



UNIVERSITI PUTRA MALAYSIA

***NEWTON-KANTOROVICH METHOD FOR SOLVING ONE- AND
TWO-DIMENSIONAL NONLINEAR VOLTERRA INTEGRAL EQUATIONS
OF THE SECOND KIND***

HAMEED HUSAM HAMEED

FS 2016 9



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TWO-DIMENSIONAL NONLINEAR VOLTERRA INTEGRAL EQUATIONS
OF THE SECOND KIND**

By

HAMEED HUSAM HAMEED

**Thesis Submitted to the School of Graduate Studies, Universiti Putra
Malaysia, in Fulfilment of the Requirements for the Degree of Doctor
of Philosophy**

January 2016



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DEDICATIONS

To Her Dignity, Hadhrat Fatima Al-Zahraa (peace upon her).



Abstract of thesis presented to the Senate of Universiti Putra Malaysia in
fulfilment of the requirement for the degree of Doctor of Philosophy

**NEWTON-KANTOROVICH METHOD FOR SOLVING ONE- AND
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January 2016

Chair: Assoc. Prof. Zainidin Eshkuvatov, PhD
Faculty: Science

The problems of nonlinear Volterra integral equations (VIEs) which consist of one dimensional nonlinear VIE, system of 2×2 nonlinear VIEs, system of $n \times n$ nonlinear VIEs, and two dimensional nonlinear VIE, with smooth unknown functions and continuous bounded given functions are discussed. The Newton-Kantorovich method (NKM) is used to linearize the problems. Then the Nystrom-type Gauss-Legendre quadrature formula (QF) is used to solve the linearized equations and systems. New majorant functions are found for some problems which lead to the increment of convergence interval. The new approach based on the subcollocation method is developed and motivation leads to high accurate approximate solutions. The existence and uniqueness of solution are proved and error estimation and rate of convergence are obtained. Numerical examples show that our results are coincided with the theoretical finding.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

**KAEDAH NEWTON-KANTOROVICH BAGI SATU- DAN DUA-BERDIMENSI
VOLTERRA PERSAMAAN KAMIRAN TAK LINEAR JENIS KEDUA**

Oleh

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Masalah persamaan kamiran Volterra (VIEs) tak linear yang mengandungi VIEs tak linear berdimensi satu, sistem 2×2 VIEs tak linear, sistem $n \times n$ VIEs tak linear dan VIEs tak linear berdimensi dua, dengan fungsi anu licin dan fungsi tersedia selanjur terbatas di bincangkan. Kaedah Newton-Kantorovich digunakan untuk melinearkan masalah. Kemudian formula quadratur Gauss-Legendre digunakan untuk menyelesaikan persamaan dan sistem terlinear. Fungsi majoran baharu ditemui untuk sesetengah masalah yang membawa kepada peningkatan selang penumpuan. Pendekatan baharu berdasarkan kepada kaedah subkolokasi yang membawa kepada penyelesaian hampiran dengan ketepatan tinggi dibina. Kewujudan dan keunikan penyelesaian dibuktikan, anggaran ralat dan kadar penumpuan diperolehi. Contoh berangka menunjukkan keputusan kami adalah sefara dengan penemuan secara teoritikal.

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I certify that a Thesis Examination Committee has met on 14 January 2016 to conduct the final examination of Hameed Husam Hameed on his thesis entitled "Newton-Kantorovich Method for Solving One- and Two-Dimensional Nonlinear Volterra Integral Equations of the Second Kind" in accordance with the Universities and University Colleges Act 1971 and the Constitution of the Universiti Putra Malaysia [P.U.(A) 106] 15 March 1998. The Committee recommends that the student be awarded the Doctor of Philosophy.

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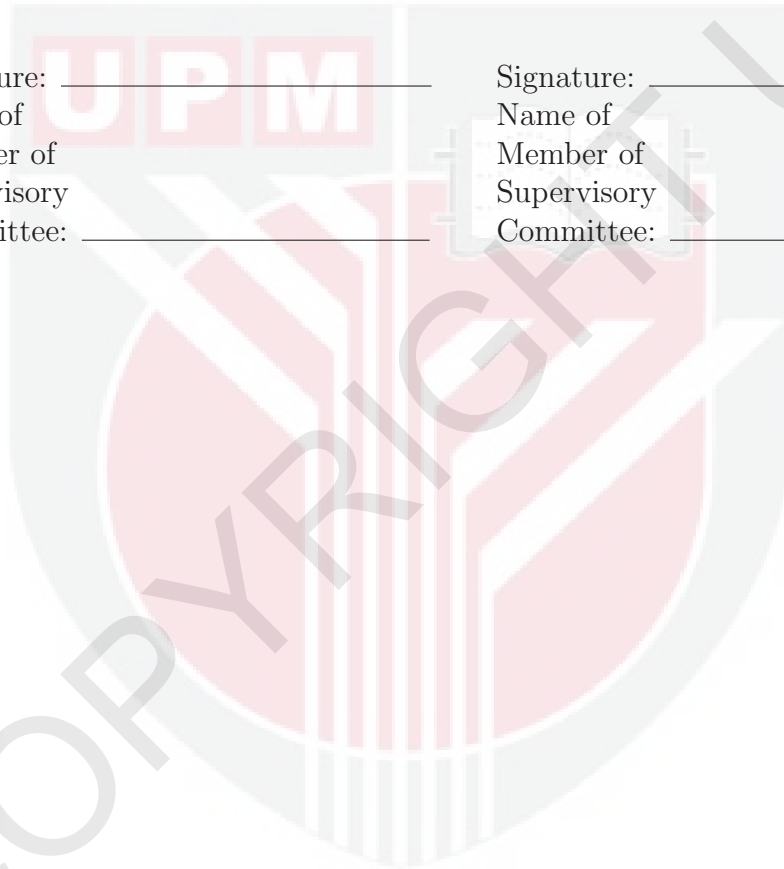


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LIST OF ABBREVIATIONS

FIE	Fredholm integral equation.
IE	Integral equation.
NKM	Newton-Kantorovich method.
NM	Newton method.
QF	Quadrature formula.
VIE	Volterra integral equatuion.
IVP	Initial value problem.



CHAPTER 1

INTRODUCTION

1.1 Integral equations

1.1.1 A brief historical introduction

The theory of abstract Volterra equations has been developed rapidly in the last few decades. The applications of this theory to many problems could be found for instance, mechanics, physics, engineering, economics and even biology, so its a good tool to deal with the problems arising in these fields. J. Fourier (1768-1830) was considered as the initiator of the theory of integral equations (IEs), [see Cor-duneanu (1991)] by obtaining the inversion formula for Fourier transform with suitable conditions on the functions f and g , where $g(t) = (2\pi)^{-\frac{1}{2}} \int_0^{\infty} e^{it\tau} f(\tau) d\tau$ and $f(\tau) = (2\pi)^{-\frac{1}{2}} \int_0^{\infty} e^{-it\tau} g(t) dt$. Of course, we can consider inversion formula as the inverse Fourier integral operator . This consideration was taken by V. Volterra during the end of 18th century, who specified the problem of solving integral equations (IEs) as the problem of finding inverses of certain integral operators. Abel (1823) dealt with the integral equation (IE) known as 'Abel's equation' of the form ,

$$\int_0^t (t - \tau)^{-\alpha} u(\tau) d\tau = F(t), \quad 0 < \alpha < 1.$$

There are two significant cases related to Abel's equation. First, the kernel $k(\tau) = \tau^{-\alpha}$ has a singularity at the origin, and second, Abel's equation is a convolution type equation and both cases had a important effect on the development of the theory of IEs. Sonine (1884) tried to find the solution of various IEs, in particular Abel's equation. Levi (1895) generalized some results wich concerned with Abel's type equations. Also, in 1895 Volterra considered more general equations such as

$$u(t) + \int_0^t k(t, \tau) u(\tau) d\tau = F(t),$$

where $u(\tau)$ is unknown function to be found. Five years after Volterra made his first well known contribution to the theory of IEs, Fredholm introduced a new theory for IEs containing a parameter of the form

$$y(t) + \lambda \int_a^b k(t, \tau) y(\tau) d\tau = f(t).$$

After Fredholm's publications, the theory of IEs was developed and enriched by many mathematician like Picard (1906), Poincar (1910), Hilbert (1912), and Heywood (1912) . The causal operator or nonanticipative operator was introduced by Tonelli (1928), that is operator U acting between function spaces, such that $x(t) = y(t)$ for $t \leq T$ leads to $(Ux)(T) = (Uy)(T)$. The significance of this kind of operators is obvious since most phenomena evaluated via mathematical models are casual phenomena. Tychnoff (1938) found a great connection between the theory of abstract Volterra operators with the many applications in mathematical

physics. He introduced a more modern approach which contributed to its development, and considered as a model for future studies. The essential connection between Hammerstein equations and the spectrum of the linear integral operator was done by Dolph (1949). Krein (1955) introduced a new method to solve the integral equations by constructing the resolvent kernel concept. From the 1960s onwards, many scientific research schools in the world are tried towards the investigation of different problems related to the theory of integral operators and IEs. By way of example, not exhaustive enumeration, Levin and Nohel (1960) referred to the study of stability of nuclear reactors by means of IEs. During the 1970s and the 1980s the mathematicians who are interested in the IEs concentrated their efforts on problems occurring in continuum mechanics, particularly in viscoelasticity, for instance, Londen and Nohel (1982) and Hrusa (1983). Most contributions to the theory of IEs that published during the last two decades, dealt with different applications. In fact many mathematicians made their effort to solve such applied problems by using a recent tools created by basic mathematical research (monotone operators, linear and nonlinear semigroups of transformation, as well as many other functional analytic methods). In this way, the theory of integral equations has been quite developed and enriched.

1.1.2 Classification of integral equations

There are a huge number of different kinds of IEs, and each has its own case of theoretical and numerical analysis. The first main division is the one dimensional and multidimensional IEs. It follows by dividing the IEs into linear IEs and nonlinear IEs. For most the part of the analysis of IEs, efforts have been put on linear equations in a single variable, however in principle numerous of the ways are usable to multi dimensional and nonlinear equations as well.

In general, for the linear IEs, it is often use two major classes, namely Volterra and Fredholm IEs. Of course, we shall classify them later. Most standard linear IE has the form

$$h(t)x(t) = f(t) + \lambda \int_{a(t)}^{b(t)} K(t, \tau)x(\tau)d\tau, \quad (1.1)$$

where $a(t)$ and $b(t)$ are the limits of integration with $a(t) < b(t)$, $t \in [t_0, T]$, λ is a constant, and $K(t, \tau)$ is a known function of two variables t and τ , called kernel of the IE, where $x(t)$ is the unknown function, that to be found with $f(t)$, $h(t)$, and $K(t, \tau)$ are given functions beforehand. The limits of integration $a(t)$ and $b(t)$ may be both variables, constants, or mixed. Two different ways that depend on the limits of integration are used to characterize IEs, namely

1. If the limits of integration are fixed, then IE is called a Fredholm integral equation (FIE) which has the form

$$h(t)x(t) = f(t) + \lambda \int_a^b K(t, \tau)x(\tau)d\tau, \quad (1.2)$$

where a and b are constants.

2. If at least one limit is a variable, the equation is called a Volterra integral

equation (VIE) which has the form

$$h(t)x(t) = f(t) + \lambda \int_a^x K(t, \tau)x(\tau)d\tau. \quad (1.3)$$

Moreover, there are three other distinct kinds, that depend on the appearance of the function $h(t)$, are defined as follows

- (i) If $h(t) = 0$, the IE is called, the first kind FIE (or VIE).
- (ii) If $h(t) = 1$, the IE is called, the second kind FIE (or VIE).
- iii If $0 \neq h(t) \neq 1$, the IE is called, the third kind FIE (or VIE).

In all FIEs or VIEs presented above, if $f(t)$ is identically to zero, the resulting equation gives

$$x(t) = \lambda \int_a^b K(t, \tau)x(\tau)d\tau, \quad (1.4)$$

or

$$x(t) = \lambda \int_a^t K(t, \tau)x(\tau)d\tau, \quad (1.5)$$

which is called homogeneous FIE or homogeneous VIE.

Example 1.1 Consider the one dimensional linear VIE of the first kind

$$\int_a^t [g(t) - g(\tau)] y(\tau)d\tau = f(t),$$

it is assumed that $f(a) = f'_t(a) = 0$ and $\frac{f'_t}{g'_t} \neq 0$.

The solution is $y(t) = \frac{d}{dt} \left[\frac{f'_t(t)}{g'_t(t)} \right]$, (Polyanin and Manzhirov, 2012).

Example 1.2 Consider the one dimensional nonlinear VIE of the first kind

$$\int_0^t y(\tau)y(t - \tau)d\tau = A^2t^\lambda.$$

Solutions: $y(t) = \pm A \frac{\sqrt{\Gamma(\lambda + 1)}}{\Gamma(\frac{\lambda+1}{2})} t^{\frac{\lambda-1}{2}}$, where $\Gamma(z)$ is the gamma function, (Polyanin and Manzhirov, 2012)

Example 1.3 Consider the one dimensional linear VIE of the second kind

$$y(t) - \lambda \int_a^t \frac{g(t)}{g(\tau)} y(\tau)d\tau = f(t).$$

Solution: $y(t) = f(t) + \lambda \int_a^t e^{\lambda(t-\tau)} \frac{g(t)}{g(\tau)} f(\tau)d\tau$, (Polyanin and Manzhirov, 2012)

Example 1.4 Consider the one dimensional nonlinear VIE of the second kind

$$y(t) + \int_a^t f(y(\tau))d\tau = At + B.$$

Solution in an implicit form, $\int_{y_0}^y \frac{d\tau}{A - f(\tau)} = t - a$, $y_0 = Aa + B$, (Polyanin and Manzhirov, 2012).

Example 1.5 Consider the one dimensional linear FIE of the first kind

$$\int_a^b [g_1(t)h_1(\tau) + g_2(t)h_2(\tau)] y(\tau)d\tau = f(t).$$

This IE has solution only if its right hand side is representable in the form

$$f(t) = A_1g_1(t) + A_2g_2(t),$$

where A_1 and A_2 are constants. In this case, any function $y = y(t)$ satisfying the normalization type conditions

$$\int_a^b h_1(\tau)y(\tau)d\tau = A_1, \quad \int_a^b h_2(\tau)y(\tau)d\tau = A_2,$$

is a solution of the IE. otherwise, the equation has no solution, (Polyanin and Manzhirov, 2012).

Example 1.6 Consider the one dimensional nonlinear FIE of the first kind

$$\int_a^b y^k(t)f(\tau, y(\tau))d\tau = g(t).$$

Solution: $y(t) = \lambda[g(t)]^{\frac{1}{k}}$, where λ is determined from the algebraic (or transcendental) equation $\lambda^k \int_a^b f(\tau, \lambda g^{\frac{1}{k}}(\tau))d\tau = 1$, (Polyanin and Manzhirov, 2012).

Example 1.7 Consider the one dimensional linear FIE of the second kind

$$y(t) - \lambda \int_a^b g(t)h(\tau)y(\tau)d\tau = f(t).$$

1. Assume that $\lambda \neq \left(\int_a^b g(\tau)h(\tau)d\tau \right)^{-1}$, then the solution is $y(t) = f(t) + \lambda kg(t)$, where $k = \left(1 - \lambda \int_a^b g(\tau)h(\tau)d\tau \right)^{-1} \int_a^b h(\tau)f(\tau)d\tau$.
2. Assume that $\lambda = \left(\int_a^b g(\tau)h(\tau)d\tau \right)^{-1}$.

(i) For $\int_a^b h(\tau)f(\tau)d\tau = 0$, the solution has the form $y(t) = f(t) + Cg(t)$, where C is an arbitrary constant.

(ii) For $\int_a^b h(\tau)f(\tau)d\tau \neq 0$, there is no solution.

(Polyanin and Manzhirov, 2012).

Example 1.8 Consider the one dimensional nonlinear FIE of the second kind

$$y(t) + \int_a^b g(t)y(t)f(\tau, y(\tau))d\tau = h(t).$$

The solution: $h(t)1 + \lambda g(t)$, where λ is determined from the algebraic (or transcendental) equation

$$\lambda - F(\lambda) = 0, \quad F(\lambda) = \int_a^b f\left(t, \frac{h(\tau)}{1 + \lambda g(\tau)}\right) d\tau.$$

(Polyanin and Manzhirov, 2012).

Example 1.9 Consider the two dimensional linear VIE the second kind

$$\begin{aligned} u(x, t) - \int_0^t \int_0^x (xy + te^z)u(y, z)dydz \\ = xe^{-t} + t - \frac{1}{3}x^4 - xt + \frac{1}{3}x^4e^{-t} - \frac{1}{2}x^2t^2 - \frac{1}{4}x^3t^2 - xte^t + xte^t, \quad x, t \in [0, t] \end{aligned}$$

The solution: $u(x, t) = xe^{-1} + t$, (Tari et al., 2009).

Example 1.10 Consider the two dimensional nonlinear VIE the second kind

$$u(x, t) - \int_0^t \int_0^x (y^2 + e^{-2z})u^2(y, z)dydz = x^2e^t + \frac{1}{14}x^7e^{2t} - \frac{1}{5}x^5t, \quad x, t \in [0, 1].$$

The solution: $u(x, t) = x^2e^t$, (Tari et al., 2009).

Example 1.11 Consider the two dimensional linear FIE the second kind

$$u(x, t) = f(x, t) + \int_0^1 \int_0^1 \left(e^{\left(\frac{x}{5}\right)^5 y} - 1 \right) u(y, z)dydz, \quad 0 \leq x, t \leq 1.$$

The solution: $u(x, t) = xt$, (Heydari et al., 2013)

Example 1.12 Consider the two dimensional nonlinear FIE the second kind

$$u(x, t) = f(x, t) + \int_0^1 \int_0^1 (y \sin(z) + 1)u^3(y, z)dydz, \quad 0 \leq x, t \leq 1, \quad (1.6)$$

where $f(x, t) = x \cos(t) + \frac{1}{20}(\cos^4(1) - 1) - \frac{1}{12} \sin(1)(\cos^2(1) + 2)$. The solution : $u(x, t) = x \cos(t)$, (Heydari et al., 2013).

1.1.3 Converting initial value problem to Volterra integral equation

In this section, we show the strategy of converting an initial value problem (IVP) to an equivalent VIE. For simplicity reasons, we will employ this procedure to the second order initial value problem of the form

$$y''(t) + p(t)y'(t) + q(t)y(t) = g(t), \quad (1.7)$$

with the initial conditions

$$y(0) = \alpha, \quad y'(0) = \beta, \quad (1.8)$$

where α and β are constants, $p(t)$ and $q(t)$ are analytic functions, and $g(t)$ is continuous through the interval of discussion. To accomplish our aim, let

$$y''(t) = u(t), \quad (1.9)$$

where $u(t)$ is a continuous function. By integration both sides of (1.9) from 0 to t we get

$$y'(t) - y'(0) = \int_0^t u(\tau) d\tau, \quad (1.10)$$

or

$$y'(t) = \beta + \int_0^t u(\tau) d\tau. \quad (1.11)$$

Integration both sides of (1.11) from 0 to t we obtain

$$y(t) - y(0) = \beta t + \int_0^t \int_0^x u(\tau) d\tau dx, \quad (1.12)$$

or

$$y(t) = \alpha + \beta t + \int_0^t (t - \tau) u(\tau) d\tau. \quad (1.13)$$

By substituting (1.9), (1.11), and (1.13) into the IVP (1.7) to get the VIE:

$$u(t) + p(t) \left[\beta + \int_0^t u(\tau) d\tau \right] + q(t) \left[\alpha + \beta t + \int_0^t (t - \tau) u(\tau) d\tau \right] = g(t). \quad (1.14)$$

The last equation can be represented in the standard VIE form:

$$u(t) = f(t) - \int_0^t K(t, \tau) u(\tau) d\tau, \quad (1.15)$$

where

$$K(t, \tau) = p(t) + q(t)(t - \tau), \quad (1.16)$$

and

$$f(t) = g(t) - [\beta p(t) + \alpha q(t) + \beta t q(t)]. \quad (1.17)$$

To generalize the steps introduced above, we consider the general IVP:

$$y^{(n)} + a_1(t)y^{(n-1)} + \dots + a_{n-1}(t)y' + a_n(t)y = g(t), \quad (1.18)$$

with initial conditions

$$y(0) = c_0, y'(0) = c_1, y''(0) = c_2, \dots, y^{(n-1)}(0) = c_{n-1}, \quad (1.19)$$

where $a_i(t), 1 \leq i \leq n$ are analytic functions, and $g(t), u(t)$ are continuous functions through the interval of discussion. Now set the transformation:

$$y^{(n)}(t) = u(t). \quad (1.20)$$

Integrating both sides with respect to t to get

$$y^{(n-1)}(t) = c_{n-1} + \int_0^t u(\tau) d\tau. \quad (1.21)$$

Also by integrating both sides again with respect to t yields

$$y^{(n-2)}(t) = c_{n-1} + c_{n-1}t + \int_0^t \int_0^x u(\tau) d\tau dx. \quad (1.22)$$

In equation (1.22) we reduce the double integral to a single integral by using the general form (Kanwal, 2013, pp.385)

$$\int_a^t \int_a^{t_{n-1}} \dots \int_a^{t_2} \int_a^{t_1} F(\tau) d\tau dt_1 \dots dt_{n-1} = \frac{1}{(n-1)!} \int_a^t (t-\tau)^{n-1} F(\tau) d\tau, \quad (1.23)$$

and get

$$y^{(n-2)}(t) = c_{n-2} + c_{n-1}t + \int_0^t (t-\tau)u(\tau)d\tau. \quad (1.24)$$

By integrating both sides of equation (1.21) we get

$$y^{(n-3)}(t) = c_{n-3} + c_{n-2}t + \frac{1}{2}c_{n-1}t^2 + \int_0^t \int_0^{t_2} \int_0^{t_1} u(\tau) d\tau dt_1 dt_2 \quad (1.25)$$

then by using the concept of equation (1.23) we obtain

$$y^{(n-3)}(t) = c_{n-3} + c_{n-2}t + \frac{1}{2}c_{n-1}t^2 + \frac{1}{2} \int_0^t (t-\tau)^2 u(\tau) d\tau. \quad (1.26)$$

Keeping the integration process yields

$$y(t) = \sum_{k=0}^{n-1} \frac{c_k}{k!} t^k + \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} u(\tau) d\tau. \quad (1.27)$$

Substituting (1.20)– (1.27) into (1.18) gives

$$u(t) = f(t) - \int_0^t K(t, \tau)u(\tau)d\tau, \quad (1.28)$$

where

$$K(t, \tau) = \sum_{k=1}^n \frac{a_k}{(k-1)!} (t-\tau)^{k-1}, \quad (1.29)$$

and

$$f(t) = g(t) - \sum_{j=1}^n a_j \left(\sum_{k=1}^j \frac{c_{n-k}}{(j-k)!} t^{j-k} \right). \quad (1.30)$$

In the following, we give an example on the applicability of the conversion methods

Example 1.13 Convert the following IVP to an equivalent VIE:

$$y'(t) - 2ty(t) = e^{t^2}, \quad y(0) = 1. \quad (1.31)$$

Now set

$$y'(t) = u(t). \quad (1.32)$$

Integrating both sides of (1.32), then using the initial condition $y(0) = 1$ leads to

$$y(t) - y(0) = \int_0^t u(\tau) d\tau, \quad (1.33)$$

or

$$y(t) = 1 + \int_0^t u(\tau) d\tau, \quad (1.34)$$

substituting (1.32) and (1.34) into (1.31) gives the equivalent VIE:

$$u(t) = 2t + e^{t^2} + 2t \int_0^t u(\tau) d\tau. \quad (1.35)$$

Example 1.14 Convert the following IVP to an equivalent VIE:

$$y'''(t) - y'' - y' + y = 0, \quad y(0) = 1, \quad y'(0) = 2, \quad y''(0) = 3. \quad (1.36)$$

We first set

$$y'''(t) = u(t), \quad (1.37)$$

integrating both sides of (1.37) then using the initial condition $y''(0) = 3$ we get

$$y''(t) = 3 + \int_0^t u(\tau) d\tau. \quad (1.38)$$

By repeating Integration once again and then use the initial condition $y'(0) = 2$ we obtain

$$y'(t) = 2 + 3t + \int_0^t \int_0^x u(\tau) d\tau dx = 2 + 3t + \int_0^t (t-\tau)u(\tau) d\tau. \quad (1.39)$$

Integrating again and using $y(0) = 1$ we find

$$\begin{aligned} y(t) &= 1 + 2t + \frac{3}{2}t^2 + \int_0^t \int_0^{t_2} \int_0^{t_1} u(\tau) d\tau dt_1 dt_2 \\ &= 1 + 2t + \frac{3}{2}t^2 + \frac{1}{2} \int_0^t (t - \tau)^2 u(\tau) d\tau. \end{aligned} \quad (1.40)$$

Substituting (1.37)– (1.40) into (1.36) yields the VIE:

$$u(t) = 4 + t + \frac{3}{2}t^2 + \int_0^t \left[1 + (t - \tau) - 12(t - \tau)^2 \right] u(\tau) d\tau. \quad (1.41)$$

(Wazwaz, 2011)

1.2 Newton-Kantorovich method

1.2.1 Idea and history of the Newton method

The basic idea of the Newton method (NM) is very simple, it is linearization method. Assume that $F : R^1 \rightarrow R^1$ is a differentiable function and we are solving the equation

$$F(x) = 0. \quad (1.42)$$

Starting with an initial point x_0 , we can formulate a linear approximation of $F(x)$ in a neighborhood of x_0 such that

$$F(x + h) \approx F(x_0) + F'(x_0)h, \quad (1.43)$$

and solve the corresponding linear equation. Thus we get the recurrent method

$$x_{k+1} = x_k - F'(x_k)^{-1}F(x_k), \quad k = 0, 1, \dots \quad (1.44)$$

This is the method initiated by Newton in 1669. In deed, Newton dealt only with polynomial equation of the form $P(x) = 0$ and he omitted higher order in h when he used the approximation of $P(x + h)$. In 1690 J. Raphson proposed the general form of method (1.44) (not assuming $F(x)$ to be a polynomial and using the notion of derivative) and the method was developed by many famous mathematicians, such as Fourier, Cauchy, and others. For instance Fourier in 1818 proved the convergence of the method is quadratically in a neighborhood of a root, while Cauchy provided a multidimensional extension of (1.44) and used the method to prove the existence of a root of an equation. Essential early contributions to the investigation of the method are due to Fine (1916) when he proved the convergence in the n -dimensional case with no assumption on the existence of a solution. Bennett (1916) extended the result to the infinite-dimensional case. Basic results on the NM and numerous references can be found in the books of Ostrowski et al. (1960), Ortega and Rheinboldt (1970), Rheinboldt (1974), and Deuffhard (2011).

1.2.2 Fréchet and Gâteaux derivative

Suppose we have an operator P mapping an open set Ω of one B-space X into a set Δ of another B-space Y . We take a fixed point $x_0 \in \Omega$ and assume that there exists a continuous linear operator $U \in B(X, Y)$ such that, for every $x \in X$,

$$\lim_{t \rightarrow 0} \frac{P(x_0 + tx) - P(x_0)}{t} = U(x). \quad (1.45)$$

In this situation we say that the linear operator U is the derivative of P at the point x_0 . Then we write

$$U = P'(x_0). \quad (1.46)$$

The derivative just defined is often called the Gâteaux derivative, or the weak derivative, and the element $U(x)$ is called the Gâteaux differential.

Let us denote by \bar{K} the set of all $x \in X$ with $\|x\| = 1$, if the limit in equation (1.45) is uniform for $x \in \bar{K}$, then P is said to be differentiable at x_0 and in this case $P'(x_0)$ is called the Fréchet derivative, or strong derivative. The differentiability of an operator P at a point x_0 means, in other words, that there exists a linear operator U such that, for every $\epsilon > 0$, we can find $\delta > 0$, whenever $\|\Delta x\| < \delta$, ($\Delta x \in X$), we have by Kantorovich and Akilov (1982)

$$\|P(x_0 + \Delta x) - P(x_0) - U(\Delta x)\| \leq \epsilon \|\Delta x\|. \quad (1.47)$$

1.2.3 Kantorovich's contribution

In 1948, L. V. Kantorovich published the seminar paper (Kantorovich, 1948a), where he suggested an extension of the NM to functional spaces. The results were also included in the survey paper Kantorovich (1948b). Further developments of the method can be found in Kantorovich (1949), Kantorovich (1957) and in the monographs Kantorovich and Akilov (1959), Kantorovich and Akilov (1982). This contribution by Kantorovich to one of the fundamental techniques in numerical analysis and functional analysis cannot be underestimated. Kantorovich analyzed the same equation as (1.42),

$$P(x) = 0, \quad (1.48)$$

but now $P : X \rightarrow Y$, where X and Y are Banach spaces. The proposed method reads as (1.44):

$$x_{k+1} = x_k - P'(x_k)^{-1}P(x_k), \quad k = 0, 1, 2, \dots, \quad (1.49)$$

where $P'(x_k)$ is the (Fréchet) derivative of the nonlinear operator $P(x)$ at the point x_k and $P'(x_k)^{-1}$ is its inverse. Now, suppose we have an operator P mapping an open subset Ω of Banach space X into Banach space Y . Assume that there is a zero of P in Ω , that is an element x^* such that

$$P(x^*) = 0. \quad (1.50)$$

Choose an arbitrary element $x_0 \in \Omega$. If we assume that P has a continuous derivative in Ω , we can replace the element $P(x_0) = P(x_0) - P(x^*)$ by the approximation $P'(x_0)(x_0 - x^*)$. We therefore have reason to suppose that a solution

of the equation

$$P'(x_0)(x_0 - x) = P(x_0), \quad (1.51)$$

will be close to x^* . But this equation is linear, so its solution is easy to find

$$x_1 = x_0 - [P'(x_0)]^{-1} (P(x_0)), \quad (1.52)$$

assuming that $[P'(x_0)]^{-1}$ exists. If we continue this process, we obtain, starting from the initial approximation x_0 a sequence $\{x_n\}$, where

$$x_{n+1} = x_n - [P'(x_n)]^{-1} (P(x_n)), \quad (n = 0, 1, \dots). \quad (1.53)$$

Each x_n is an approximate solution to the equation

$$P(x) = 0. \quad (1.54)$$

NM is clearly not always feasible. Firstly, x_n may pass beyond the set Ω for some value of n , and secondly, $[P'(x_n)]^{-1}$ may not exist.

If the sequence $\{x_n\}$ converges to a root x^* and x_0 is chosen close enough to x^* , then, by the continuity of P' , the operators $P'(x_n)$ and $P'(x_0)$ will only differ a small amount. Thus equation (1.53) can be replaced by the simplified formula

$$x_{n+1} = x_n - [P'(x_0)]^{-1} (P(x_n)), \quad (n = 0, 1, \dots), \quad (1.55)$$

which is significantly simpler than formula (1.53), although in general they yield poorer approximations. We shall call this method of generating the sequence $\{x_n\}$ in (1.55) the Newton-Kantorovich method (NKM).

Definition 1.1 *Kantorovich and Akilov (1982) The majorant function:*

Let the equation (1.54) be written as

$$x = S(x), \quad (1.56)$$

where S is defined on the ball $\|x - x_0\| < R$ and $S(x) = x - \Gamma_0 P(x)$ with

$$\Gamma_0 = [P'(x_0)]^{-1} \text{ for NKM and } \Gamma_0 = [P'(x)]^{-1} \text{ for NM.}$$

As well as we consider the real equation

$$t = \phi(t) \quad (1.57)$$

where the function ϕ is defined in the interval $[t_0, T]$, with $T = t_0 + r < t_0 + R$.

We say that equation (1.57) (or the function ϕ) majorizes equation (1.56) (or the operator S) if

- $\|S(x_0) - x_0\| \leq \phi(t_0) - t_0;$
- $\|S'(x)\| \leq \phi'(x)$ whenever $\|x - x_0\| \leq t - t_0.$

The following theorem, i.e the Kantorovich theorem which deals with the majorant function

$$\psi(t) = Kt^2 - 2t + 2\eta, \quad (1.58)$$

where k, η are real nonnegative real numbers.

Theorem 1.1 . (Kantorovich and Akilov, 1982, pp. 532). Suppose that the operator P is defined in open ball Ω , where $\|x - x_0\| \leq R$ and has a continuous second derivative in closed ball Ω_0 , whih $\|x - x_0\| \leq r, r \leq R$. Suppose, in addition, that

- 1) $\Gamma_0 = [P'(x_0)]^{-1}$ exists and is a continuous linear operator;
- 2) $\|\Gamma_0(P(x_0))\| \leq \eta$;
- 3) $\|\Gamma_0 P''(x)\| \leq K, (x \in \Omega_0)$,

where K and η are given in equation (1.58). Then, provided $h = K\eta \leq \frac{1}{2}$ and $r \geq r_0 = \frac{1 - \sqrt{1 - 2h}}{h}\eta$, equation (1.48) has a solution x^* , and the Newton-Kantorovich process converges to this solution. Furthermore, $\|x - x_0\| \leq r_0$. Also, if for $h < \frac{1}{2}$ we have $r < r_1 = \frac{1 + \sqrt{1 - 2h}}{h}\eta$, while for $h = \frac{1}{2}$ we have $r \leq r_1$, then x^* is the unique solution in Ω_0 . The rate of convergence is given by

$$\|x^* - x_n\| \leq \frac{\eta}{h} \left(1 - \sqrt{1 - 2h}\right)^{n+1}, \quad (n = 0, 1, 2, \dots).$$

1.3 Research aims and objectives

The objectives of the thesis are listed as follow:

- a) To solve one dimensional nonlinear Volterra type IE of the second kind by NKM, establish the convergence of the proposed method and obtain the error estimation.
- b) To solve 2×2 system of nonlinear Volterra type IEs of the second kind by NKM, establish the convergence of method and obtain the error estimation.
- c) To solve $n \times n$ system of nonlinear Volterra type IEs of the second kind by NKM, establish the convergence of method and obtain error estimation.
- d) To solve two dimensional nonlinear Volterra type IE of the second kind by NKM, establish the convergence of method and obtain the error estimation.

1.4 Organization of the thesis

This thesis has seven chapters. Chapter 1 contains a brief introduction to the general integral equations, historical introduction, classification of IEs, the meaning of Fréchet and Gâteaux derivatives and the concept of majorant function. The literature reviews are given in chapter 2. Chapter 3 focuses on finding the approximate solution of one dimensional nonlinear Volterra type IE via NKM. NKM, the majorant function and the concept of Gauss-Legendre formula are discussed in chapter 4 to find the approximate solution of system of 2×2 nonlinear Volterra type IE. In chapter 5, we deal with the system of $n \times n$ nonlinear Volterra type integral equation, then finding the approximate solution by NKM. Chapter

6 discusses the NKM and two dimensional Gauss-Legendre quadrature formula (QF) to find the approximate solution of two dimensional nonlinear VIE. Chapter 7 gives some future works as an extension to this research.



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