UNIVERSITI PUTRA MALAYSIA

PHASE-FITTED AND AMPLIFICATION-FITTED RUNGE-KUTTA TYPE METHODS FOR SOLVING LINEAR DIFFERENTIAL EQUATIONS WITH OSCILLATORY SOLUTIONS

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By

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PHASE-FITTED AND AMPLIFICATION-FITTED RUNGE-KUTTA TYPE METHODS FOR SOLVING LINEAR DIFFERENTIAL EQUATIONS WITH OSCILLATORY SOLUTIONS

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August 2016

Chairman : Professor Fudziah Binti Ismail, PhD
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New phase-fitted and amplification-fitted Runge-Kutta methods (RK) based on the existing Runge-Kutta methods of order four and five were derived to solve second order ordinary differential equations with oscillatory solutions. The new method has the property of zero phase-lag and zero dissipation. The effects of phase-fitted and amplification-fitted relations are tested over a large interval on homogeneous and non-homogeneous problems which have oscillatory solutions and the numerical results proved that the new methods are more accurate compared to the existing methods. Then, the first order Fuzzy Differential Equations (FDEs) are solved using the RK methods with phase-fitted and amplification-fitted and the numerical results show that the methods are more accurate than the existing methods.

The technique of phase-fitted and amplification-fitted is then extended to diagonally implicit Runge-Kutta methods (DIRK) for solving second order ordinary differential equations (ODEs) with oscillatory solution. We derived the phase-fitted and amplification-fitted fourth DIRK method based on the fourth order existing DIRK methods. Numerical results show that the DIRK with phase-fitted and amplification-fitted is more accurate and efficient for solving oscillatory problems.

In the next part of the thesis, we derived the order conditions of Runge-Kutta Nystrom method purposely for solving linear second order ordinary differential equations. Based on the order conditions we derived the new fifth order four-stage and sixth order five-stage explicit Runge-Kutta-Nyström methods for linear ordinary differential equations (LODEs). Then the methods are phase-fitted and amplification-fitted so that they will have zero-dispersion and zero-dissipation. The fifth order four-stage RKN method for LODEs has the property of First Same As Last (FSAL). Numerical results proved that the methods with phase-fitted and amplification-fitted are much more efficient than the existing methods with the same algebraic order.

Next we used the RKN methods for solving Hyperbolic partial differential equations (PDEs) that is the second order wave equations. We very well know that the second order PDEs can be converted to second order linear ODEs using the methods of lines. Thus we applied the RKN methods for LODEs to solve the resulting second order linear ODEs. Numerical results show that the RKN methods for LODEs are accurate and reliable for solving the second order wave equation.

As a conclusion, in this thesis, we have derived phase-fitted and amplification-fitted RK and RKN methods for solving first and second order oscillatory problems. The phase-fitted and amplification-fitted RK method is also applied to first order fuzzy differential equations (FDEs). The non fitted RKN method for LODEs is also used for solving hyperbolic partial differential equations. Numerical results show that all the methods are more accurate then the existing methods in the secientific literature.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk Ijazah Doktor Falsafah

SUAI-FASA DAN SUAI-AMPLIFIKASI JENIS KAEDAH RUNGE-KUTTA UNTUK MENYELESAIKAN MASALAH BERAYUN BAGI PERSAMAAN PEMBEZAAN LINEAR

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Kaedah Baru Runge-Kutta (RK) yang di suai-fasa dan suai-amplifikasi berdasarkan kaedah Runge-Kutta yang sedia ada peringkat keempat dan kelima diterbitkan untuk menyelesaikan persamaan pembezaan biasa dengan penyelesaian berayun. Kaedah baru ini mempunyai sifat serakan sifar dan lesapan sifar. Kesan hubungan suai-fasa dan suai- amplifikasi diuji untuk selang yang lebih besar keatas masalah homogen dan tak homogen yang mempunyai penyelesaian berayun dan keputusan berangka membuktikan bahawa kaedah baru tersebut adalah lebih tepat berbanding dengan kaedah yang sedia ada. Kemudian, Persamaan Pembezaan Kabur (PPK) Peringkat pertama diselesaikan dengan menggunakan kaedah RK dengan suai-fasa dan suai-amplifikasi tersebut dan keputusan berangka menunjukkan bahawa kaedah ini adalah lebih tepat daripada kaedah yang sedia ada.

Seterusnya teknik suai-fasa dan suai-amplifikasi diperluaskan pula kepada kaedah Runge-Kutta Pepenjuru tersirat (RKPT) untuk menyelesaikan persamaan pembezaan biasa (PPB) peringkat pertama dengan penyelesaian berayun. Kami terbitkan kaedah suai-fasa dan dan suai-amplifikasi berdasarkan kaedah RKPT peringkat empat yang sedia ada. Keputusan berangka menunjukkan bahawa kaedah RKPT dengan suai-fasa dan dan suai-amplifikasi adalah lebih tepat untuk menyelesaikan masalah berayun.

Dalam bahagian seterusnya tesis ini, kami terbitkan syarat peringkat kaedah Runge-Kutta Nystrom khas untuk menyelesaikan persamaan pembezaan biasa linear peringkat kedua. Berdasarkan syarat peringkat tersebut, kami terbitkan kaedah tak tersirat Runge-Kutta-Nystrom baharu peringkat kelima tahap-empat dan peringkat keenam tahap-lima untuk menyelesaikan persamaan pembezaan biasa linear (PPBL) peringkat kedua. Kemudian kaedah ini telah di suai-fasa dan suai-implifikasi agar ianya mempunyai serakan sifar dan lesapan sifar. Kaedah RKN peringkat kelima tahap-empat untuk PPBL ini mempunyai ciri yang pertama sama dengan yang terakhir (PSDA). Keputusan berangka membuktikan bahawa kaedah baharu dengan suai-fasa

dan suai-implifikasi ini adalah lebih cekap berbanding kaedah sedia ada dengan peringkat aljabar yang sama.

Seterusnya kami menggunakan kaedah RKN yang telah diterbitkan untuk menyelesaikan persamaan pembezaan separa (PPS) hiperbolik iaitu persamaan gelombang peringkat kedua. Kita sedia ketahui bahawa persamaan pembezaan separa peringkat kedua boleh diubah kepada persamaan pembezaan biasa linear peringkat kedua dengan menggunakan kaedah garis. Oleh itu kami gunakan kaedah RKN untuk PPBL bagi menyelesaikan persamaan pembezaan yang terhasil. Keputusan berangka menunjukkan bahawa kaedah RKN untuk PPBL adalah lebih jitu bagi menyelesaikan persamaan gelombang peringkat kedua.

Kesimpulannya, dalam tesis ini, kami telah menerbitkan kaedah RK dan RKN dengan suai-fasa dan suai-amplifikasi untuk menyelesaikan masalah berayun peringkat pertama dan kedua masing-masingnya. Kaedah RK dengan suai-fasa dan suai-amplifikasi juga digunakan untuk untuk menyelesaikan persamaan pembezaan kabur peringkat pertama. Kaedah RKN tanpa suai-fasa untuk PPBL juga digunakan untuk menyelesaikan persamaan pembezaan separa hiperbola. Keputusan berangka menunjukkan bahawa semua kaedah ini adalah lebih jitu dari kaedah sedia ada dalam literatur saintifik.

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LIST OF ABBREVIATIONS

DEs	Differential Equations
ODEs	Ordinary Differential Equations
IVPs	Initial Value Problems
RK	Runge-Kutta method
RKN	Runge-Kutta-Nyström method
FDEs	Fuzzy Differential Equations
FIVP	Fuzzy Initial Value Problem
PDEs	Partial Differential Equations
HPDEs	Hyperbolic Partial Differential Equations
LTE	Local Truncation Error
FSAL	The First Same as Last
LODEs	Linear Ordinary Differential Equations
DIRK	Diagonally Implicit Runge-Kutta method
PL	Phase-lag or dispersion
DS	Dissipation
E	Absolute error
MAXE	Maximum of absolute error
FCN	Number of function evaluations
TIME	Execution time
New ERK4 PLDS	New Classical fourth order Runge-Kutta method with phase-fitted and amplification-fitted
Classical RK4	Classical Runge-Kutta method of order four
New ERK5 PLDS	New Runge-Kutta method of order five for Linear Ordinary Differential Equations proposed from Zingg (1999) with phase-fitted and amplification-fitted

RK5 Zingg	Fifth order Runge-Kutta method for Linear Ordinary Differential Equations proposed from Zingg (1999)
FRK4 PLDS	Classical Runge-Kutta method of order four with phase-fitted and amplification-fitted for solving Fuzzy Differential Equations
FRK4	Classical Runge-Kutta method of order four for solving Fuzzy Differential Equations
DIRK4	Fourth order Diagonally Implicit Runge-Kutta method
Hairer DIRK4 PLDS	Fourth order five-stage diagonally implicit Runge-Kutta method proposed from Hairer (2010) with phase-fitted and amplification-fitted
Hairer DIRK4	Fourth order five-stage diagonally implicit Runge-Kutta method proposed from Hairer (2010)
Izzati DIRK4 PLDS	Fourth order four-stage diagonally implicit Runge-Kutta method proposed from Che Jawias (2010) with phase-fitted and amplification-fitted
Izzati DIRK4	Fourth order four-stage diagonally implicit Runge-Kutta method proposed from Che Jawias (2010)
New RKN5	New four-stage fifth order Runge-Kutta-Nyström of Linear Ordinary Differential Equations
RKN5 PLDS	New four-stage fifth order Runge-Kutta-Nyström of Linear Ordinary Differential Equations with phase-fitted and amplification-fitted
VRKN5	Fifth order four-stage Runge-Kutta-Nyström proposed from Van der Houwen et.al (1987) with dispersion of order two and dissipation of order eight
SRKN5	Fifth order four-stage Runge-Kutta-Nyström proposed from Simos et.al (1994) with dispersion of order four and dissipation of order eight
New RKN6	New five-stage sixth order Runge-KuttaNyström method for Linear Ordinary Differential Equations
RKN6 PLDS	New five-stage sixth order Runge-Kutta-Nyström method for Linear Ordinary Differential Equations with phase-fitted and amplification-fitted

VRKN6	Sixth order five-stage Runge-Kutta-Nyström proposed from Van der Houwen et.al (1987) with dispersion of order two and dissipation of order eight
RKN5 LODEs	New four-stage fifth order Runge-Kutta-Nyström of Linear Ordinary Differential Equations
RKN6 LODEs	New five-stage sixth order Runge-KuttaNyström method for Linear Ordinary Differential Equations

CHAPTER 1

INTRODUCTION

Ordinary differential equations (ODEs) are equations that involve an unknown function with independent variable and one or more of its derivatives. ODEs arise in many contexts of engineering and science such as fluid dynamics, radioactive decay and population growth.

Many theoretical and numerical studies for such equations have appeared in literature. The analytical way to solve ODEs is via application of integration technique. However, the anti-derivatives for most realistic systems of ODEs are difficult or impossible to find. Thus, numerical methods for ODEs have attracted considerable attention.

The initial value problems (IVPs) of special second order ODEs are defined as follows:

$$y'' = f(x, y), \qquad y(x_0) = y_0, \qquad y'(x_0) = y'_0 \tag{1.1}$$

where $f: R \times R \rightarrow R^n, and\ y_0, y'_0 \in R^n$. The solution of (1.1) exhibits a pronounced oscillatory character. One way to solve oscillatory problems is by Runge-Kutta methods. In this study, we are focusing on solving problem (1.1) by using RK methods and RKN methods for oscillating problems.

1.1 Ordinary Differential Equations

Any research on ordinary differential equations gives a number of methods for explicitly finding solutions to first-order initial-value problems (IVPs). In practice, however, few of the problems originating from the study of physical phenomena can be solved exactly.

The *n*-th order ODEs can be written as:

$$y^{(n)} = f\left(x, y, \dots, y^{(n-1)}\right), \text{ where } n = 2,3,4$$

with initial conditions:

$$y(a) = y_0 \text{ and } y^{(i)}(a) = \eta_i, 0 < i < n-1, x \in [a, b]$$

while the first order ODEs can be written as:

$$\frac{dy}{dx} = f\big(x, y(x)\big), y(a) = y_0 \quad \text{for} \quad x \in [a, b] \tag{1.2}$$

In (1.2), the quantity being differentiated, y is named as the dependent variable, while the quantity with respect to which y is differentiated, x is named as independent variable.

Below, we state the existence and uniqueness theorem for finding the solution to the first order IVPs.

Theorem 1.1: (Existence and Uniqueness)

Let $f(x, y)$ be defined and continuous for all points (x, y) in the region D defined by $a \le x \le b, -\infty < y < \infty$, where a and b are finite, and let there exists a constant L such that for any $x \in [a, b]$ and any two numbers y_1 and y_2,

$$|f(x, y_1) - f(x, y_2)| \le L|y_1 - y_2|$$

This condition is known as *Lipschitz condition*. Then there exists exactly one function $y(x)$ with the following three properties:

i. $y(x)$ is continuous and differentiable for $x \in [a, b]$,
ii. $y' = f\big(x, y(x)\big), x \in [a, b]$,
iii. $y(a) = \eta$.

The proof is given by Henrici (1962).

Basically, the numerical methods for ODEs are classified as one-step method and multistep method. One-step method requires the information from only one previous point x_n to find the approximation at the mesh point x_{n+1}. On the other hand, multistep method requires the usage of information from more than one previous points to find the next approximation.

Many differential equations which appear in practice are systems of second order IVPs (1.1) in which the derivative does not appear explicitly. Such a system can be transformed into first order differential equations of doubled dimension by considering the vector (y, y') as the new variable. In this study, we are focusing on solving special second order equation (1.1) for which it is known in advance that their solution is oscillating. Consider the second order linear differential equation:

$$y'' = Ay \tag{1.3}$$

where A is a continuous real-valued function. A solution of (1.3) is said to be *oscillatory* if it has arbitrarily large zeros, and otherwise it is said to be *nonoscillatory*. Equation (1.3) is called oscillatory if all its solutions are oscillatory.

When dealing with the oscillatory problems of (1.1), we need to consider the dispersion (phase-lag) and dissipation (amplification error) properties of the methods developed.

1.2 Fuzzy Differential Equations

In many cases of the modeling of real world problem, knowledge or information about the behavior of the physical system is often incomplete, uncertain or vague. For example, values of parameter, functional relationship or initial conditions, may not be accurate. These uncertainties have to be considered for obtaining a more realistic model. Fuzzy Differential Equations (FDEs) are utilized for the purpose of the modeling of uncertainty and processing vague or incomplete information in mathematical models.

Most of the problems in science and engineering require the solution of a Fuzzy Differential Equation (FDE) which are satisfied in Fuzzy initial conditions. Due to the large potential of FDEs involved in various fields, a lot of research has been focused in this area. Sometimes it is too complicated to obtain the exact solution of (FDE) which models the science and engineering problems, hence numerical methods are used.

Below, we state some definitions and theorems for FDEs.

Definition 1.2.1:

An arbitrary fuzzy number is represented by ordered pair of functions $\left(\underline{u}(\alpha), \overline{u}(\alpha)\right), 0 \leq \alpha \leq 1$, satisfying the following requirements:

- $\underline{u}(\alpha)$ is a bounded left continuous nondecreasing function over [0,1].
- $\overline{u}(\alpha)$ is a bounded left continuous nonincreasing function over [0,1].
- $\underline{u}(\alpha) \leq \overline{u}(\alpha), 0 \leq \alpha \leq 1$.

Definition 1.2.2:

Let I be a real interval. A mapping $y: I \rightarrow E$ is called a fuzzy process and its α-level set is denoted by:

$$[y(t)]^{\alpha} = \left[\underline{y}^{\alpha}(t), \overline{y}^{\alpha}(t)\right], t \in I, \alpha \in (0,1] \tag{1.4}$$

Definition 1.2.3:

Let R_F denote the class of fuzzy numbers and let $x, y \in R_F$. If there exists $z \in R_F$ such that $x = y + z$, then z is called the Hukuhara difference of x and y and it is denoted by $x \ominus y$.

Note: the " $\ominus$ " sign stands always for Hukuhara difference and $x \ominus y \neq x + (-1)y$.

Definition 1.2.4:

Let $f: (a, b) \to R_F$ and $x_0 \in (a, b)$. We say that f is Hukuhara differentiable at x_0, if there exists an element $f'(x_0) \in R_F$, such that for all $h > 0$ sufficiently small:

$\exists f(x_0 + h) \ominus f(x_0), f(x_0) \ominus f(x_0 - h)$ and the limits:

$$\lim_{h \to 0} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h \to 0} \frac{f(x_0) \ominus f(x_0 - h)}{h} = f'(x_0) \qquad (1.5)$$

Theorem 1.2.1:

Let $F: (a, b) \to R_F$ be Hukuhara differentiable and denote $[F(t)]^\alpha = \left[\underline{F}^\alpha(t), \overline{F}^\alpha(t)\right]$. Then the boundary functions $\underline{F}^\alpha(t)$ and $\overline{F}^\alpha(t)$ are differentiable and:

$$[F'(t)]^\alpha = \left[\left(\underline{F}^\alpha\right)'^{(t)}, (\overline{F}^\alpha)'(t)\right], \alpha \in [0,1] \qquad (1.6)$$

The proof is given by Kaleva (2006).

Definition 1.2.5:

Let us consider the fuzzy initial value problem (FIVP):

$$y' = f(t, y), y(t_0) = y_0 \qquad (1.7)$$

where $f: [t_0, t_0 + \alpha] \times R_F \to R_F$ and $y_0 \in R_F$. Then the theorem (1.2.1) shows us a way how to translate FIVP (1.7) into a system of ODEs.

Definition 1.2.6:

Let $[y(t)]^\alpha = \left[\underline{y}^\alpha(t), \overline{y}^\alpha(t)\right]$. If $y(t)$ is Hukuhara differentiable then $[y'(t)]^\alpha = \left[\left(\underline{y}^\alpha\right)'(t), (\overline{y}^\alpha)'(t)\right]$. So equation (4.4) translates into the following systems of ODEs:

$$
\begin{cases}
((\underline{y}^{\alpha})'(t)) = \underline{f}^{\alpha}(t, \underline{y}^{\alpha}(t), \overline{y}^{\alpha}(t)) \\
((\overline{y}^{\alpha})'(t) = \overline{f}^{\alpha}(t, \underline{y}^{\alpha}(t), \overline{y}^{\alpha}(t)) \\
\underline{y}^{\alpha}(t_0) = \underline{y}_0^{\alpha} \\
\overline{y}^{\alpha}(t_0) = \overline{y}_0^{\alpha}
\end{cases}
\tag{1.8}
$$

where:

$$
[f(t,y)]^{\alpha} = \left[\underline{f}^{\alpha}\left(t, \underline{y}^{\alpha}, \overline{y}^{\alpha}\right), \overline{f}^{\alpha}(t, \underline{y}^{\alpha}, \overline{y}^{\alpha})\right] \tag{1.9}
$$

Definition 1.2.7:

The Seikkala derivative $y'(t)$ of a fuzzy process *y* (defined by equation (1.9)) is defined by:

$$
[y'(t)]^{\alpha} = \left[(\underline{y}^{\alpha})'(t), (\overline{y}^{\alpha})'(t)\right], 0 < \alpha \leq 1 \tag{1.10}
$$

Remark 1.2.1:

If $y: I \rightarrow E$ is Seikkala differentiable and its Seikkala derivative *y'* is integrable over [0,1], then:

$$
y(t) = y(t_0) + \int_{t_0}^{t} y'(s)ds \tag{1.11}
$$

for all values of t_0, t where $t_0, t \in I$.

1.3 Partial Differential Equations

Many problems in applied science, physics, and engineering can be modeled as partial differential equations (PDEs), that is differential equation which involves more than one independent variable.

A PDE for the function $u(x_1, \ldots, x_n)$ is an equation of the form:

$$
f\left(x_1, \ldots, x_n, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}, \frac{\partial^2 u}{\partial x_1 \partial x_1}, \ldots, \frac{\partial^2 u}{\partial x_n \partial x_n}, \ldots\right) = 0 \tag{1.12}
$$

Below we listed some of the common partial differential equations:

Elliptic equations:

The PDE involves $u_{xx}(x, y) + u_{yy}(x, y)$ is an *elliptic equation*. The particular elliptic equation we will consider in this thesis is known as the *Poisson equation*:

$$\frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y) = f(x, y) \tag{1.13}$$

This equation arise in the study various time-independent physical problems such as the steady-state distribution of heat in a plane region, the potential energy of a point in a plane acted on by gravitational forces in the plane and two dimensional steady-state problems involving incompressible fluids.

Parabolic equations:

Another type of PDE is the *parabolic* partial differential equation of the form:

$$\frac{\partial u}{\partial t}(x, t) - \alpha^2 \frac{\partial^2 u}{\partial x^2}(x, t) = 0 \tag{1.14}$$

The PDE concerns the flow of heat along a rod of length *l,* which has a uniform temperature within each cross-sectional element. The parameter α is determined by the heat conductivity of the rod.

Hyperbolic equations:

The hyperbolic equation is the one-dimensional *wave equation*. Suppose an elastic string of length *l* is stretched between two supports at the same horizontal level.

If the string is set to vibrate in a vertical plane, the vertical displacement $u(x, t)$ of a point *x* at time *t* satisfies the PDE:

$$\alpha^2 \frac{\partial^2 u}{\partial x^2}(x, t) - \frac{\partial^2 u}{\partial t^2}(x, t) = 0 \quad \text{for } 0 < x < 1 \text{ and } 0 < t \tag{1.15}$$

1.4 Runge-Kutta (RK) methods

An s-stage explicit Runge-Kutta method can be expressed by the following relations:

$$y_{n+1} = y_n + h \sum_{i=1}^{s} b_i k_i \tag{1.16}$$

where

$$k_i = f\left(x_n + c_i, y_n + h\sum_{j=1}^{i} a_{ij}k_j\right) \tag{1.17}$$

The method is used for the computation of the approximation of y_{n+1}, when y_n is known. The method shown above can also be presented using the Butcher table below:

Table 1.1 : Butcher Table

$$\begin{array}{c|cccc}
c_2 & a_{21} & & & \\
c_3 & a_{31} & a_{32} & & \\
. & . & . & . & \\
c_s & a_{s1} & a_{s2} & . & a_{s,s-1} \\
\hline
 & b_1 & b_2 & . & b_s
\end{array}$$

where the coefficients $c_2, \ldots\ldots, c_s$ must satisfy the equations:

$$c_i = \sum_{j=1}^{i-1} a_{ij}, \quad i = 2, \ldots., s \tag{1.18}$$

1.4.1 Algebraic Order Condition for RK method

General order conditions for RK method can be attained from direct expansion of Taylor series by using Local Truncation Error (LTE). The s-stage up to order six RK methods are given as follows:

$$order\ 1: \quad \sum b_i = 1 \tag{1.19}$$

$$order\ 2: \quad \sum b_i c_i = \frac{1}{2} \tag{1.20}$$

$$order\ 3: \quad \frac{1}{2}\sum b_i c_i^2 = \frac{1}{6} \tag{1.21}$$

$$order\ 4: \quad \frac{1}{6}\sum b_i c_i^3 = \frac{1}{24} \tag{1.22}$$

$$\sum b_i a_{ij} c_j = \frac{1}{24} \tag{1.23}$$

$$order\ 5: \quad \frac{1}{24}\sum b_i c_i^4 = \frac{1}{120} \tag{1.24}$$

$$\frac{1}{4}\sum b_i c_i a_{ij} c_j = \frac{1}{120} \tag{1.25}$$

$$\frac{1}{2}\sum b_i a_{ij} c_j^2 = \frac{1}{120} \tag{1.26}$$

$$order\ 6: \quad \frac{1}{120}\sum b_i c_i^5 = \frac{1}{720} \tag{1.27}$$

$$\frac{1}{20}\sum b_i c_i^2 a_{ij} c_j = \frac{1}{720} \tag{1.28}$$

$$\frac{1}{10}\sum b_i c_i a_{ij} c_j^2 = \frac{1}{720} \tag{1.29}$$

$$\frac{1}{6}\sum b_i a_{ij} c_j^3 = \frac{1}{720} \tag{1.30}$$

$$\sum b_i a_{ij} a_{jk} c_k = \frac{1}{720} \tag{1.31}$$

where the coefficients may be made dependent by imposing the Butcher conditions (1963):

$$\sum_{i=1}^{s} b_i a_{ij} = b_j(1 - c_j), \qquad j = 2,3,\dots,s \tag{1.32}$$

Additionally, if $b_2 = 0$, we have:

$$\sum_{j=1}^{i-1} a_{ij} c_j = \frac{c_i^2}{2}, \qquad i = 3,4,\dots,s \tag{1.33}$$

These valuable relations actually reduce the total number of conditions arising in higher orders. So, the conditions (1.34) and (1.35) are supplemented by extra simplifying relations.

1.4.2 Local Truncation Error for RK method

Dormand (1996) proposed that having achieved a particular order of accuracy. The best strategy for practical purposes would be choosing the free RK parameters is to minimize the error norm:

$$A^{(p+1)} = \left\|\tau^{(p+1)}\right\|_2 = \sqrt{\left(\sum_{j=1}^{n_{p+1}} \tau_j^{(p+1)}\right)^2} \tag{1.34}$$

The error coefficients up to order six for RK methods are as follows:

$$order\ 1: \quad \tau_1^{(1)} = \sum b_i - 1 \tag{1.35}$$

$$order\ 2: \quad \tau_1^{(2)} = \sum b_i c_i - \frac{1}{2} \tag{1.36}$$

$$order\ 3: \quad \tau_1^{(3)} = \frac{1}{2}\sum b_i c_i^2 - \frac{1}{6} \tag{1.37}$$

$$order\ 4: \quad \tau_1^{(4)} = \frac{1}{6}\sum b_i c_i^3 - \frac{1}{24} \tag{1.38}$$

$$\tau_2^{(4)} = \sum b_i a_{ij} c_j - \frac{1}{24} \tag{1.39}$$

$$order\ 5: \quad \tau_1^{(5)} = \frac{1}{24}\sum b_i c_i^4 - \frac{1}{120} \tag{1.40}$$

$$\tau_2^{(5)} = \frac{1}{4}\sum b_i c_i a_{ij} c_j - \frac{1}{120} \tag{1.41}$$

$$\tau_3^{(5)} = \frac{1}{2}\sum b_i a_{ij} c_j^2 - \frac{1}{120} \tag{1.42}$$

$$order\ 6: \quad \tau_1^{(6)} = \frac{1}{120}\sum b_i c_i^5 - \frac{1}{720} \tag{1.43}$$

$$\tau_2^{(6)} = \frac{1}{20}\sum b_i c_i^2 a_{ij} c_j - \frac{1}{720} \tag{1.44}$$

$$\tau_3^{(6)} = \frac{1}{10}\sum b_i c_i a_{ij} c_j^2 - \frac{1}{720} \tag{1.45}$$

$$\tau_4^{(6)} = \frac{1}{6}\sum b_i a_{ij} c_j^3 - \frac{1}{720} \tag{1.46}$$

$$\tau_5^{(6)} = \sum b_i a_{ij} a_{jk} c_k - \frac{1}{720} \tag{1.47}$$

1.4.3 Analysis of Absolute Stability for RK method

Consider the standard test problem of differential equation:

$$y' = f(t, y) = \lambda y \;\; and \;\; y(x_n) = y_n \tag{1.48}$$

which has the true solution:

$$y(x) = y_n e^{\lambda(x - x_n)} \tag{1.49}$$

Applying the test equation (1.48) to the RK formula (1.16) and by setting $v = \lambda h$, we obtain:

$$y_{n+1} = R(v) y_n, \quad where\ |R(v)| < 1 \tag{1.50}$$

$$y_{n+1} = [1 + v b^T (I - vA)^{-1}] y_n \tag{1.51}$$

where $b = b_i$ and $A = a_{ij}$, $i, j = 1, \dots, s$ are parameters of RK method and $R(v)$ is said to be the stability polynomial. The stability polynomial for RK method of order p is:

$$R(v) = 1 + v + \frac{v^2}{2!} + \frac{v^3}{3!} + \frac{v^4}{4!} + \dots + \frac{v^p}{p!} \tag{1.52}$$

1.5 Runge-Kutta-Nyström (RKN) methods

Nyström (1925) introduced RK methods for second order ODEs which has been called RKN methods. The form of s-stage RKN method of order p is:

$$y_{n+1} = y_n + h y'_n + h^2 \sum_{i=1}^{s} b_i k_i$$

$$y'_{n+1} = y_n + h \sum_{i=1}^{s} \bar{b}_i k_i \tag{1.53}$$

where

$$k_i = f\left(x_n + c_i h, y_n + c_i h y'_n + h^2 \sum_{i=1}^{s-1} a_{ij} k_j \right), i = 1, 2, \dots, s \tag{1.54}$$

1.5.1 Algebraic Condition for RKN method

The order condition for RKN methods can be attained from direct expansion of the Taylor series by using the local Truncation Error (LTE). The order conditions for RKN methods up to order six obtained from Dormand (1996) are given as follows:

for y:

$$order\ 2: \quad \sum b_i = \frac{1}{2} \tag{1.55}$$

$$order\ 3: \quad \sum b_i c_i = \frac{1}{6} \tag{1.56}$$

$$order\ 4: \quad \frac{1}{2}\sum b_i c_i^2 = \frac{1}{24} \tag{1.57}$$

$$order\ 5: \quad \frac{1}{6}\sum b_i c_i^3 = \frac{1}{120} \tag{1.58}$$

$$\sum b_i a_{ij} c_j = \frac{1}{120} \tag{1.59}$$

$$order\ 6: \quad \frac{1}{24}\sum b_i c_i^4 = \frac{1}{720} \tag{1.60}$$

$$\frac{1}{2}\sum b_i c_i a_{ij} c_j = \frac{1}{720} \tag{1.61}$$

$$\frac{1}{2}\sum b_i a_{ij} c_j^2 = \frac{1}{720} \tag{1.62}$$

for *y'*:

$$order\ 1: \quad \sum \bar{b}_i = 1 \tag{1.63}$$

$$order\ 2: \quad \sum \bar{b}_i c_i = \frac{1}{2} \tag{1.64}$$

$$order\ 3: \quad \frac{1}{2}\sum \bar{b}_i c_i^2 = \frac{1}{6} \tag{1.65}$$

$$order\ 4: \quad \frac{1}{6}\sum \bar{b}_i c_i^3 = \frac{1}{24} \tag{1.66}$$

$$\sum \bar{b}_i a_{ij} c_j = \frac{1}{24} \tag{1.67}$$

$$order\ 5: \quad \frac{1}{24}\sum \bar{b}_i c_i^4 = \frac{1}{120} \tag{1.68}$$

$$\frac{1}{4}\sum \bar{b}_i c_i a_{ij} c_j = \frac{1}{120} \tag{1.69}$$

$$\frac{1}{2}\sum \bar{b}_i a_{ij} c_j^2 = \frac{1}{120} \tag{1.70}$$

$$order\ 6: \quad \frac{1}{120}\sum \bar{b}_i c_i^5 = \frac{1}{720} \tag{1.71}$$

$$\frac{1}{20}\sum \bar{b}_i c_i^2 a_{ij} c_j = \frac{1}{720} \tag{1.72}$$

$$\frac{1}{10}\sum \bar{b}_i c_i a_{ij} c_j^2 = \frac{1}{720} \tag{1.73}$$

$$\frac{1}{6}\sum \bar{b}_i a_{ij} c_j^3 = \frac{1}{720} \tag{1.74}$$

$$\sum \bar{b}_i a_{ij} a_{jk} c_k = \frac{1}{720} \tag{1.75}$$

The Nyström row sum conditions that need to be satisfied are:

$$\sum_{i=1}^{s} a_{ij} = \frac{c_i^2}{2}, \qquad i = 1, \dots, s \tag{1.76}$$

The simplifying assumption given by Butcher (2003) which is used in order to reduce the number of equations:

$$b_i = \bar{b}_i(1 - c_i) \tag{1.77}$$

The First Same as Last (FSAL) property where the last stage is evaluated at the same point as the first stage of the next step is used to reduce function evaluation. For RKN method to be FSAL, it has to satisfy:

$$c_1 = 0, c_s = 1\ and\ a_{sj} = b_j, j = 1, \dots, s-1 \tag{1.78}$$

1.5.2 Local Truncation Error for RKN method

We need to use the quantity (1.36) for *y* and the quantity (1.81) for *y'* in order to find the norms of (LTE) coefficients for RKN method. The quantity for *y'* is known as:

$$\left\|\tau'^{(p+1)}\right\|_2 = \sqrt{\left(\sum_{j=1}^{n_{p+1}} \tau'^{(p+1)}_j\right)^2}\ for\ y'_n \tag{1.79}$$

The error coefficients for RKN methods up to order six are:
for y:

$$order\ 2: \quad \tau_1^{(2)} = \sum b_i - \frac{1}{2} \tag{1.80}$$

$$order\ 3: \quad \tau_1^{(3)} = \sum b_i c_i - \frac{1}{6} \tag{1.81}$$

$$order\ 4: \quad \tau_1^{(4)} = \frac{1}{2} \sum b_i c_i^2 - \frac{1}{24} \tag{1.82}$$

$$order\ 5: \quad \tau_1^{(5)} = \frac{1}{6} \sum b_i c_i^3 - \frac{1}{120} \tag{1.83}$$

$$\tau_2^{(5)} = \sum b_i a_{ij} c_j - \frac{1}{120} \tag{1.84}$$

$$order\ 6: \quad \tau_1^{(6)} = \frac{1}{24} \sum b_i c_i^4 - \frac{1}{720} \tag{1.85}$$

$$\tau_2^{(6)} = \frac{1}{2} \sum b_i c_i a_{ij} c_j - \frac{1}{720} \tag{1.86}$$

$$\tau_3^{(6)} = \frac{1}{2} \sum b_i a_{ij} c_j^2 - \frac{1}{720} \tag{1.87}$$

for *y'*:

$$order\ 1: \quad \tau'^{(1)}_1 = \sum \bar{b}_i - 1 \tag{1.88}$$

$$order\ 2: \quad \tau'^{(2)}_1 = \sum \bar{b}_i c_i - \frac{1}{2} \tag{1.89}$$

$$order\ 3: \quad \tau'^{(3)}_1 = \frac{1}{2} \sum \bar{b}_i c_i^2 - \frac{1}{6} \tag{1.90}$$

$$order\ 4: \quad \tau'^{(4)}_1 = \frac{1}{6} \sum \bar{b}_i c_i^3 - \frac{1}{24} \tag{1.91}$$

$$\tau'^{(4)}_2 = \sum \bar{b}_i a_{ij} c_j - \frac{1}{24} \tag{1.92}$$

$$order\ 5: \quad \tau'^{(5)}_1 = \frac{1}{24} \sum \bar{b}_i c_i^4 - \frac{1}{120} \tag{1.93}$$

$$\tau'^{(5)}_2 = \frac{1}{4} \sum \bar{b}_i c_i a_{ij} c_j - \frac{1}{120} \tag{1.94}$$

$$\tau'^{(5)}_3 = \frac{1}{2}\sum \bar{b}_i a_{ij} c_j^2 - \frac{1}{120} \tag{1.95}$$

$$order\ 6: \quad \tau'^{(6)}_1 = \frac{1}{120}\sum \bar{b}_i c_i^5 - \frac{1}{720} \tag{1.96}$$

$$\tau'^{(6)}_2 = \frac{1}{20}\sum \bar{b}_i c_i^2 a_{ij} c_j - \frac{1}{720} \tag{1.97}$$

$$\tau'^{(6)}_3 = \frac{1}{10}\sum \bar{b}_i c_i a_{ij} c_j^2 - \frac{1}{720} \tag{1.98}$$

$$\tau'^{(6)}_4 = \frac{1}{6}\sum \bar{b}_i a_{ij} c_j^3 - \frac{1}{720} \tag{1.99}$$

$$\tau'^{(6)}_5 = \sum \bar{b}_i a_{ij} a_{jk} c_k - \frac{1}{720} \tag{1.100}$$

1.6 Objectives of the Thesis

In this study, we developed new and more efficient methods based on RK methods and RKN methods for solving oscillatory problems, first order fuzzy differential equations and second order PDE particularly the wave equation. The main objectives of the thesis are:

i. To derive phase-fitted and amplification-fitted explicit RK methods for the fourth order (classical) method and fifth order method of Zingg (1999) for linear ordinary differential equations (LODEs) for solving ordinary differential equations with oscillatory solutions.
ii. To discover the effect of the phase-fitted and amplification-fitted of explicit RK methods on the first order fuzzy differential equations (FDEs).
iii. To derive phase-fitted and amplification-fitted fourth order five-stage diagonally implicit RK method (DIRK4) proposed from Hairer et.al (2010) and phase-fitted and amplification-fitted fourth order four-stage diagonally implicit RK method proposed from Che Jawias et.al (2010) for ordinary differential equations with oscillatory solutions.
iv. To derive fifth order four-stage and sixth order five-stage RKN methods for LODEs for solving ordinary differential equations with oscillatory solutions.
v. To derive the phase-fitted and amplification-fitted RKN methods for LODEs for solving ordinary differential equations with oscillatory solutions.
vi. To discover the effect of RKN methods for LODEs on hyperbolic partial differential equations (PDEs).

1.7 Outline of Thesis

In Chapter 1, a brief introduction on ordinary differential equations, fuzzy differential equations and partial differential equations are given. The development of numerical methods, basic theory on algebraic order of RK method and RKN method and LTE for RK and RKN methods are also discussed in this chapter.

Chapter 2 focused on the literature reviews on explicit RK methods, DIRK methods, RKN methods, (FDEs) and hyperbolic partial differential equations.

In Chapter 3, we discuss the phase-fitted and amplification-fitted for explicit RK methods. We derive phase-fitted and amplification-fitted for fourth order classical RK method and phase-fitted and amplification-fitted for fifth order RK method for LODEs proposed from Zingg (1999). The stability region of the methods will be determined. The results of the new methods have been compared with the existing methods for solving LODEs.

In Chapter 4, we used the methods derived in Chapter 3 for solving first order fuzzy differential equations (FDEs). The numerical results for method with phase-fitted and amplification-fitted have been compared with the method without phase-fitted and amplification-fitted for solving FDEs, which clearly shown the advantage of the method with phase-fitted and amplification-fitted.

In Chapter 5, we derived the phase-fitted and amplification-fitted fourth order five-stage diagonally implicit RK method proposed by Hairer et.al (2010) and fourth order four-stage diagonally implicit RK method for LODEs proposed by Che Jawias et.al (2010). The results of the new methods have been compared with the original methods without phase-fitted and amplification-fitted.

In Chapter 6, we derived the order conditions of Runge-Kutta-Nyström method purposely for solving linear second order ordinary differential equations. Based on the order conditions we derived the new fifth order four-stage and sixth order five-stage explicit Runge-Kutta-Nyström methods for linear ordinary differential equations (LODEs). Then the methods are phase-fitted and amplification-fitted so that they will have zero-dispersion and zero-dissipation. The fifth order four-stage RKN method for LODEs has the property of First Same As Last (FSAL). Numerical results proved that the methods with phase-fitted and amplification-fitted are much more efficient than the existing methods with the same algebraic order.

Chapter 7, focused on the application of RKN method for solving hyperbolic partial differential equations (PDEs) that is the second order wave equations. The PDE is first converted to a system of second order linear ODEs using the methods of lines. The resulting second order linear ODEs are then solved using the RKN method derived in Chapter 6. Numerical results show that the RKN methods for LODEs are accurate and reliable for solving the second order wave equation.

Finally, we summarized and conclude our studies in the last chapter that is Chapter 8. Future studies are also suggested in this chapter.

1.8 Problem Statement

Initial value problem (IVPs) for second order ODEs where the first derivative does not appear explicitly often arise in many scientific areas of engineering and applied science such as celestial mechanics, molecular dynamics and quantum mechanics. The solution of these IVPs often exhibits a pronounced oscillatory character and it is well known that it is always difficult to get the accurate numerical results if the initial value problems are oscillatory in nature. To address the problem we will focus on developing RK type methods with reduced phase-lag and dissipation errors. Phase-lag or dispersion error is the angle between the true and the approximated solution and dissipation is the distance of the approximate solution from the standard cyclic solution. The aim of this research is to derive methods with phase-fitted and amplification-fitted so that the methods have zero phase-lag and zero dissipation; hence they are suitable for solving oscillatory problems.

1.9 Scope of Study

The main purpose of this research is to solve the second order ordinary differential equations (1.1) in which the solutions exhibit a pronounced oscillatory character. In this study we are focusing on solving problem (1.1) by using Runge-Kutta methods and Runge-Kutta-Nyström methods for oscillatory problems with phase-fitted and amplification-fitted techniques. The research also includes numerical solutions of first order fuzzy differential equations by using the phase-fitted and amplification fitted explicit RK methods and numerical solutions of hyperbolic partial differential equations by using RKN methods for LODEs.

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