UNIVERSITI PUTRA MALAYSIA

DOMINATING SETS AND DOMINATION POLYNOMIALS OF GRAPHS

SAEID ALIKHANI

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By

SAEID ALIKHANI

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia in Fulfilment of the Requirements for the Degree of Doctor of Philosophy

March 2009

DEDICATION

To

My wife and my son

Elahe and Amir Hossein

For their great patience

My Parents

For their encouragement

and

My Dear Teachers

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfillment of the requirement for the degree of Philosophy of Doctor

DOMINATING SETS AND DOMINATION POLYNOMIALS OF GRAPHS

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SAEID ALIKHANI

March 2009

Chair: Prof. Dr. Peng Yee Hock, PhD

Faculty: Institute for Mathematical Research

This thesis introduces domination polynomial of a graph. The domination polynomial of a graph G of order n is the polynomial $D(G,x) = \sum_{i=\gamma(G)}^{n} d(G,i)x^i$, where $d(G,i)$ is the number of dominating sets of G of size i, and $\gamma(G)$ is the domination number of G. We obtain some properties of this polynomial, and establish some relationships between the domination polynomial of a graph G and geometrical properties of G.

Since the problem of determining the dominating sets and the number of dominating sets of an arbitrary graph has been shown to be NP-complete, we study the domination polynomials of classes of graphs with specific construction. We introduce graphs with specific structure and study the construction of the family of all their dominating sets. As a main consequence, the relationship between the domination polynomials of graphs containing a simple path of length at least three, and the domination polynomial of related graphs obtained by replacing the path by a shorter path is, $D(G,x) =$

$x\Big[D(G*e_1,x)+D(G*e_1*e_2,x)+D(G*e_1*e_2*e_3,x)\Big]$, where $G*e$ is the graph obtained from G by contracting the edge e, and e_1, e_2 and e_3 are three edges of the path. As an example of graphs which contain no simple path of length at least three, we study the family of dominating sets and the domination polynomials of centipedes. We extend the result of the domination polynomial of centipedes to the graphs $G \circ K_1$, where $G \circ K_1$ is the corona of the graph G and the complete graph K_1.

As is the case with other graph polynomials, such as the chromatic polynomials and the independence polynomials, it is natural to investigate the roots of domination polynomial. In this thesis we study the roots of the domination polynomial of certain graphs and we characterize graphs with one, two and three distinct domination roots.

Two non-isomorphic graphs may have the same domination polynomial. We say that two graphs G and H are dominating equivalence (or simply $\mathcal{D}$-equivalence) if $D(G,x) = D(H,x)$. We study the $\mathcal{D}$-equivalence classes of some graphs. We end the thesis by proposing some conjectures and some questions related to this polynomial.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

SET DOMINASI DAN POLINOMIAL DOMINASI BAGI GRAF

Oleh

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Tesis ini perkenalkan polinomial dominasi bagi satu graf. Polinomial dominasi bagi suatu graf G berperingkat n ialah polynomial $D(G,x) = \sum_{i=\gamma(G)}^{n} d(G,i)x^i$, dimana $d(G,i)$ ialah bilangan set dominasi bagi G yang bersaiz i, dan $\gamma(G)$ ialah nombor dominasi bagi G. Kita perolehi beberapa sifat polinomial ini, dan beberapa hubungan di antara polinomial dominasi suatu graf G dan sifat geometri daripada G.

Oleh sebab masalah menentukan set dominasi dan bilangan set dominasi bagi sebarang graf adalah diketahui NP-lengkap, kita kaji polinomial dominasi bagi kelas-kelas graf dengan pembinaan tertentu. Kita perkenalkan graf dengan struktur tertentu dan mengkaji pembinaan famili semua set dominasinya. Sebagai akibat utama, hububgan antara polinomial dominasi graf yang mengandungi satu lintasan ringkas panjang sekurang-kurangnya tiga, dan polinomial dominasi graf berkaitan yang diperolehi dengan menggantikan lintasan itu dengan lintasan lebih pendek ialah $D(G,x) = x\Big[D(G*e_1,x) + D(G*e_1*e_2,x) + D(G*e_1*e_2*e_3,x)\Big]$, dimana $G*e$ ialah graf yang diperolehi daripada G dengan

mengecutkan garis e, dan e_1, e_2 dan e_3 adalah tiga garis dalam lintasan tersebut. Sebagai contoh graf yang tidak mengandungi lintasan ringkas panjang sekurang-kurangnya tiga pula, kita kaji famili set dominasi dan polinomial dominasi bagi "centipede". Kita perluaskan keputusan mengenai polinomial dominasi centipedes kepada graph $G \circ K_1$, dimana $G \circ K_1$ ialah corona graf G dan graf lengkap K_1.

Sebagaimana dengan polinomial graf lain, seperti polinomial kromatik dan polinomial takbersandar, mengkaji punca polinomial dominasi adalah lazim. Dalam tesis ini, kita kaji punca polinomial dominasi bagi graf tertentu dan kita cirikan graf-graf dengan satu, dua dan tiga punca dominasi berbeza.

Dua graf tak isomorfik mungkin menpunyai polinomial dominasi yang sama. Kita kata dua graf G dan H adalah setara dominasi (atau $\mathcal{D}$-setara) jika $D(G, x) = D(H, x)$. Kita mengkaji kelas $\mathcal{D}$-setara bagi beberapa graf. Kita juga mencadangkan konjektur dan menggariskan beberapa soalan yang berkaitan dengan polinomial ini.

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I certify that a Thesis Examination Committee has met on (31 March 2009) to conduct the final examination of (Saeid Alikhani) on his thesis entitled "**Dominating sets and Domination Polynomials of Graphs**" in accordance with the Universities and University Colleges Act 1971 and the Constitution of the Universiti Putra Malaysia [P.U.(A) 106] 15 March 1998. The Committee recommends that the student be awarded the (Doctor of Philosophy).

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This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of **Doctor of Philosophy**. The members of Supervisory Committee were as follows:

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Date: 14 May 2009

DECLARATION

I declare that the thesis is my original work except for quotations and citations which have been duly acknowledged. I also declare that it has not been previously, and is not concurrently, submitted for any other degree at University Putra Malaysia or at any other institution.

Saeid Alikhani
Date:

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CHAPTER 1
INTRODUCTION AND LITERATURE REVIEW

1.1 Basic definitions and knowledge

A graph $G = (V, E)$ is a finite nonempty set $V(G)$ of objects called vertices together with a (possibly empty) set $E(G)$ of unordered pairs of distinct vertices of G called edges. We will only consider simple graphs, those without multiple edges or loops. Two vertices u and v in G are *adjacent* if there exists an edge between them, that is, if $\{u, v\} \in E(G)$. We often write uv instead of $\{u, v\}$. The vertices u and v are the *ends* of $e = uv$, and e is *incident* with both u and v; both u and v are incident with e. The *degree* of a vertex $v \in V(G)$, written $deg(v)$, is the number of edges in G which are incident with v. The minimum and maximum degree of vertices in G is denoted by $\delta(G)$ and $\Delta(G)$, respectively. If every vertex in G has degree k, then G is *k-regular*. A graph G is said to be *isomorphism* with a graph H, if there is a bijection

$$\phi : V(G) \to V(H)$$

which preserve the adjacency and non-adjacency, that is $uv \in E(G)$ if and only if $\phi(u)\phi(v) \in E(H)$.

A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If H is a subgraph of G such that for all pair of vertices u and v in $V(H)$ it is true that $uv \in E(H)$ if and only if $uv \in E(G)$, then H is an *induced subgraph* of G.

If G and H are simple graphs such that $V(G) = V(H) = V$, and, for all distinct u and v in V, $uv \notin E(G)$ if and only if $uv \in E(H)$, then H (resp.,

G) is the *complement* of G (resp., H). We write $H = \overline{G}$ (or $G = \overline{H}$). The *union* $G = G_1 \cup G_2$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph with vertices $V(G) = V(G_1) \cup V(G_2)$ and edges $E(G) = E(G_1) \cup E(G_2)$. If, in addition, $V_1 \cap V_2 = \emptyset$, then G is the *disjoint union* of G_1 and G_2, written $G = G_1 \dot{\cup} G_2$.

The simple graph G on n vertices having n edge between every pair vertices is the *complete graph* K_n. Its complement is the *null graph* of order n, written $\overline{K}_n$.

For vertices $u, v \in V(G)$, a $u - v$ *path* is an alternating sequence of vertices and edges that begins with the vertex u and ends with the vertex v in which each edge of the sequence joins the vertex that precedes it in the sequence to the vertex that follows it in the sequence. Moreover, no vertex is repeated in this sequence. The number of edges in the sequence is considered the *length* of the path. A graph G is *connected* if for every pair of vertices in $V(G)$, there exists a path between them. A graph G is *disconnected* if it is not connected. A maximal connected subgraph of a graph G is called a *component* of G. A *cycle* on n vertices, denoted C_n, is a path which originates and concludes at the same vertex. The length of a cycle is the number of edges in the cycle. A *wheel* W_n is a graph with n vertices, obtained from a cycle C_{n-1} by adding a new vertex and edges joining it to all vertices of the cycle. An *end vertex* (or *pendant vertex*) is any vertex of degree 1 (that is, a vertex adjacent to exactly one other vertex). For a graph $G = (V, E)$, a subset U of V is *independent* if, for all u and v in U, edge uv is not in E. A *bipartite graph* G is a graph with independent sets V_1 and V_2 where V_1 and V_2 partition $V(G)$. A *complete bipartite* graph is a bipartite graph with partite (disjoint) sets V_1 and V_2 having the added property that every vertex of V_1 is adjacent to every vertex of V_2. Complete bipartite graphs are denoted $K_{m,n}$ where $|V_1| = m$ and $|V_2| = n$. The bipartite graph $K_{1,n}$ is called a *star graph*.

The *corona* of two graphs G_1 and G_2, as defined by Frucht and Harary in [23], is the graph $G = G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2, where the ith vertex of G_1 is adjacent to every vertex in the ith copy of G_2. The corona $G \circ K_1$, in particular, is the graph constructed from a copy of G, where for each vertex $v \in V(G)$, a new vertex v' and a pendant edge vv' are added.

The *join* $G = G_1 \vee G_2$ of two graphs G_1 and G_2 has $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \Big\{uv|u \in V(G_1) \text{ and } v \in V(G_2)\Big\}$.

The *open neighborhood* of a vertex u, denoted as $N(u)$, consists of all vertices in $V(G)$ which are adjacent to u. The *closed neighborhood* of a vertex v, is $N[v] = N(v) \cup \{v\}$. A set $S \subseteq V$ is a *dominating set* of G, if $N[S] = V$, or equivalently, every vertex in $V - S$ is adjacent to at least one vertex in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set in G. A dominating set with cardinality $\gamma(G)$ is called a γ-*set*, and the family of γ-sets is denoted by $\Gamma(G)$. For a detailed treatment of this parameter, the reader is referred to [22]. We denote the family of dominating sets of graph G with cardinality i by $\mathcal{D}(G, i)$.

A subset M of $E(G)$ is called a *matching* in G if its elements are not loops and no two of them are adjacent in G; the two ends of an edge in M are said to be *matched* under M. A matching M *saturates* a vertex v, and v is said to be M-*saturated* if some edges of M is incident with v; otherwise v is

M-unsaturated.

A *simple path* is a path where all its internal vertices have degree two. In Figure 1.1 we show a graph G which contains a simple path of length k with vertices labeled $1, 2, \ldots, k, k+1$.

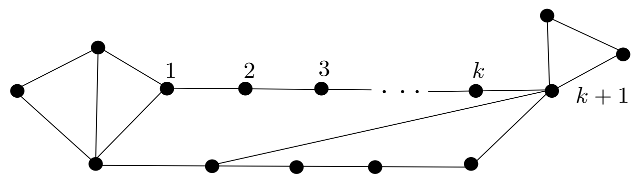

Figure 1.1. Graph G contains a simple path of length k.

A finite sequence of real numbers $(a_0, a_1, a_2, \ldots, a_n)$ is said to be *unimodal* if there is some $k \in \{1, \ldots, n-1\}$, called the *mode* of sequence, such that

$$a_0 \leq \ldots \leq a_{k-1} \leq a_k \geq a_{k+1} \geq \ldots \geq a_n;$$

the mode is *unique* if $a_{k-1} < a_k > a_{k+1}$. A polynomial is called *unimodal* if the sequence of its coefficients is unimodal.

As usual we use $\lceil x \rceil$, $\lfloor x \rfloor$ for the smallest integer greater than or equal to x and the largest integer less than or equal to x, respectively. In this thesis we denote the set $\{1, 2, \ldots, n\}$ simply by $[n]$.

1.2 Literature review

There are two parts in this dissertation: the dominating sets and the domination polynomials of graphs. First, in this section, we review the history of domination theory.

Domination is an area in graph theory with an extensive research activity. The dominating set problem asks to determine the domination number of a given graph. Formal study of the dominating set problem began in the 1960, the term itself first appearing in the 1967 book on graph theory [31] by Ore. However, the problem has historical roots in the dominating queens problem, which we extract from [22], written by well-known authorities: T.W. Haynes, S.T. Hedetniemi, and P.J. Slaster.

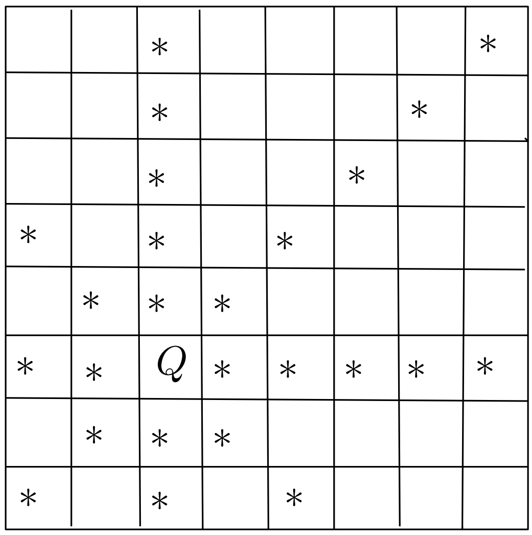

Figure 1.2. Squares attacked by a Queen.

The historical roots of domination is said to be the following chess problem. Consider an 8×8 chessboard on which a queen can move any number of squares, horizontally, vertically, or diagonally (assuming that no other chess piece lies in this way). Figure 1.2 shows the squares that a queen attack or dominate. One is interested to find the minimum number of queens needed on chessboard such that all squares are either occupied or can attack by a queen. In Figure 1.3, five queens are shown who dominate all the squares.

To model the queens problem on a graph, let G represent the chessboard such that every vertex corresponds to a square, and there is an edge connecting two vertices if and only if the corresponding squares are separated by any number of squares horizontally, vertically, or diagonally. Such a set of queens in fact represents a dominating set.

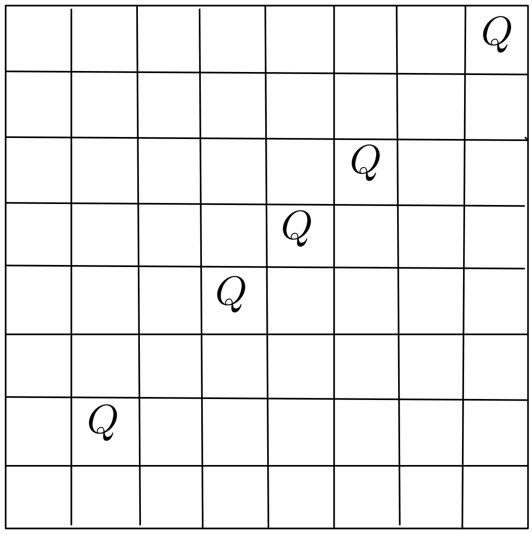

Figure 1.3. Five dominating queens.

For another motivation of this concept, consider a bipartite graph where one part represent people, the other part represents jobs, and the edges represent the skills of each person. One is interested to find the minimum number of people such that jobs are occupied. As shown in Figure 1.4, {Jane, John} form a minimum size dominating set.

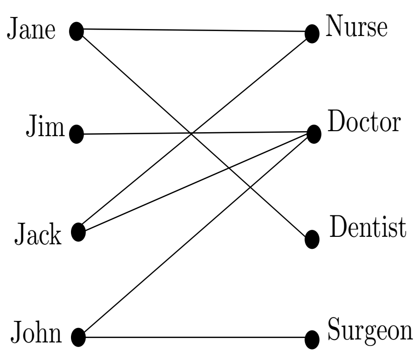

Figure 1.4. Dominated jobs.

The concept of dominating set occurs in a variety of problems. A number of these problems were motivated by communication network problems. The communication network includes a set of nodes, where one node can communicate with another if it is directly connected to that node. In order to send a message directly from a set of nodes to all others, one need to choose this set such that all other nodes are connected to at least one node in the set. Now, such a set is a dominating set in a graph which represent the network. Other applications of domination are, the facility location problem, land surveying and routings. ([22]).

We introduce some specific graphs and construct the family of all dominating sets of these graphs by recursive method.

There are people who feel that combinatorial and graph theory results should be given "purely combinatorial" proof, but I am not one of them. For me, the most interesting parts of graph theory and combinatorics have always been those overlapping with other areas of Mathematics. (See preface of [19]). Algebraic graph theory is one of this area which has been extended by many mathematicians. Let us review the history of graph polynomials:

The edge-difference polynomial. The historically first polynomial in graph theory was introduced by J.J. Sylvester [36] in 1878, and further studied by J. Petersen [33]. It is a multivariate polynomial depending on the ordering of the vertices of a graph.

The chromatic polynomial. Let $\chi(G, \lambda)$ denotes the number of proper vertex colourings of G with at most λ colours. G. Birkhoff [7], observed in 1912 that $\chi(G, \lambda)$ is, for a fixed graph G, a polynomial in λ, which is now called the chromatic polynomial of G. The chromatic polynomial is the oldest graph polynomial to appear in the literature. The book by F.M. Dong, K.M. Koh and K.L. Teo [15] gives an excellent and extensive survey of this polynomial.

The Tutte polynomial. An interesting generalization of the chromatic polynomial were introduced by H. Whitney in 1932 and W.T. Tutte in 1947. The most prominent among them is now called the Tutte polynomial $T(G, x, y)$ which is a two variables polynomial from which the chromatic polynomial can be obtained via a simple substitution and multiplication with a pre-factor. For a modern exposition, the reader is referred to [20], [39] or chapter X of [8].

The characteristic polynomial. This polynomial denoted by $P(G, \lambda)$, is the characteristic polynomial of the adjacency matrix M_G of the graph G, that is $P(G, \lambda) = det(\lambda \mathbf{1} - M_G)$ and it is completely determined by the eigenvalues of M_G, which are all real, as the matrix is symmetric. An excellent survey is [14].

The matching polynomials. The *acyclic polynomial* of graph G of order n is the polynomial $m(G, \lambda) = \sum_k (-1)^k m_k(G) \lambda^{n-2k}$, where the coefficients $m_k(G)$ count k-matchings. A close relative of this polynomial is the *matching generating polynomial* of a graph G defined as $g(G, \lambda) = \sum_k m_k(G) \lambda^k$. An excellent survey on these two matching polynomials may be found in [19, 29].

The independence polynomial. This polynomial denoted by $I(G, \lambda)$, is the polynomial $I(G, \lambda) = \sum_{i=0}^{\beta} i(G, k) \lambda^k$, where $i(G, k)$ is the number of independent sets of G with cardinality i and β is the independence number of G. An excellent survey of this polynomial are [24, 28].

The Interlace Polynomial. Two of the more interesting recent graph polynomials were introduced by R. Arratia, B. Bollobs and G. Sorkin in [3, 4]. They are called interlace polynomials and there is a univariate and a two-variable version. M. Aigner and H. van der Holst [1] studied these polynomials from a matrix point of view and derived various combinatorial interpretations.

The Cover Polynomial of Directed Graphs. An interesting recent graph polynomial on directed graphs is the cover polynomial introduced by F.R.K. Chung and R.L. Graham [13], and independently in the context of rook polynomials, by I. Gessel, [18]. In [13] it is presented as an attempt to create a Tutte-like polynomial for directed graphs, and is closely related to the chromatic polynomial.

The collection of graph polynomials looks like a zoo. There are prominent animals like the elephant, the giraffe, the gorilla, and there are exotic animals defying classification, like the lamprey (petromyzon marinus, not really a fish) or platypus (ornithorhynchus anatinus, not really a water bird, not really a mammal). Some animals look different, but are related, like the elephant and the rock hyrax (procavia capensis); some look alike, but are not related, like the hedgehog (erinaceus europus) and the echidna (tachyglossus aculeatus). ([30]). In this thesis we introduce a new animal in this zoo!

We introduce the domination polynomial of a graph. We translate the properties of graphs and results on the structure of the families of all dominating sets of graphs, into algebraic properties and then, using the results of algebra, to obtain results on graphs.

In Chapter 2, we introduce the domination polynomial of graphs and obtain some of its properties. Also, in this chapter, we investigate some relationships between domination polynomials and graphs.

In Chapters 3 and 4, we study the dominating sets and domination polynomial of two types of graphs which contain a simple path of length at least three (or simply call non-P_4-free), denoted by $G(m)$ and $G'(m)$. The main result of