) which consist of one dimensional nonlinear VIE, system of 2×2 nonlinear VIEs, system of $n \times n$ nonlinear VIEs, and two dimensional nonlinear VIE, with smooth unknown functions and continuous bounded given functions are discussed. The Newton-Kantorovich method (NKM) is used to linearize the problems. Then the Nystrom-type Gauss-Legendre quadrature formula (QF) is used to solve the linearized equations and systems. New majorant functions are found for some problems which lead to the increment of convergence interval. The new approach based on the subcollocation method is developed and motivation leads to high accurate approximate solutions. The existence and uniqueness of solution are proved and error estimation and rate of convergence are obtained. Numerical examples show that our results are coincided with the theoretical finding.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

KAEDAH NEWTON-KANTOROVICH BAGI SATU- DAN DUA-BERDIMENSI VOLTERRA PERSAMAAN KAMIRAN TAK LINEAR JENIS KEDUA

Oleh

HAMEED HUSAM HAMEED

Januari 2016

Pengerusi: Profesor Madya . Zainidin Eshkuvatov, PhD
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Masalah persamaan kamiran Volterra (VIEs) tak linear yang mengandungi VIEs tak linear berdimensi satu, sistem 2×2 VIEs tak linear, sistem $n \times n$ VIEs tak linear dan VIEs tak linear ber dimensi dua, dengan fungsi anu licin dan fungsi tersedia selanjar terbatas di bincangkan. Kaedah Newton-Kantorovich digunakan untuk melinearkan masalah. Kemudian formula quadratur Gauss-Legendre digunakan untuk menyelesaikan persamaan dan sistem terlinear. Fungsi majoran baharu ditemui untuk sesetengah masalah yang membawa kepada peningkatan selang penumpuan. Pendekatan baharu berdasarkan kepada kaedah sub-kolokasi yang membawa kepada penyelesaian hampiran dengan ketepatan tinggi dibina. Kewujudan dan keunikan penyelesaian dibuktikan, anggaran ralat dan kadar penumpuan diperolehi. Contoh berangka menunjukkan keputusan kami adalah sefara dengan penemuan secara teoritikal.

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This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Doctoral of philosophy. The members of the Supervisory Committee were as follows:

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LIST OF ABBREVIATIONS

FIE	Fredholm integral equation.
IE	Integral equation.
NKM	Newton-Kantorovich method.
NM	Newton method.
QF	Quadrature formula.
VIE	Volterra integral equatuion.
IVP	Initial value problem.

CHAPTER 1

INTRODUCTION

1.1 Integral equations

1.1.1 A brief historical introduction

The theory of abstract Volterra equations has been developed rapidly in the last few decades. The applications of this theory to many problems could be found for instance, mechanics, physics, engineering, economics and even biology, so its a good tool to deal with the problems arising in these fields. J. Fourier (1768-1830) was considered as the initiator of the theory of integral equations (IEs), [see Corduneanu (1991)] by obtaining the inversion formula for Fourier transform with suitable conditions on the functions f and g, where $g(t) = (2\pi)^{-\frac{1}{2}} \int_0^\infty e^{it\tau} f(\tau)d\tau$ and $f(\tau) = (2\pi)^{-\frac{1}{2}} \int_0^\infty e^{-it\tau} g(t)dt$. Of course, we can consider inversion formula as the inverse Fourier integral operator . This consideration was taken by V. Volterra during the end of 18^{th} century, who specified the problem of solving integral equations (IEs) as the problem of finding inverses of certain integral operators. Abel (1823) dealt with the integral equation (IE) known as 'Abel's equation' of the form ,

$$\int_0^t (t-\tau)^{-\alpha} u(\tau)d\tau = F(t), \quad 0 < \alpha < 1.$$

There are two significant cases related to Abel's equation. First, the kernel $k(\tau) = \tau^{-\alpha}$ has a singularity at the origin, and second, Abel's equation is a convolution type equation and both cases had a important effect on the development of the theory of IEs. Sonine (1884) tried to find the solution of various IEs, in particular Abel's equation. Levi (1895)generalized some results wich concerned with Abel's type equations. Also, in 1895 Volterra considered more general equations such as

$$u(t) + \int_0^t k(t,\tau)u(\tau)d\tau = F(t),$$

where $u(\tau)$ is unknown function to be found. Five years after Volterra made his first well known contribution to the theory of IEs, Fredholm introduced a new theory for IEs containing a parameter of the form

$$y(t) + \lambda \int_a^b k(t,\tau)y(\tau)d\tau = f(t).$$

After Fredholm's publications, the theory of IEs was developed and enriched by many mathematician like Picard (1906), Poincar (1910), Hilbert (1912), and Heywood (1912) . The causal operator or nonanticipative operator was introduced by Tonelli (1928), that is operator U acting between function spaces, such that $x(t) = y(t)$ for $t \le T$ leads to $(Ux)(T) = (Uy)(T)$. The significance of this kind of operators is obvious since most phenomena evaluated via mathematical models are casual phenomena. Tychnoff (1938) found a great connection between the theory of abstract Volterra operators with the many applications in mathematical physics. He introduced a more modern approach which contributed to its development, and considered as a model for future studies. The essential connection between Hammerstein equations and the spectrum of the linear integral operator was done by Dolph (1949). Kreın (1955) introduced a new method to solve the integral equations by constructing the resolvent kernel concept. From the 1960s onwards, many scientific research schools in the world are tried towards the investigation of different problems related to the theory of integral operators and IEs By way of example, not exhaustive enumeration, Levin and Nohel (1960) referred to the study of stability of nuclear reactors by means of IEs. During the 1970s and the 1980s the mathematicians who are interested in the IEs concentrated their efforts on problems occuring in continuum mechanics, particularly in viscoelasticity, for instance, Londen and Nohel (1982) and Hrusa (1983). Most contributions to the theory of IEs that published during the last two decades, dealt with different applications. In fact many mathematicians made their effort to solve such applied problems by using a recent tools created by basic mathematical research (monotone operators, linear and nonlinear semigroups of transformation, as well as many other functional analytic methods). In this way, the theory of integral equations has been quite developed and enriched.

1.1.2 Classification of integral equations

There are a huge number of different kinds of IEs, and each has its own case of theoretical and numerical analysis. The first main division is the one dimensional and multidimensional IEs. It follows by dividing the IEs into linear IEs and nonlinear IEs. For most the part of the analysis of IEs, efforts have been put on linear equations in a single variable, however in principle numerous of the ways are usable to multi dimensional and nonlinear equations as well.
In general, for the linear IEs, it is often use two major classes, namely Volterra and Fredholm IEs. Of course, we shall classify them later. Most standard linear IE has the form

$$h(t)x(t) = f(t) + \lambda \int_{a(t)}^{b(t)} K(t,\tau)x(\tau)d\tau, \tag{1.1}$$

where $a(t)$ and $b(t)$ are the limits of integration with $a(t) < b(t)$, $t \in [t_0, T]$, λ is a constant, and $K(t,\tau)$ is a known function of two variables t and τ, called kernel of the IE, where $x(t)$ is the unknown function, that to be found with $f(t)$, $h(t)$, and $K(t,\tau)$ are given functions beforehand. The limits of integration $a(t)$ and $b(t)$ may be both variables, constants, or mixed. Two different ways that depend on the limits of integration are used to characterize IEs, namely

1. If the limits of integration are fixed, then IE is called a Fredholm integral equation (FIE) which has the form

$$h(t)x(t) = f(t) + \lambda \int_{a}^{b} K(t,\tau)x(\tau)d\tau, \tag{1.2}$$

   where a and b are constants.

2. If at least one limit is a variable, the equation is called a Volterra integral

equation (VIE) which has the form

$$h(t)x(t) = f(t) + \lambda \int_a^x K(t,\tau)x(\tau)d\tau. \quad (1.3)$$

Moreover, there are three other distict kinds, that depend on the appearance of the function $h(t)$, are defined as follows

(i) If $h(t) = 0$, the IE is called, the first kind FIE (or VIE).

(ii) If $h(t) = 1$, the IE is called, the second kind FIE (or VIE).

iii If $0 \neq h(t) \neq 1$, the IE is called, the third kind FIE (or VIE).

In all FIEs or VIEs presented above, if $f(t)$ is identically to zero, the resulting equation gives

$$x(t) = \lambda \int_a^b K(t,\tau)x(\tau)d\tau, \quad (1.4)$$

or

$$x(t) = \lambda \int_a^t K(t,\tau)x(\tau)d\tau, \quad (1.5)$$

which is called homogeneous FIE or homogeneous VIE.

Example 1.1 *Consider the one dimensional linear VIE of the first kind*

$$\int_a^t [g(t) - g(\tau)]\, y(\tau)d\tau = f(t),$$

it is assumed that $f(a) = f_t^{'}(a) = 0$ *and* $\frac{f_t^{'}}{g_t^{'}} \neq 0$.

The solution is $y(t) = \frac{d}{dt}\left[\frac{f_t^{'}(t)}{g_t^{'}(t)}\right]$, *(Polyanin and Manzhirov, 2012).*

Example 1.2 *Consider the one dimensional nonlinear VIE of the first kind*

$$\int_0^t y(\tau)y(t-\tau)d\tau = A^2 t^{\lambda}.$$

Solutions: $y(t) = \pm A \frac{\sqrt{\Gamma(\lambda+1)}}{\Gamma(\frac{\lambda+1}{2})} t^{\frac{\lambda-1}{2}}$, *where* $\Gamma(z)$ *is the gamma function, (Polyanin and Manzhirov, 2012)*

Example 1.3 *Consider the one dimensional linear VIE of the second kind*

$$y(t) - \lambda \int_a^t \frac{g(t)}{g(\tau)} y(\tau)d\tau = f(t).$$

Solution: $y(t) = f(t) + \lambda \int_a^t e^{\lambda(t-\tau)} \frac{g(t)}{g(\tau)} f(\tau)d\tau$, *(Polyanin and Manzhirov, 2012)*

Example 1.4 *Consider the one dimensional nonlinear VIE of the second kind*

$$y(t) + \int_a^t f(y(\tau))d\tau = At + B.$$

Solution in an implicit form, $\int_{y_0}^{y} \frac{d\tau}{A - f(\tau)} = t - a, \quad y_0 = Aa + B,$ *(Polyanin and Manzhirov, 2012).*

Example 1.5 *Consider the one dimensional linear FIE of the first kind*

$$\int_a^b [g_1(t)h_1(\tau) + g_2(t)h_2(\tau)]\, y(\tau)d\tau = f(t).$$

This IE has solution only if its right hand side is representable in the form

$$f(t) = A_1 g_1(t) + A_2 g_2(t),$$

where A_1 *and* A_2 *are constants. In this case, any function* $y = y(t)$ *satisfying the normalization type conditions*

$$\int_a^b h_1(\tau)y(\tau)d\tau = A_1, \quad \int_a^b h_2(\tau)y(\tau)d\tau = A_2,$$

is a solution of the IE. otherwise, the equation has no solution, (Polyanin and Manzhirov, 2012).

Example 1.6 *Consider the one dimensional nonlinear FIE of the first kind*

$$\int_a^b y^k(t)f(\tau, y(\tau))d\tau = g(t).$$

Solution: $y(t) = \lambda[g(t)]^{\frac{1}{k}}$*, where* λ *is determined from the algebraic (or transcendental) equation* $\lambda^k \int_a^b f(\tau, \lambda g^{\frac{1}{k}}(\tau))d\tau = 1$*, (Polyanin and Manzhirov, 2012).*

Example 1.7 *Consider the one dimensional linear FIE of the second kind*

$$y(t) - \lambda \int_a^b g(t)h(\tau)y(\tau)d\tau = f(t).$$

1. *Assume that* $\lambda \neq \left(\int_a^b g(\tau)h(\tau)d\tau\right)^{-1}$*, then the solution is* $y(t) = f(t) + \lambda k g(t)$*, where* $k = \left(1 - \lambda \int_a^b g(\tau)h(\tau)d\tau\right)^{-1} \int_a^b h(\tau)f(\tau)d\tau.$

2. *Assume that* $\lambda = \left(\int_a^b g(\tau)h(\tau)d\tau\right)^{-1}.$

(i) *For* $\int_a^b h(\tau)f(\tau)d\tau = 0$, *the solution has the form* $y(t) = f(t) + Cg(t)$, *where* C *is an arbitrary constant.*

(ii) *For* $\int_a^b h(\tau)f(\tau)d\tau \neq 0$, *there is no solution.*

(Polyanin and Manzhirov, 2012).

Example 1.8 *Consider the one dimensional nonlinear FIE of the second kind*

$$y(t) + \int_a^b g(t)y(t)f(\tau, y(\tau))d\tau = h(t).$$

The solution: $h(t)1 + \lambda g(t)$, *where* λ *is determined from the algebraic (or transcendental) equation*

$$\lambda - F(\lambda) = 0, \quad F(\lambda) = \int_a^b f\left(t, \frac{h(\tau)}{1 + \lambda g(\tau)}\right) d\tau.$$

(Polyanin and Manzhirov, 2012).

Example 1.9 *Consider the two dimensional linear VIE the second kind*

$$\begin{aligned} u(x,t) - &\int_0^t \int_0^x (xy + te^z)u(y,z)dydz \\ &= xe^{-t} + t - \frac{1}{3}x^4 - xt + \frac{1}{3}x^4e^{-t} - \frac{1}{2}x^2t^2 - \frac{1}{4}x^3t^2 - xte^t + xte^t, \quad x,t \in [0,t] \end{aligned}$$

The solution: $u(x,t) = xe^{-1} + t$, *(Tari et al., 2009).*

Example 1.10 *Consider the two dimensional nonlinear VIE the second kind*

$$u(x,t) - \int_0^t \int_0^x (y^2 + e^{-2z})u^2(y,z)dydz = x^2e^t + \frac{1}{14}x^7e^{2t} - \frac{1}{5}x^5t, \quad x,t \in [0,1].$$

The solution: $u(x,t) = x^2e^t$, *(Tari et al., 2009).*

Example 1.11 *Consider the two dimensional linear FIE the second kind*

$$u(x,t) = f(x,t) + \int_0^1 \int_0^1 \left(e^{\left(\frac{x}{5}\right)^5 y} - 1\right) u(y,z)dydz, \quad 0 \leq x,t \leq 1.$$

The solution: $u(x,t) = xt$, *(Heydari et al., 2013)*

Example 1.12 *Consider the two dimensional nonlinear FIE the second kind*

$$u(x,t) = f(x,t) + \int_0^1 \int_0^1 (y\sin(z) + 1)u^3(y,z)dydz, \quad 0 \leq x,t \leq 1, \tag{1.6}$$

where $f(x,t) = x\cos(t) + \frac{1}{20}(\cos^4(1) - 1) - \frac{1}{12}\sin(1)(\cos^2(1) + 2)$. *The solution :* $u(x,t) = x\cos(t)$, *(Heydari et al., 2013).*

1.1.3 Converting initial value problem to Volterra integral equation

In this section, we show the strategy of converting an initial value problem (IVP) to an equivalent VIE. For simplicity reasons, we will employ this procedure to the second order initial value problem of the form

$$y''(t) + p(t)y'(t) + q(t)y(t) = g(t), \tag{1.7}$$

with the initial conditions

$$y(0) = \alpha, \quad y'(0) = \beta, \tag{1.8}$$

where α and β are constants, $p(t)$ and $q(t)$ are analytic functions, and $g(t)$ is continuous through the interval of discussion. To accomplish our aim, let

$$y''(t) = u(t), \tag{1.9}$$

where $u(t)$ is a continuous function. By integration both sides of (1.9) from 0 to t we get

$$y'(t) - y'(0) = \int_0^t u(\tau)d\tau, \tag{1.10}$$

or

$$y'(t) = \beta + \int_0^t u(\tau)d\tau. \tag{1.11}$$

Integration both sides of (1.11) from 0 to t we obtain

$$y(t) - y(0) = \beta x + \int_0^t \int_0^x u(\tau)d\tau dx, \tag{1.12}$$

or

$$y(t) = \alpha + \beta x + \int_0^t (t-\tau)u(\tau)d\tau. \tag{1.13}$$

By substituting (1.9), (1.11), and (1.13) into the IVP (1.7) to get the VIE:

$$u(t) + p(t)\left[\beta + \int_0^t u(\tau)d\tau\right] + q(t)\left[\alpha + \beta t + \int_0^t (t-\tau)u(\tau)d\tau\right] = g(t). \tag{1.14}$$

The last equation can be represented in the standard VIE form:

$$u(t) = f(t) - \int_0^t K(t,\tau)u(\tau)d\tau, \tag{1.15}$$

where

$$K(t,\tau) = p(t) + q(t)(t-\tau), \tag{1.16}$$

and

$$f(t) = g(t) - \left[\beta p(t) + \alpha q(t) + \beta t q(t)\right]. \tag{1.17}$$

To generalize the steps introduced above, we consider the general IVP:

$$y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_{n-1}(t)y' + a_n(t)y = g(t), \tag{1.18}$$

with initial conditions

$$y(0) = c_0, y^{'}(0) = c_1, y^{''}(0) = c_2, \cdots, y^{(n-1)}(0) = c_{n-1}, \tag{1.19}$$

where $a_i(t), 1 \leq i \leq n$ are analytic functions, and $g(t)$, $u(t)$ are continuous functions through the interval of discussion. Now set the transformation:

$$y^{(n)}(t) = u(t). \tag{1.20}$$

Integrating both sides with respect to t to get

$$y^{(n-1)}(t) = c_{n-1} + \int_0^t u(\tau)d\tau. \tag{1.21}$$

Also by integrating both sides again with respect to t yields

$$y^{(n-2)}(t) = c_{n-1} + c_{n-1}t + \int_0^t \int_0^x u(\tau)d\tau dx. \tag{1.22}$$

In equation (1.22) we reduce the double integral to a single integral by using the general form (Kanwal, 2013, pp.385)

$$\int_a^t \int_a^{t_{n-1}} \cdots \int_a^{t_2} \int_a^{t_1} F(\tau)d\tau dt_1 \cdots dt_{n-1} = \frac{1}{(n-1)!}\int_a^t (t-\tau)F(\tau)d\tau, \tag{1.23}$$

and get

$$y^{(n-2)}(t) = c_{n-2} + c_{n-1}t + \int_0^t (t-\tau)u(\tau)d\tau. \tag{1.24}$$

By integrating both sides of equation (1.21) we get

$$y^{(n-3)}(t) = c_{n-3} + c_{n-2}t + \frac{1}{2}c_{n-1}t^2 + \int_0^t \int_0^{t_2} \int_0^{t_1} u(\tau)d\tau dt_1 dt_2 \tag{1.25}$$

then by using the concept of equation (1.23) we obtain

$$y^{(n-3)}(t) = c_{n-3} + c_{n-2}t + \frac{1}{2}c_{n-1}t^2 + \frac{1}{2}\int_0^t (t-\tau)^2 u(\tau)d\tau. \tag{1.26}$$

Keeping the integration process yields

$$y(t) = \sum_{k=0}^{n-1} \frac{c_k}{k!}t^k + \frac{1}{(n-1)!}\int_0^t (t-\tau)^{n-1}u(\tau)d\tau. \tag{1.27}$$

Substituting (1.20)− (1.27) into (1.18) gives

$$u(t) = f(t) - \int_0^t K(t,\tau)u(\tau)d\tau, \tag{1.28}$$

where

$$K(t,\tau)=\sum_{k=1}^{n}\frac{a_k}{(k-1)!}(t-\tau)^{k-1}, \tag{1.29}$$

and

$$f(t)=g(t)-\sum_{j=1}^{n}a_j\left(\sum_{k=1}^{j}\frac{c_{n-k}}{(j-k)!}t^{j-k}\right). \tag{1.30}$$

In the following, we give an example on the applicability of the conversion methods

Example 1.13 *Convert the following IVP to an equivalent VIE:*

$$y^{'}(t)-2ty(t)=e^{t^2},\quad y(0)=1. \tag{1.31}$$

Now set

$$y^{'}(t)=u(t). \tag{1.32}$$

Integrating both sides of (1.32), then using the initial condition $y(0)=1$ leads to

$$y(t)-y(0)=\int_0^t u(\tau)d\tau, \tag{1.33}$$

or

$$y(t)=1+\int_0^t u(\tau)d\tau, \tag{1.34}$$

substituting (1.32) and (1.34) into (1.31) gives the equivalent VIE:

$$u(t)=2t+e^{t^2}+2t\int_0^t u(\tau)d\tau. \tag{1.35}$$

Example 1.14 *Convert the following IVP to an equivalent VIE:*

$$y^{'''}(t)-y^{''}-y^{'}+y=0,\quad y(0)=1,\quad y^{'}(0)=2,\quad y^{''}(0)=3. \tag{1.36}$$

We first set

$$y^{'''}(t)=u(t), \tag{1.37}$$

integrating both sides of (1.37) then using the initial condition $y^{''}(0)=3$ we get

$$y^{''}(t)=3+\int_0^t u(\tau)d\tau. \tag{1.38}$$

By repeating Integration once again and then use the initial condition $y^{'}(0)=2$ we obtain

$$y^{'}(t)=2+3t+\int_0^t\int_0^x u(\tau)d\tau dx=2+3t+\int_0^t(t-\tau)u(\tau)d\tau. \tag{1.39}$$

Integrating agian and using $y(0) = 1$ we find

$$y(t) = 1 + 2t + \frac{3}{2}t^2 + \int_0^t \int_0^{t_2} \int_0^{t_1} u(\tau)d\tau dt_1 dt_2 \qquad (1.40)$$
$$= 1 + 2t + \frac{3}{2}t^2 + \frac{1}{2}\int_0^t (t-\tau)^2 u(\tau)d\tau.$$

Substituting (1.37)– (1.40) into (1.36) yields the VIE:

$$u(t) = 4 + t + \frac{3}{2}t^2 + \int_0^t \left[1 + (t-\tau) - 12(t-\tau)^2\right] u(\tau)d\tau. \qquad (1.41)$$

(Wazwaz, 2011)

1.2 Newton-Kantorovich method

1.2.1 Idea and history of the Newton method

The basic idea of the Newton method (NM) is very simple, it is linearization method. Assume that $F : R^1 \to R^1$ is a differentiable function and we are solving the equation

$$F(x) = 0. \qquad (1.42)$$

Starting with an initial point x_0, we can formulate a linear approximation of $F(x)$ in a neighborhood of x_0 such that

$$F(x+h) \approx F(x_0) + F'(x_0)h, \qquad (1.43)$$

and solve the corresponding linear equation. Thus we get the recurrent method

$$x_{k+1} = x_k - F'(x_k)^{-1}F(x_k), \quad k = 0, 1, ... \qquad (1.44)$$

This is the method initiated by Newton in 1669. In deed, Newton dealt only with polynomial equation of the form $P(x) = 0$ and he omitted higher order in h when he used the approximation of $P(x+h)$. In 1690 J. Raphson proposed the general form of method (1.44) (not assuming $F(x)$ to be a polynomial and using the notion of derivative) and the method was developed by many famous mathematicians, such as Fourier, Cauchy, and others. For instance Fourier in 1818 proved the convergence of the method is quadratically in a neighborhood of a root, while Cauchy provided a multidimensional extension of (1.44) and used the method to prove the existence of a root of an equation. Essential early contributions to the investigation of the method are due to Fine (1916) when he proved the convergence in the n-dimensional case with no assumption on the existence of a solution. Bennett (1916) extended the result to the infinite-dimensional case. Basic results on the NM and numerous references can be found in the books of Ostrowski et al. (1960), Ortega and Rheinboldt (1970), Rheinboldt (1974), and Deuflhard (2011).

1.2.2 Fréchet and Gâteaux derivative

Suppose we have an operator P mapping an open set Ω of one B-space X into a set Δ of another B-space Y. We take a fixed point $x_0 \in \Omega$ and assume that there exists a continuous linear operator $U \in B(X, Y)$ such that, for every $x \in X$,

$$\lim_{t \to 0} \frac{P(x_0 + tx) - P(x_0)}{t} = U(x). \tag{1.45}$$

In this situation we say that the linear operator U is the derivative of P at the point x_0. Then we write

$$U = P'(x_0). \tag{1.46}$$

The derivative just defined is often called the Gâteaux derivative, or the weak derivative, and the element $U(x)$ is called the Gâteaux differential.
Let us denote by $\bar{K}$ the set of all $x \in X$ with $\|x\| = 1$, if the limit in equation (1.45) is uniform for $x \in \bar{K}$, then P is said to be differentiable at x_0 and in this case $P'(x_0)$ is called the Fréchet derivative, or strong derivative. The differentiability of an operator P at a point x_0 means, in other words, that there exists a linear operator U such that, for every $\epsilon > 0$, we can find $\delta > 0$, whenever $\|\Delta x\| < \delta$, $(\Delta x \in X)$, we have by Kantorovich and Akilov (1982)

$$\|P(x_0 + \Delta x) - P(x_0) - U(\Delta x)\| \leq \epsilon \|\Delta x\|. \tag{1.47}$$

1.2.3 Kantorovich's contribution

In 1948, L. V. Kantorovich published the seminar paper (Kantorovich, 1948a), where he suggested an extension of the NM to functional spaces. The results were also included in the survey paper Kantorovich (1948b). Further developments of the method can be found in Kantorovich (1949), Kantorovich (1957) and in the monographs Kantorovich and Akilov (1959), Kantorovich and Akilov (1982). This contribution by Kantorovich to one of the fundamental techniques in numerical analysis and functional analysis cannot be underestimated. Kantorovich analyzed the same equation as (1.42),

$$P(x) = 0, \tag{1.48}$$

but now $P : X \to Y$, where X and Y are Banach spaces. The proposed method reads as (1.44):

$$x_{k+1} = x_k - P'(x_k)^{-1} P(x_k), \quad k = 0, 1, 2, ..., \tag{1.49}$$

where $P'(x_k)$ is the (Fréchet) derivative of the nonlinear operator $P(x)$ at the point x_k and $P'(x_k)^{-1}$ is its inverse. Now, suppose we have an operator P mapping an open subset Ω of Banach space X into Banach space Y. Assume that there is a zero of P in Ω, that is an element x^* such that

$$P(x^*) = 0. \tag{1.50}$$

Choose an arbitrary element $x_0 \in \Omega$. If we assume that P has a continuous derivative in Ω, we can replace the element $P(x_0) = P(x_0) - P(x^*)$ by the approximation $P'(x_0)(x_0 - x^*)$. We therefore have reason to suppose that a solution of the equation

$$P'(x_0)(x_0 - x) = P(x_0), \tag{1.51}$$

will be close to x^*. But this equation is linear, so its solution is easy to find

$$x_1 = x_0 - \left[P'(x_0)\right]^{-1}(P(x_0)), \tag{1.52}$$

assuming that $\left[P'(x_0)\right]^{-1}$ exists. If we continue this process, we obtain, starting from the initial approximation x_0 a sequence $\{x_n\}$, where

$$x_{n+1} = x_n - \left[P'(x_n)\right]^{-1}(P(x_n)), \quad (n = 0, 1, \cdots). \tag{1.53}$$

Each x_n is an approximate solution to the equation

$$P(x) = 0. \tag{1.54}$$

NM is clearly not always feasible. Firstly, x_n may pass beyond the set Ω for some value of n, and secondly, $\left[P'(x_n)\right]^{-1}$ may not exist.
If the sequence $\{x_n\}$ converges to a root x^* and x_0 is chosen close enough to x^*, then, by the continuity of P', the operators $P'(x_n)$ and $P'(x_0)$ will only differ a small amount. Thus equation (1.53) can be replaced by the simplified formula

$$x_{n+1} = x_n - \left[P'(x_0)\right]^{-1}(P(x_n)), \quad (n = 0, 1, \cdots), \tag{1.55}$$

which is significantly simpler than formula (1.53), although in general they yield poorer approximations. We shall call this methos of generating the sequence $\{x_n\}$ in (1.55) the Newton-Kantorovich method (NKM).

Definition 1.1 *Kantorovich and Akilov (1982) The majorant function:*
Let the equation (1.54) be written as

$$x = S(x), \tag{1.56}$$

where S is defined on the ball $\|x - x_0\| < R$ and $S(x) = x - \Gamma_0 P(x)$ with $\Gamma_0 = \left[P^{'}(x_0)\right]^{-1}$ for NKM and $\Gamma_0 = \left[P^{'}(x)\right]^{-1}$ for NM.
As well as we consider the real equation

$$t = \phi(t) \tag{1.57}$$

where the function ϕ is defined in the interval $[t_0, T]$, with $T = t_0 + r < t_0 + R$. We say that equation (1.57) (or the function ϕ) majorizes equation (1.56) (or the operator S) if

- $\|S(x_0) - x_0\| \leq \phi(t_0) - t_0$;
- $\|S'(x)\| \leq \phi'(x)$ *whenever* $\|x - x_0\| \leq t - t_0$.

The following theorem, i.e the Kantorovich theorem which deals with the majorant function

$$\psi(t) = Kt^2 - 2t + 2\eta, \tag{1.58}$$

where k, η are real nonnegative real numbers.

Theorem 1.1 . *(Kantorovich and Akilov, 1982, pp. 532). Suppose that the operator P is defined in open ball Ω, where $\|x - x_0\| \leq R$ and has a continuous second derivetive in closed ball Ω_0, whih $\|x - x_0\| \leq r, r \leq R$. Suppose, in addition, that*

1) $\Gamma_0 = \left[P'(x_0)\right]^{-1}$ *exists and is a continuous linear operator;*

2) $\|\Gamma_0(P(x_0))\| \leq \eta$;

3) $\|\Gamma_0 P''(x)\| \leq K, (x \in \Omega_0)$,

where K and η are given in equation (1.58). Then, provided $h = K\eta \leq \frac{1}{2}$ and $r \geq r_0 = \frac{1 - \sqrt{1-2h}}{h}\eta$, equation (1.48) has a solution x^*, and the Newton-Kantorovich process converges to this solution. Furthermore, $\|x - x_0\| \leq r_0$. Also, if for $h < \frac{1}{2}$ we have $r < r_1 = \frac{1+\sqrt{1-2h}}{h}\eta$, while for $h = \frac{1}{2}$ we have $r \leq r_1$, then x^* is the unique solution in Ω_0.
The rate of convergence is given by

$$\|x^* - x_n\| \leq \frac{\eta}{h}\left(1 - \sqrt{1-2h}\right)^{n+1}, \quad (n = 0, 1, 2, \cdots).$$

1.3 Research aims and objectives

The objectives of the thesis are listed as follow:

a) To solve one dimensional nonlinear Volterra type IE of the second kind by NKM, establish the convergence of the proposed method and obtain the error estimation.

b) To solve 2×2 system of nonlinear Volterra type IEs of the second kind by NKM, establish the convergence of method and obtain the error estimation.

c) To solve $n \times n$ system of nonlinear Volterra type IEs of the second kind by NKM, establish the convergence of method and obtain error estimation.

d) To solve two dimensional nonlinear Volterra type IE of the second kind by NKM, establish the convergence of method and obtain the error estimation.

1.4 Organization of the thesis

This thesis has seven chapters. Chapter 1 contains a brief introduction to the general integral equations, historical introduction, classification of IEs, the meaning of Fréchet and Gâteaux derivatives and the concept of majorant function. The literature reviews are given in chapter 2. Chapter 3 focuses on finding the approximate solution of one dimensional nonlinear Volterra type IE via NKM. NKM, the majorant function and the concept of Gauss-Legendre formula are discussed in chapter 4 to find the approximate solution of system of 2×2 nonlinear Volterra type IE. In chapter 5, we deal with the system of $n \times n$ nonlinear Volterra type integral equation, then finding the approximate solution by NKM. Chapter

6 discusses the NKM and two dimensional Gauss-Legendre quadrature formula (QF) to find the approximate solution of two dimensional nonlinear VIE. Chapter 7 gives some future works as an extension to this research.

REFERENCES

Agarwal, R. P., ORegan, D., Tisdell, C. C. and Wong, P. J. 2007. Constant-sign solutions of a system of Volterra integral equations. *Computers and Mathematics with Applications* 54 (1): 58–75.

Agrawal, O. P. and Khan, Y. 2015. Analytical and Numerical Methods for Solving Partial Differential Equations and Integral Equations Arising in Physical Models 2014. *Abstract and Applied Analysis* .

Akyüz-Daşcıoğlu, A. and Yaslan, H. C. 2006. An approximation method for the solution of nonlinear integral equations. *Applied mathematics and computation* 174 (1): 619–629.

Aleksandrov, V. and Manzhirov, A. 1987. Two-dimensional integral equations in applied mechanics of deformable solids. *Journal of Applied Mechanics and Technical Physics* 28 (5): 781–786.

Argyros, I. K. 2004. On the Newton–Kantorovich hypothesis for solving equations. *Journal of Computational and Applied Mathematics* 169 (2): 315–332.

Argyros, I. K. and Hilout, S. 2010. Extending the Newton–Kantorovich hypothesis for solving equations. *Journal of computational and applied mathematics* 234 (10): 2993–3006.

Argyros, I. K. and Khattri, S. K. 2014. Weaker Kantorovich type criteria for inexact Newton methods. *Journal of Computational and Applied Mathematics* 261: 103–117.

Atkinson, K. E. 1997. *The numerical solution of integral equations of the second kind*. Cambridge university press.

Aziz, I., Khan, F. et al. 2014. A new method based on Haar wavelet for the numerical solution of two-dimensional nonlinear integral equations. *Journal of Computational and Applied Mathematics* 272: 70–80.

Babolian, E., Bazm, S. and Lima, P. 2011. Numerical solution of nonlinear two-dimensional integral equations using rationalized Haar functions. *Communications in Nonlinear Science and Numerical Simulation* 16 (3): 1164–1175.

Badr, A. 2001. On some parameter methods for nonlinear Volterra integral equation. *Applied mathematics and computation* 117 (1): 15–22.

Bahyrycz, A., Brzdk, J. and Leśniak, Z. 2014. On approximate solutions of the generalized Volterra integral equation. *Nonlinear Analysis: Real World Applications* 20: 59–66.

Balakumar, V. and Murugesan, K. 2014. Single-Term Walsh Series method for systems of linear Volterra integral equations of the second kind. *Applied Mathematics and Computation* 228: 371–376.

Beesack, P. R. 1985. Systems of multidimensional Volterra integral equations and inequalities. *Nonlinear Analysis: Theory, Methods and Applications* 9 (12): 1451–1486.

Belbas, S. 2006. On series solutions of Volterra equations. *Applied mathematics and computation* 181 (2): 1287–1304.

Bennett, A. A. 1916. Newton's method in general analysis. *Proceedings of the National Academy of Sciences of the United States of America* 2 (10): 592.

Berenguer, M., Gámez, D., Garralda-Guillem, A., Galán, M. R. and Pérez, M. S. 2011. Biorthogonal systems for solving Volterra integral equation systems of the second kind. *Journal of Computational and Applied Mathematics* 235 (7): 1875–1883.

Biazar, J. and Ebrahimi, H. 2012. Chebyshev wavelets approach for nonlinear systems of Volterra integral equations. *Computers and Mathematics with Applications* 63 (3): 608–616.

Biazar, J. and Ghazvini, H. 2009. Hes homotopy perturbation method for solving systems of Volterra integral equations of the second kind. *Chaos, Solitons and Fractals* 39 (2): 770–777.

Boikov, I. and Tynda, A. 2003. Approximate solution of nonlinear integral equations of the theory of developing systems. *Differential Equations* 39 (9): 1277–1288.

Brunner, H. and Yatsenko, Y. 1996. Spline collocation methods for nonlinear Volterra integral equations with unknown delay. *Journal of computational and applied mathematics* 71 (1): 67–81.

Chen, H. and Zhang, C. 2011. Boundary value methods for Volterra integral and integro-differential equations. *Applied Mathematics and Computation* 218 (6): 2619–2630.

Chen, H. and Zhang, C. 2012. Block boundary value methods for solving Volterra integral and integro-differential equations. *Journal of Computational and Applied Mathematics* 236 (11): 2822–2837.

Chen, H.-Y. 1979. Solutions for certain nonlinear Volterra integral equations. *Journal of Mathematical Analysis and Applications* 69 (2): 475–488.

Chen, Z. and Jiang, W. 2015. An efficient algorithm for solving nonlinear Volterra–Fredholm integral equations. *Applied Mathematics and Computation* 259: 614–619.

Conte, D., DAmbrosio, R. and Paternoster, B. 2012. Two-step diagonally-implicit collocation based methods for Volterra Integral Equations. *Applied Numerical Mathematics* 62 (10): 1312–1324.

Conte, D., Jackiewicz, Z. and Paternoster, B. 2008. Two-step almost collocation methods for Volterra integral equations. *Applied Mathematics and Computation* 204 (2): 839–853.

Conte, D. and Paternoster, B. 2009. Multistep collocation methods for Volterra integral equations. *Applied numerical mathematics* 59 (8): 1721–1736.

Corduneanu, C. 1991. *Integral equations and applications.* , vol. 148. Cambridge University Press Cambridge.

Crisci, M., Russo, E. and Vecchio, A. 1997. Discrete-time waveform relaxation Volterra-Runge-Kutta methods: Convergence analysis. *Journal of computational and applied mathematics* 86 (2): 359–374.

Curtain, R. F. and Pritchard, A. J. 1977. *Functional analysis in modern applied mathematics.* , vol. 132. IMA.

Deuflhard, P. 2011. *Newton methods for nonlinear problems: affine invariance and adaptive algorithms.* , vol. 35. Springer Science and Business Media.

Dobriţoiu, M. and Şerban, M.-A. 2014. Step method for a system of integral equations from biomathematics. *Applied Mathematics and Computation* 227: 412–421.

Dolph, C. L. 1949. Nonlinear integral equations of the Hammerstein type. *Transactions of the American Mathematical Society* 289–307.

El-Sayed, A. and Hashem, H. 2012. A coupled system of integral equations in reflexive banach spaces. *Acta Mathematica Scientia* 32 (5): 2021–2028.

Elliott, D., Johnston, P. R. and Johnston, B. M. 2011. Estimates of the error in Gauss–Legendre quadrature for double integrals. *Journal of Computational and Applied Mathematics* 236 (6): 1552–1561.

Eshkuvatov, Z., Ahmedov, A., Long, N. N. and Shafiq, O. 2010. Approximate solution of the system of nonlinear integral equation by Newton–Kantorovich method. *Applied Mathematics and Computation* 217 (8): 3717–3725.

Ezquerro, J., Gonzalez, D. and Hernandez, M. 2012. A variant of the Newton–Kantorovich theorem for nonlinear integral equations of mixed Hammerstein type. *Applied Mathematics and Computation* 218 (18): 9536–9546.

Ezquerro, J., González, D. and Hernández, M. 2013a. A modification of the classic conditions of Newton–Kantorovich for Newtons method. *Mathematical and Computer Modelling* 57 (3): 584–594.

Ezquerro, J., González, D. and Hernández, M. 2013b. On the local convergence of Newtons method under generalized conditions of Kantorovich. *Applied Mathematics Letters* 26 (5): 566–570.

Ezquerro, J. A., González, D. and Hernández-Verón, M. 2014. A semilocal convergence result for Newtons method under generalized conditions of Kantorovich. *Journal of Complexity* 30 (3): 309–324.

Fazeli, S., Hojjati, G. and Shahmorad, S. 2012. Super implicit multistep collocation methods for nonlinear Volterra integral equations. *Mathematical and Computer Modelling* 55 (3): 590–607.

Ferreira, O. P. and Svaiter, B. F. 2012. A robust Kantorovichs theorem on the inexact Newton method with relative residual error tolerance. *Journal of Complexity* 28 (3): 346–363.

Fine, H. B. 1916. On Newton's method of approximation. *Proceedings of the National Academy of Sciences of the United States of America* 2 (9): 546.

Galperin, E., Kansa, E., Makroglou, A. and Nelson, S. 2000. Variable transformations in the numerical solution of second kind Volterra integral equations with continuous and weakly singular kernels; extensions to Fredholm integral equations. *Journal of Computational and applied Mathematics* 115 (1): 193–211.

Goghary, H. S., Javadi, S. and Babolian, E. 2005. Restarted Adomian method for system of nonlinear Volterra integral equations. *Applied mathematics and computation* 161 (3): 745–751.

Golbabai, A., Mammadov, M. and Seifollahi, S. 2009. Solving a system of nonlinear integral equations by an RBF network. *Computers and Mathematics with Applications* 57 (10): 1651–1658.

Guoqiang, H., Hayami, K., Sugihara, K. and Jiong, W. 2000. Extrapolation method of iterated collocation solution for two-dimensional nonlinear Volterra integral equations. *Applied mathematics and computation* 112 (1): 49–61.

Hacia, L. and Kaczmarek, L. 1998. On the boundedness of solutions for system of two-dimensional Volterra integral equations. *Fasciculi Mathematici* 43–55.

He, J.-H. 1999. Homotopy perturbation technique. *Computer methods in applied mechanics and engineering* 178 (3): 257–262.

Heath, M. T. 2002. *Scientific computing*. McGraw-Hill New York.

Heydari, M., Avazzadeh, Z., Navabpour, H. and Loghmani, G. 2013. Numerical solution of Fredholm integral equations of the second kind by using integral mean value theorem II. High dimensional problems. *Applied Mathematical Modelling* 37 (1): 432–442.

Hrusa, W. J. 1983. A nonlinear functional differential equation in Banach space with applications to materials with fading memory. *Archive for rational mechanics and analysis* 84 (2): 99–137.

Izzo, G., Jackiewicz, Z., Messina, E. and Vecchio, A. 2010. General linear methods for Volterra integral equations. *Journal of computational and applied mathematics* 234 (9): 2768–2782.

Izzo, G., Russo, E. and Chiapparelli, C. 2012. Highly stable Runge–Kutta methods for Volterra integral equations. *Applied Numerical Mathematics* 62 (8): 1002–1013.

Jang, B. 2009. Comments on solving a class of two-dimensional linear and nonlinear Volterra integral equations by the differential transform method. *Journal of computational and applied mathematics* 233 (2): 224–230.

Jeffrey, A. and Dai, H. H. 2008. *Handbook of mathematical formulas and integrals*. Academic Press.

Kamont, Z. and Kwapisz, M. 1981. On non-linear Volterra integral-functional equations in several variables. In *Annales Polonici Mathematici*, 1–29. Institute of Mathematics Polish Academy of Sciences.

Kangro, R. and Oja, P. 2008. Convergence of spline collocation for Volterra integral equations. *Applied Numerical Mathematics* 58 (10): 1434–1447.

Kantorovich, L. 1948a. On Newtons method for functional equations. In *Dokl. Akad. Nauk SSSR*, 1237–1240.

Kantorovich, L. 1957. On some further applications of the Newton approximation method. *Vestnik Leningrad University. Mathematics* 12 (7): 68–103.

Kantorovich, L. and Akilov, G. 1959, Functional analysis in normed spaces [Funktsional'nyi analiz v normirovannykh prostranstvakh] Fizmatgiz.

Kantorovich, L. and Akilov, G. 1982. *Functional Analysis.* , vol. 10. Pergamon Press, Oxford.

Kantorovich, L. V. 1948b. Functional analysis and applied mathematics. *Uspekhi Matematicheskikh Nauk* 3 (6): 89–185.

Kantorovich, L. V. 1949. On Newton's method. *Trudy Matematicheskogo Instituta im. VA Steklova* 28: 104–144.

Kanwal, R. P. 2013. *Linear integral equations*. Springer Science and Business Media.

Karoui, A. and Jawahdou, A. 2010. Existence and approximate Lp and continuous solutions of nonlinear integral equations of the Hammerstein and Volterra types. *Applied Mathematics and Computation* 216 (7): 2077–2091.

Khanh, B. D. 1994. Hermite predictorcorrector scheme for regular Volterra integral equations and for some integro-differential equations of turbulent diffusion. *Journal of computational and applied mathematics* 51 (3): 305–316.

Kreın, M. 1955. On a new method of solution of linear integral equations of first and second kinds. *Doklady Akademii Nauk SSSR* 100: 413–416.

Kreyszig, E. 1989. *Introductory functional analysis with applications.* , vol. 81. wiley New York.

Kwapisz, M. 1991. Bielecki's method, existence and uniqueness results for Volterra integral equations in L p space. *Journal of Mathematical Analysis and Applications* 154 (2): 403–416.

Ladopoulos, E. 2003. Nonlinear unsteady flow problems by multidimensional singular integral representation analysis. *International Journal of Mathematics and Mathematical Sciences* 2003 (50): 3203–3216.

Levin, J. and Nohel, J. 1960. Global asymptotic stability for nonlinear systems of differential equations and applications to reactor dynamics. *Archive for Rational Mechanics and Analysis* 5 (1): 194–211.

Londen, S.-O. and Nohel, J. A. 1982, A Nonlinear Volterra Integrodifferential Equation Occurring in Heat Flow., Tech. rep., DTIC Document.

Long, N. T. et al. 2013. On a nonlinear Volterra-Hammerstein integral equation in two variables. *Acta Mathematica Scientia* 33 (2): 484–494.

Maleknejad, K. and JafariBehbahani, Z. 2012. Applications of two-dimensional triangular functions for solving nonlinear class of mixed Volterra–Fredholm integral equations. *Mathematical and Computer Modelling* 55 (5): 1833–1844.

Maleknejad, K. and Mahdiani, K. 2011. Solving nonlinear mixed Volterra–Fredholm integral equations with two dimensional block-pulse functions using direct method. *Communications in Nonlinear Science and Numerical Simulation* 16 (9): 3512–3519.

Maleknejad, K. and Najafi, E. 2011. Numerical solution of nonlinear volterra integral equations using the idea of quasilinearization. *Communications in Nonlinear Science and Numerical Simulation* 16 (1): 93–100.

Maleknejad, K., Torabi, P. and Mollapourasl, R. 2011. Fixed point method for solving nonlinear quadratic Volterra integral equations. *Computers and Mathematics with Applications* 62 (6): 2555–2566.

Manzhirov, A. 1987. Contact problems of the interaction between viscoelastic foundations subjected to ageing and systems of stamps not applied simultaneously. *Journal of Applied Mathematics and Mechanics* 51 (4): 523–535.

McKee, S. and Cuminato, J. 2015. Nonlocal diffusion, a Mittag-Leffler function and a two-dimensional Volterra integral equation. *Journal of Mathematical Analysis and Applications* 423 (1): 243–252.

McKee, S., Tang, T. and Diogo, T. 2000. An Euler-type method for two-dimensional Volterra integral equations of the first kind. *IMA journal of numerical analysis* 20 (3): 423–440.

Mirzaee, F. and Hoseini, A. A. 2014. A computational method based on hybrid of block-pulse functions and Taylor series for solving two-dimensional nonlinear integral equations. *Alexandria Engineering Journal* 53 (1): 185–190.

Mirzaee, F. and Rafei, Z. 2011. The block by block method for the numerical solution of the nonlinear two-dimensional Volterra integral equations. *Journal of King Saud University-Science* 23 (2): 191–195.

Mosleh, M. and Otadi, M. 2015. Least squares approximation method for the solution of Hammerstein–Volterra delay integral equations. *Applied Mathematics and Computation* 258: 105–110.

Mydlarczyk, W., Okrasiński, W. and Roberts, C. 2005. Blow-up solutions to a system of nonlinear Volterra equations. *Journal of mathematical analysis and applications* 301 (1): 208–218.

Nemati, S., Lima, P. M. and Ordokhani, Y. 2013. Numerical solution of a class of two-dimensional nonlinear Volterra integral equations using Legendre polynomials. *Journal of Computational and Applied Mathematics* 242: 53–69.

Ortega, J. M. and Rheinboldt, W. C. 1970. *Iterative solution of nonlinear equations in several variables.* , vol. 30. Siam.

Ostrowski, A. M., Ostrowski, A., Ostrowski, A. and Ostrowski, A. 1960. *Solution of equations and systems of equations.* , vol. 9. Academic Press New York.

Pachpatte, B. 1983. On some applications of Wazewski method for multiple Volterra integral equations. *An. Sti. Univ. Al. I. Cuza Iasi Math* 29: 75–83.

Pachpatte, B. 2007. On Volterra-Fredholm integral equation in two variables. *Demonstratio Mathematica. Warsaw Technical University Institute of Mathematics* 40 (4): 839–852.

Pachpatte, B. 2010. On a nonlinear Volterra integral equation in two variables. *Sarajevo J. Math* 6 (18): 59–73.

Pachpatte, B. G. 2011. *Multidimensional integral equations and inequalities.* , vol. 9. Springer Science and Business Media.

Pachpatte, B. G. and Talkies, N. A. 2009. Volterra integral and integrodifferential equations in two variables. *J Inequal Pure and Appl Math* 10 (4): 10.

Pilipović, S. and Stojanović, M. 1999. Generalized solutions to nonlinear Volterra integral equations with non-Lipschitz nonlinearity. *Nonlinear Analysis: Theory, Methods and Applications* 37 (3): 319–335.

Polyanin, A. D. and Manzhirov, A. V. 2012. *Handbook of integral equations*. CRC press.

Rheinboldt, W. C. 1974. *Methods for solving systems of nonlinear equations*. SIAM.

Roodaki, M. and Almasieh, H. 2012. Delta basis functions and their applications to systems of integral equations. *Computers and Mathematics with Applications* 63 (1): 100–109.

Saberi-Nadjafi, J. and Heidari, M. 2010. Solving nonlinear integral equations in the Urysohn form by Newton–Kantorovich–quadrature method. *Computers and Mathematics with Applications* 60 (7): 2058–2065.

Shen, W. and Li, C. 2009. Kantorovich-type convergence criterion for inexact Newton methods. *Applied Numerical Mathematics* 59 (7): 1599–1611.

Shidfar, A. and Molabahrami, A. 2011. Solving a system of integral equations by an analytic method. *Mathematical and Computer Modelling* 54 (1): 828–835.

Sloss, B. and Blyth, W. 2003. A Walsh function method for a non-linear Volterra integral equation. *Journal of the Franklin Institute* 340 (1): 25–41.

Soloviev, O. 1998. Low-frequency radio wave propagation in the earth-ionosphere waveguide disturbed by a large-scale three-dimensional irregularity. *Radiophysics and quantum electronics* 41 (5): 392–402.

Suryanarayana, M. 1972. On multidimensional integral equations of Volterra type. *Pacific Journal of Mathematics* 41 (3): 809–828.

Tari, A., Rahimi, M., Shahmorad, S. and Talati, F. 2009. Solving a class of two-dimensional linear and nonlinear Volterra integral equations by the differential transform method. *Journal of Computational and Applied Mathematics* 228 (1): 70–76.

Tsokos, C. P. and Padgett, W. J. 1974. *Random integral equations with applications to life sciences and engineering.* , vol. 108. Academic Press.

Uko, L. U. and Argyros, I. K. 2008. A weak Kantorovich existence theorem for the solution of nonlinear equations. *Journal of Mathematical Analysis and Applications* 342 (2): 909–914.

Wazwaz, A.-M. 2011. *Linear and nonlinear integral equations*. Springer.

Yang, L.-H., Li, H.-Y. and Wang, J.-R. 2013. Solving a system of linear Volterra integral equations using the modified reproducing kernel method. In *Abstract and Applied Analysis*. Hindawi Publishing Corporation.

Yoshida, K. 1960. *Lectures on differential and integral equations.* , vol. 10. Courier Corporation.

Zeidler, E. 1989. *Nonlinear functional analysis and its applications.* , vol. 2. Springer Science and Business Media.

Zeidler, E. 1995. *Applied functional analysis, Main principles and their application, Contents of AMS*. Springer Verlag.

Zhang, L. and Ma, F. 2015. Pouzet–Runge–Kutta–Chebyshev method for Volterra integral equations of the second kind. *Journal of Computational and Applied Mathematics* 288: 323–331.