UNIVERSITI PUTRA MALAYSIA

EXTENSION OF LAPLACE TRANSFORM TO MULTI-DIMENSIONAL FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS

WASAN AJEEL AHMOOD

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EXTENSION OF LAPLACE TRANSFORM TO MULTI-DIMENSIONAL FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS

By

WASAN AJEEL AHMOOD

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Doctor of Philosophy

May 2017

DEDICATIONS

To my father and Mother
To my husband and my son
For their great patience
My brothers, Ammar, Ali, Yasser
My sisters, Zamman, Amaal
thank you very much for every things

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Doctor of Philosophy

EXTENSION OF LAPLACE TRANSFORM TO MULTI-DIMENSIONAL FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS

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May 2017

Chair: Professor Adem Kılıçman, PhD
Faculty: Science

The main focus of this thesis is to extend the study of one-dimensional fractional to multi-dimensional fractional calculus and study of multi-dimensional Laplace transformation with their respective applications. This extension will be used to solve special types of multi-dimensional fractional calculus such as space-time partial fractional derivative. The multi-dimensional Laplace transforms method used to solve the multi-dimensional fractional calculus with constant and variable coefficients and the multi-dimensional modification of Hes variational iteration method to solve the multi-dimensional fractional integro-differential equations with non-local boundary conditions are developed. The study of multi-dimensional space-time fractional derivative with their applications and also, new fractional derivative and integral including Riemann-Liouville having a non-local and non-singular kernel are detailed. Finally, we obtained the exact solution of multi-dimensional fractional calculus, space-time partial fractional derivative and the system of matrix fractional differential equation in Riemann- Liouville sense of matrices but there are some problems that cannot be solved analytically, thus we solved them by multi-dimensional variational iteration method. This study shows that integral transform can be used to present new solutions to problems by certain applications for solving them.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

PERLUASAN TRANSFORMASI LAPLACE KEPADA PERSAMAAN KAMIRAN-PEMBEZAAN PECAHAN BERMULTI-DIMENSI

Oleh

WASAN AJEEL AHMOOD

Mei 2017

Pengerusi: Profesor Adem Kılıçman, PhD
Fakulti: Sains

Fokus utama tesis ini adalah untuk memperluas kajian terhadap kalkulus pecahan satu dimensi kepada multi-dimensi dan kajian terhadap transformasi Laplace bermulti-dimensi dengan aplikasi masing-masing. Perluasan ini akan digunakan untuk menyelesaikan jenis khas dari kalkulus pecahan bermulti-dimensi seperti terbitan pecahan separa ruang-masa. Kaedah multi-dimensi transformasi Laplace digunakan untuk menyelesaikan kalkulus pecahan bermulti-dimensi dengan pekali pemalar dan boleh berubah, dan kaedah lelaran berubah pengubahsuaian bermulti-dimensi He untuk menyelesaikan persamaan kamiran-terbitan pecahan bermulti-dimensi dengan syarat-syarat sempadan bukan setempat dikembangkan. Kajian tentang pecahan terbitan ruang-masa multi-dimensi dengan aplikasinya, dan juga terbitan dan kamiran pecahan baru termasuk Riemann-Liouville mempunyai inti bukan setempat dan tak-singular diperincikan. Akhirnya, kami memperolehi penyelesaian yang tepat dari kalkulus pecahan bermulti-dimensi, terbitan pecahan separa ruang-masa, dan sistem persamaan pembezaan pecahan matriks dalam erti-kata matriks Riemann-Liouville, tetapi terdapat beberapa masalah tidak dapat diselesaikan secara analisis, oleh itu, kita menyelesaikannya melalui kaedah lelaran berubah bermulti-dimensi. Kajian ini menunjukkan bahawa jelmaan kamiran dapat digunakan untuk mengemukakan penyelesaian baru terhadap beberapa masalah dengan aplikasi-aplikasi tertentu.

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I certify that a Thesis Examination Committee has met on 12 May 2017 to conduct the final examination of Wasan Ajeel Ahmood on her thesis entitled "Extension of Laplace Transform to Multi-Dimensional Fractional Integro-Differential Equations" in accordance with the Universities and University Colleges Act 1971 and the Constitution of the Universiti Putra Malaysia [P.U.(A) 106] 15 March 1998. The Committee recommends that the student be awarded the Doctor of Philosophy.

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LIST OF ABBREVIATIONS

Γ	Gamma Function
β	Beta Function
Erf	Error Function
Erfc	Complementary Error Function
E_t	Mellin-Ross Function
$E_{\alpha,\beta}$	Mittag-Leffler Function
FC	Fractional Calculus
ODEs	Ordinary Differential Equations
PDEs	Partial Differential Equations
FPDEs	Fractional Partial Differential Equations
MDIEs	Multi-Dimensional Integral Equations
FMDIEs	Fractional Multi-Dimensional Integral Equations
PIDEs	Partial Integro-Differential Equations
FPIDEs	Fractional Partial Integro-Differential Equations
MDFC	Multi-Dimensional Fractional Calculus
λ	Scalar Parameter
LTM	Laplace Transform Method
MDLTM	Multi-Dimensional Laplace Transform Method
FMDLTM	Fractional Multi-Dimensional Laplace Transform Method
VIM	Variational Iteration Method
FVIM	Fractional Variational Iteration Method
MVIM	Multi-Dimensional Variational Iteration Method
ADM	Adomian Decomposition Method
C	Space
X	Banach Space
H	Hilbert Space
HAM	Homotopy Analysis Method
HPM	Homotopy Perturbation Method
B(u,v)	Boussinesq-like
FW(h)	Forward operator
$H_i(t)$	Holomorphic functions
R_+	Set of all Positive Real Numbers
$\|f\|$	Norm of a function f
B $\|f\|$	Set of all Borel Linear operators on a Function f
B	Closure of a set B
D^α	Differential operator of Fractional order α
I^α	Integral operator of Fractional order α

CHAPTER 1

INTRODUCTION

1.1 Brief History of Fractional Calculus

Fractional calculus is a field in mathematical analysis which deals with the integrals and derivatives of random order with their applications. It is considered an old topic, starting from some conjectures of Leibnitz (1695, 1697) and Euler (1730), industrialized and developed up to nowadays.

According to Hall (1980) the beginning of fractional calculus back to Leibnitz and Newton in 17th century. Pertz and Gerhardt (1849) states that L'Hopital and Leibnitz had discussed the one-half order derivatives. Pertz and Gerhardt (1850) Leibnitz and Johann Bernoulli had discussed correspondence where they discussed the meaning to the derivatives of non-integer (arbitrary order) by a letter written to Leibnitz in 1695 and Johann reiterates the problem of fractional derivatives. In fact, the fractional calculus is nearly 300 years old started by a letter to Leibnitz, Bernoulli put him a question about the meaning of a non-integer derivative order.

In the middle of last century, many mathematicians have provided important contributions such as, Laplace in 1812 defined a fractional derivative in terms of an integer and Lacroix (1819) introduced the first defined called fractional derivative. Lacroix in the same year became the first mathematician to publish a paper that mentioned a fractional derivative. Also, the derivatives of non-integer order was discussed by Fourier (1822). In 1823 Abel had idea to determine the shape of a friction-less wire within a vertical plan by using the tautochrone problem. Thus, Abel problem was the first one to lead to the study of a linear Volterra integral equations of the first kind that arose in his study, when generalizing the tautochrone problem he derived the fractional equation.

The first important study of fractional calculus was introduced by Riemann–Liouville in 1832 based on Abel's idea where Abel's solution involved the consideration of Liouville type study of fractional calculus, one can use the infinite series and a definite integral related to Euler's Gamma integral

$$\int_0^\infty u^{a-1}e^{-tu}du = t^{-a}\int_0^\infty x^{a-1}e^{-x}dx = \frac{\Gamma(a)}{t^a}.$$

Then

$$t^{-a} = \frac{1}{\Gamma(a)}x^{a-1}e^{-x}dx.$$

In 1847 Riemann–Liouville used a Taylor series generalization to develop an alternative theory of fractional operators and in 1892, Riemann Liouville type theory of fractional integration and derived a generalization of Taylor series

$$D^{-\alpha}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\xi)^{\alpha-1} f(\xi)d\xi + \Psi(x).$$

There are many mathematicians including, Grunwald (1867)–Grunwald(1872), Letnikov (1868)–Letnikov(1872), Heaviside(1892–1912), Weyl(1917), Erdelyi(1939–1965). The end of eighteen century and in nineteen and twentieth centuries the theory of fractional calculus was developed as a purely theoretical field. Al-Bassam, Davis, Littlewood, Riesz, Samko, Sneddon, Weyl and Zygmund in the period 1900-1970 had published work seemed on the topic of the fractional calculus.

The fractional derivatives see an excellent implementation for the description and explanation of memory, Ford and Simpson (2001) and hereditary properties of various materials and processes. There are also some researches which used the fractional calculus in statistics, Phillips and some application fractional calculus such as electrochemistry of corrosion by Bagley. This is considered as the main advantage of fractional derivatives in evaluation with classical integer-order models. Also, fractional calculus was found many applications in the other fields of science and engineering including control theory, (Petras et al. (2002); Hwang and Cheng (2006)), fractals theory (Chen (2004); Bultheel and Martinez-Sulbaran (2007)), fluid flow Podlubny (1999), diffusion (Yuste et al. (2004); Shen and Liu (2005)), electromagnetic theory (Rizvi (1999)), probability (Li et al. (2003)), electrical networks, viscoelasticity, biology and chemistry.

Integration of to an arbitrary order which is the past of fractional calculus has also long history, see Oldham and Spanier (1974); Samko et al. (1987); Turski et al. (2004). The concept of non-integer order of integration can be traced back to the genesis of differential calculus itself. The philosopher and creator of modern calculus, the Newtons rival Leibnitz made some remarks on the meaning and possibility of fractional derivative of order in the late of 17-th century. However, a rigorous investigation was first carried out by Liouville in a serious of papers from 1832-1837, where he defined the first outcast of an operator of fractional integration.

The fractional Calculus traditional definitions of calculus integral and derivative operators in much the same way fractional exponents see (Oldham and Spanier (1978), Miller and Ross (1993), Rahimy (2010)).

Today the fractional differential equations are used in many branches of sciences, mathematics, physics, chemistry and engineering. During the past three decades, the fractional calculus has gained importance in different fields of science and engineering due to its applicability. Thus, the one-dimensional fractional Laplace transform is defined for functions of only one independent variable. The fractional Laplace transform is a special case of linear canonical transform, when it was first introduced in 1970 and found to be useful in many applications. In 2009 Jumarie G. defined the one-dimensional fractional Laplace transform for functions of one independent variable with inversion and studied some properties that were concerned with this definition. Recently some new integral transforms were introduced and applied to solve some ordinary differential equations, integral equation and integro-differential equations as well as partial differential equations, multi-dimensional integral and integro-differential equations.

The discrete and distribute methods are two main categories for vibration in dynamical systems. In the first systems depended on time, where as the variables in distributed systems depend on time and space such as in beams, plates etc, the ordinary differential equations describe the motion equations of discrete systems, while the partial differential equations describe the motion of equations in the distributed systems, see (Meirovitch (1980)). In the literature, there are many studies and works that have been made in the area of vibration problems and several techniques such as; finite element, finite difference method, perturbation techniques, series solutions(DTM), etc. have been used to handle the related problems. The variational iteration method(VIM) was first proposed by (He (1997))–(He et al. (2014)) and after that the method has being applied to study many nonlinear partial differential equations, autonomous and singular ordinary differential equations such as solitary wave solutions, rational solutions, compacton solutions and other types of solutions, for example, some could be found in Abdou and Soliman (2005).

Thus the fractional calculus described and were used to model physical processes in many areas by constructing the fractional differential equations. In the past, the investigation of travelling-wave solutions for non-linear equations has played an important role in the study of non-linear physical phenomena, see Caputo (1967); Kilbas et al. (2006); Kilicman and Al Zhour (2007); Magin and Ovadia (2008). Further, fractional differential equations were also used to model various important physical phenomena in porous media, colloid materials, disordered, random, amorphous, geology, finance, medicine etc., see Atanackovic and Pilipovic (2011), Mainardi and Pagnini (2001). In Blumen et al. (1989); Chaves (1998); Metzler and Klafter (2000); Meerschaert and Scheffler (2001) many new mathematical models were developed for anomalous diffusion to derive limiting distribution of a specified stochastic problems and the models were developed successfully by employing the fractional derivatives in the diffusion equation. Also, many physical situation are observed by anomalous diffusion.

Recently, (Klafter et al. (1987); Meerschaert et al. (2002)) used fractional differential equations to govern the limiting particle distribution for models. (Metzler and Klafter (2000); Meerschaert and Scheffler (2001)) used semigroups of operators as technical tools and (Arendt et al. (2001), Hwang and Cheng (2006)) used the theory of operator stable probability distributions. (Meerschaert et al. (2002)) investigated with scaling and similarity properties to find fundamental solutions for Cauchy problems of space-time fractional diffusion equation and starting from composite Fourier-Laplace representation. (Mainardi (1996)) made a tutorial like survey for linear of fractional differential equations (Rieamann-Liouville sense and Caputo sense) by realization of processing in basic theory of relaxation processes.

Further, Mainardi (1996) also used Laplace transform to consider the time fractional diffusion-wave equation and obtained the fundamental solutions. In Mainardi et al. (2001) were used the Fourier, Laplace integral transforms and Mittag-Leffler functions to discuss the fundamental solution of the fractional space-time diffusion equation when the fundamental solution can only be expressed as a convolution form of the Green function and the initial value problem then the solution were computed with some amount of difficulty.

Then after, many authors and researchers studied the fractional calculus and derived the exact solution by many methods, some of these method are Laplace transform method (Kreysig (1983); Farjo (2007); Kimeu (2009); Wazwaz (2011); Jiwen (2012)), fractional Laplace transform, Jumarie (2009). Several methods including the Laplace transform are discussed in introducing the Riemann–Liouville fractional integral. homotopy perturbation method, Adomian decomposition numerical method by Momani and Noor (2006); Odibat and Momani (2008); Hesameddini and Fotros (2012), variational iteration method (Tatari and Dehghan (2007); Maha and Fadhel (2009); Nawaz (2011); Irandoust-Pakchin and Abdi-Mazraeh (2013)), differential transform method (Mirzaee (2011); Alquran (2012)), (Momani and Noor (2006); Imran and Mohyud-Din (2013)) solved the fractional partial differential equations by applying Adomian's decomposition method coupled with Laplace transform and many other methods.

1.2 Basic Concepts

In this section, we will give some well known definitions and concepts in fractional calculus that are used in this thesis, we will discuss some useful mathematical different of classical definitions and properties that are inherently tied to fractional calculus and will commonly be encountered. These include:

Definition 1.1 : *(Grunwald-Letnikov Fractional Derivative)*
Let f be a function of t by using the Cauchy formula is defined:

$$ {}_aD_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau, $$

where $f(t)$ has $m+1$ continuous derivatives in the closed interval $[a,t]$, then we get the fractional integral of order α:

$$
\begin{aligned}
{}_aD_t^{-\alpha} f(t) &= \sum_{k=0}^{m} \frac{f^{(k)}(a)(t-a)^{\alpha+k}}{\Gamma(\alpha+k+1)} \\
&\quad + \frac{1}{\Gamma(\alpha+m+1)} \int_a^t (t-\tau)^{\alpha-m} f^{(m+1)}(\tau) d\tau, \;\; m < \alpha < m+1 \quad (1.1)
\end{aligned}
$$

By replacing each α by $-\alpha$ of the eq. (1.1), can get:

$$
{}_aD_t^{\alpha} f(t) = \sum_{k=0}^{m} \frac{f^{(k)}(a)(t-a)^{-\alpha+k}}{\Gamma(-\alpha+k+1)} + \frac{1}{\Gamma(-\alpha+m+1)} \int_a^t (t-\tau)^{-\alpha-m} f^{(m+1)}(\tau) d\tau,
$$

this is the fractional derivative of the Grunwald-Letnikov sense.

Example 1.1 : Let the fractional derivative of the power function:
$f(t) = (t-a)^v, \;\; v > -1$, where v is a real number.

Solution: From the Cauchy formula and replacing α by $-\alpha$:

$$
{}_aD_t^{\alpha}(t-a)^v = \frac{1}{\Gamma(-\alpha)} \int_a^t (t-\tau)^{-\alpha-1} (\tau-a)^v d\tau.
$$

Letting $\tau = a + \xi(1-a)$ and by the definition of the beta function, we get:

$$
\begin{aligned}
{}_aD_t^{\alpha}(t-a)^v &= \frac{1}{\Gamma(-\alpha)} (t-a)^{v-\alpha} \int_0^1 \xi^v (t-\xi)^{-\alpha-1} d\xi \\
&= \frac{1}{\Gamma(-\alpha)} \beta(-\alpha, v+1)(t-a)^{v-\alpha} \\
&= \frac{\Gamma(v+1)}{\Gamma(v-\alpha+1)} (t-a)^{v-\alpha}, \;\; (\alpha < 0, v > 0).
\end{aligned}
$$

Some properties of Grunwald-Letnikov Fractional Derivative:

(i) If $p < 0$ and q is any real number, then:

$$
{}_aD_t^q({}_aD_t^p f(t)) = {}_aD_t^{p+q} f(t), \;\; (m < p < m+1, \;\; n < q < n+1).
$$

(ii) If $p > 0$ and q is any real number when $f^{(k)}(a) = 0, \;\; (k = 0, 1, ..., m-1)$, then:

$$
{}_aD_t^q({}_aD_t^p f(t)) = {}_aD_t^{p+q} f(t).
$$

The next definition is the Riemann-Liouville fractional derivative, see Ziada and El-Sayed (2010).

Definition 1.2 : *(Riemann–Liouville Fractional Derivative)*
Consider the most widely known definition of the fractional derivative is:

$$ {}_aD_t^{\alpha} f(t) = \left(\frac{d}{dt}\right)^{m+1} \int_a^t (t-\tau)^{m-\alpha} f(\tau)d\tau, \ \ (m \le \alpha < m+1). $$

It can be also written as:

$$ {}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(k-\alpha)} \frac{d^k}{dt^k} \int_a^t (t-\tau)^{k-\alpha-1} f(\tau)d\tau, \ \ (k-1 \le \alpha < k). $$

Some properties of Riemann-Liouville fractional derivative:

(i) If p, q are two positive real number and $t > a$, then:

$$ {}_aD_t^{p}({}_aD_t^{-q} f(t)) = {}_aD_t^{p-q} f(t). $$

(ii) If $0 \le k-1 \le q < k$, then:

$$ {}_aD_t^{-p}({}_aD_t^{q} f(t)) = {}_aD_t^{q-p} f(t) - \sum_{j=1}^{k} [{}_aD_t^{q-j} f(t)]_{t=a} \frac{(t-a)^{p-j}}{\Gamma(1+p-j)}. $$

(iii) If f is a continuous for $t \ge a$, then one can obtain,

$$ {}_aD_t^{-p}({}_aD_t^{-q} f(t)) = {}_aD_t^{-p-q} f(t). $$

The next definition is the Caputo's fractional derivative.

Definition 1.3 : *(The Caputo's Fractional Derivative)*
Let f be a function of t, the Caputo's fractional derivative is defined by:

$$ {}_a^cD_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha-n)} \int_a^t \frac{f^{(n)}\tau d\tau}{(t-\tau)^{\alpha+1-n}}, \ \ (n-1 < \alpha < n). $$

Some properties of the Caputo's fractional derivative:

(i) If $\mu, \alpha \ge 0$ and $t > 0$, then:

$$ {}_a^cD_t^{-\mu}({}_a^cD_t^{\alpha} f(t)) = {}_a^cD_t^{\alpha-\mu} f(t) - \sum_{k=0}^{l-1} f^{(k)}(0^+) \frac{t^{k+\mu-\alpha}}{\Gamma(\mu-\alpha+k+1)}, $$

$$ 0 < \alpha < \mu, \ \ m-1 < \mu < m, \ \ l-1 < \alpha \le l < m-1, \ \ (m,l) \in N. $$

(ii) by taking $\alpha = \mu$ for the above equation, we get:

$$ {}_a^cD_t^{-\alpha}({}_a^cD_t^{\alpha} f(t)) \quad = \quad f(t) - \sum_{k=0}^{m-1} f^{(k)}(0^+)\frac{t^k}{k!}, \ \ (m-1 < \alpha \leq m). $$

The next part is an important application to the above two definitions criterion, here we state these to show the difference between them.

The comparison between Caputo's sense and Riemann-Lioville sense fractional derivative: (Li et al. (2011))

(i) In the Caputo sense, the derivative acts first on the function after we evaluate the integral and in the Riemann-Liouville sense, the derivatives acts on the integral i.e., we first evaluate the integral and after we calculate the derivative. The derivative in the Caputo sense is more restrictive than the Riemann-Liouville one.

We also note that, both derivatives are defined by means of the Riemann-Liouvile fractional integral. The importance of this derivative is that, the derivative in the Caputo sense can be used, for example, in the case of a fractional differential equation with initial conditions which have a well known interpretation, as in the calculus of integer order.

Example 1.2 : Let $f \in AC^n[a,b]$, where $-\infty < a < b < \infty, \ \ \alpha \in C$ for $Re(\alpha) \geq 0$ and $n = [Re(\alpha)] + 1$.

Then if left fractional derivatives, ${}_cD^{\alpha}_{a+}c$, and the right, ${}_cD^{\alpha}_{b-}c$, denotes the the Caputo sense respectively, they can be defined in terms of the f on integral operator by using Riemann-Liouville as:

$$ ({}_cD^{\alpha}_{a+}f)(x) = I^{n-\alpha}_{a+}f^{(n)})(x) $$

and

$$ ({}_cD^{\alpha}_{b-}f)(x) = (-1)^n I^{n-\alpha}_{b-}f^{(n)})(x). $$

In particular if $\alpha = 0$ we have ${}_cD^0_{a+} =_c D^{0a}_{b-} = I$.
More general, if $\alpha = n \in N^*$, then we have

$$ ({}_cD^{n}_{a+}f)(x) = f^{(n)}(x) $$

and

$$ ({}_cD^{n}_{b-}f)(x) = (-1)^n f^{(n)}(x). $$

Example 1.3 : Let $\alpha \in C$, with $Re(\alpha) \geq 0$ and $n = [Re(\alpha) + 1]$, where $[\mu]$ denotes the integer part of μ, the fractional derivatives in the Riemann-Liouville sense, on the left and on the right, are defined by:

$$(D^{\alpha}_{a+}f)(x) := \frac{d^n}{dx^n}(I^{n-\alpha}_{a^+}f(x))$$

and

$$(D^{\alpha}_{b-}f)(x) := (-1)^n \frac{d^n}{dx^n}(I^{n-\alpha}_{b^-}f(x)).$$

respectively.

If $\alpha = n \in N^*$, then we have

$$(D^{n}_{a+}f)(x) = f^{(n)}(x)$$

and

$$(D^{n}_{b-}f)(x) = (-1)^n f^{(n)}(x).$$

(ii) The Riemann-Liouville and Caputo fractional derivatives from the result of the example that was obtained by Caputo operator with the result of example that can also be obtained by Riemann-Liouville integral operator on the function f, we see in particular that the two operators have different kernels and different domains.

Example 1.4 : Let $\alpha > 0,\ \ m = \lceil \alpha \rceil$ and $f(x) = (x-a)^{\gamma}$ for some $\gamma \geq 0$. Then

$$\begin{aligned} {}^{c}D^{\alpha}_{a}f(x) &= 0,\ \ \gamma \in 0,1,2,...,m-1 \\ &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)}(x-a)^{\gamma-\alpha},\ \ \gamma \in N\ \ and\ \ \gamma > m\ \ or\ \ \gamma \notin N\ \ and\ \ \gamma > m-1. \end{aligned}$$

Example 1.5 : Let $f(x) = (x-a)^{\gamma}$ for some $\gamma \geq -1$ and $\alpha > 0$ Then

$$J^{\alpha}_{a}f(x) = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}(x-a)^{\alpha+\gamma}.$$

(iii) We noted that the Caputo derivative is left inverse of Riemann-Liouville integral, but it is not the right inverse of Riemann-Liouville integral.

Theorem 1.1 : *If f is continuous and* $\alpha \geq 0$, *then* ${}^{c}D^{\alpha}_{a}J^{\alpha}_{a}f(x) = f(x)$.

Proof:
Suppose that $g = J^{\alpha}_{a}f$, we have $D^k g(a) = 0$ for $k = 0,1,...,m-1$, and thus from ${}^{R}D^{\alpha}_{a}f(x) = {}^{c}D^{\alpha}_{a}f(x)$, we have:

$${}^{c}D^{\alpha}_{a}J^{\alpha}_{a}f = {}^{c}D^{\alpha}_{a} = {}^{R}D^{\alpha}_{a} = {}^{R}D^{\alpha}_{a}J^{\alpha}_{a}f = f.$$

Theorem 1.2 : *Assume that* $\alpha \geq 0,\ \ m = \lceil \alpha \rceil$, *and* $f \in A^n[a,b]$. *Then*

$$J_a^{\alpha c} D_a^{\alpha} f(x) = f(x) - \sum_{k=0}^{m-1} \frac{D^k f(a)}{k!}(x-a)^k.$$

(iv) The derivative of a constant by using Caputo's definitions is equal to zero, whereas the Riemann-Liouville derivative of a constant is not equal to zero. That is

$$ {}_0D_t^{\alpha} c = \frac{ct^{-\alpha}}{\Gamma(1-\alpha)},$$

where c is a constant.

The next definition is known as the Sequential fractional derivative.

Definition 1.4 : *(The Sequential Fractional Derivative)*
Let f be a function of t, n-th order differentiation is simply a series of first-order differentiations and replacing each first-order derivative by fractional derivatives of orders. Then:

$$\begin{aligned} D^{\alpha} f(t) &= D^{\alpha_1} D^{\alpha_2} \ldots D^{\alpha_n} f(t), \\ \alpha &= \alpha_1 + \alpha_2 + \ldots + \alpha_n. \end{aligned}$$

Some Properties of the Fractional Derivatives, see Ziada and El-Sayed (2010):

(i) Linearity: The fractional differential operation is linear

$$D^p(C_1 f_1(t) + C_2 f_2(t) + \ldots + C_n f_n(t)) = C_1 D^p f_1(t) + C_2 D^p f_2(t) + \ldots + C_n D^p f_n(t),$$

where D^p is any mutation for the above equation.

(ii) The Leibnitz rule for fractional derivatives:
Let f be a continuous function of τ in the interval $[a,t]$ and $\varphi(t)$ has $n+1$ continuous derivatives in this interval. Then

$$ {}_aD_t^p(\varphi(t) f(t)) = \sum_{k=0}^{\infty} \binom{p}{k} \varphi^k(t) {}_aD_t^{p-k} f(t).$$

(iii) Fractional derivative of a composite function:
Let an analytic composite function $\varphi(t) = F(h(t))$, and by using the Leibnitz rule:

$$\begin{aligned} {}_aD_t^p F(h(t)) &= \frac{(t-a)^{-p}}{\Gamma(1-p)} \varphi(t) \\ &\quad + \sum_{k=0}^{\infty} \binom{p}{k} \frac{k!(t-a)^{k-p}}{\Gamma(k-p+1)} \sum_{m=1}^{k} F^{(m)}(h(t)) \sum \prod_{r=1}^{k} \frac{1}{a_r!} \left(\frac{h^{(r)}(t)}{r!}^{a_r} \right), \end{aligned}$$

where the sum extends over all combinations of non-negative integer values of $a_1, a_2, ..., a_k$ such that,

$$\sum_{r=1}^{k} r a_r = k, \ \ \sum_{r=1}^{k} a_r = m.$$

There exist different definitions of the fractional integral some of them as follows:

Definition 1.5 : *(The Riemann–Liouville Fractional Integral) (Miller and Ross (1993)) Let v be a real non-negative number. Let f be piecewise continuous on $J' = (0,\infty)$ and integrable on any finite sub-interval of $J = [0,\infty]$. Then*

$${}_cD_x^{-v} = \frac{1}{\Gamma(v)} \int_c^x (x-t)^{v-1} f(t)dt, \ \ v > 0,$$

where f is a known function of t.

Example 1.6 : Lets evaluate $D^{-v}x^{\mu}$, where $Re(v) > 0, \ \mu > -1$.

Solution: By definition of the Riemann–Liouville fractional integral:

$$\begin{aligned}
D^{-v}x^{\mu} &= \frac{1}{\Gamma(v)} \int_0^x (x-t)^{v-1} t^{\mu} dt \\
&= \frac{1}{\Gamma(v)} \int_0^x (1-\frac{t}{x})^{v-1} x^{v-1} t^{\mu} dt \\
&= \frac{1}{\Gamma(v)} \int_0^1 (1-u)^{v-1} x^{v-1} (xu)^{\mu} x du \\
&= \frac{1}{\Gamma(v)} x^{\mu+v} \int_0^1 u^{\mu} (1-u)^{v-1} du \\
&= \frac{1}{\Gamma(v)} x^{\mu+v} B(\mu+1, v) \\
&= \frac{\Gamma(\mu)+1}{\Gamma(\mu+v+1)} x^{\mu+v}.
\end{aligned}$$

In the above example, we have established that

$$D^{-v}x^{\mu} = \frac{\Gamma(\mu)+1}{\Gamma(\mu+v+1)} x^{\mu+v}, \ \ v > 0, \ \ \mu > -1, \ \ x > 0.$$

It is also known as the Power Rule.

Definition 1.6 : *(Weyl Fractional Integral) (Weyl (1917)) Let $f \in L^p(\Re/2\pi)Z, \ \ 1 \le p < \infty$ be periodic with period 2π and such that its integral* *over a period vanishes. The Weyl fractional integral of order* α *is defined as:*

$$\left(I_{\pm}^{\alpha} f\right)(x) = (\Psi_{2\pi}^{\alpha})(x)\frac{1}{2\pi}\int_{0}^{2\pi}(x-y)f(y)dy,$$

where

$$\Psi_{2\pi}^{\alpha} = \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{e^{ikx}}{(\pm ik)^{\alpha}} \;\; for \;\; 0 < \alpha < 1.$$

Definition 1.7 : *(Riesz Fractional Integral) (Riesz (1949))*
Let $f \in L_{1oc}^{1}(\Re)$ *be locally integrable. The Riesz fractional integral or Riesz potential of order* $\alpha > 0$ *is defined as the linear combination:*

$$(I^{\alpha} f)(x) = \frac{(I_{+}^{\alpha} f)(x) - (I_{-}^{\alpha} f)(x)}{2cos(\alpha\pi/2)} \frac{1}{2\Gamma(\alpha)cos(\alpha\pi/2)} \int_{-\infty}^{\infty} \frac{f(y)}{\left|x-y^{1-\alpha}\right|}dy$$

of right-and left-sided Weyl fractional integrals.

In the next we recall some definitions and properties of the fractional calculus theory that we use in this thesis:

Definition 1.8 : *A real function* $f(x), \;\; x > 0$*, is said to be in the space* $C_{\mu}, \;\; m \in R$*, if there exist a real number* $P > \mu$ *such that* $f(x) = x^{p} f_{1}(x)$*, where* $f_{1}(x) \in C[0,1)$*. Clearly* $C_{\mu} \subset C_{\beta}$ *if* $\beta \leq \mu$*.*

Definition 1.9 : *A real function* $f(x), \;\; x > 0$*, is said to be in the space* $C_{\mu}^{m}, \;\; m \in N \cup 0$*, if* $f^{m} \in C_{\mu}$*.*

Definition 1.10 : *The left sided Riemann-Liouville fractional integral operator of order* $\alpha \geq 0$ *of a function* $f \in C_{\mu}, \;\; \mu \geq -1$ *is defined as Gorenflo and Mainardi (1997).*

$$\begin{aligned} J^{\alpha} f(x) &= \frac{1}{\Gamma(\alpha)} \int_{0}^{x} \frac{f(t)}{(x-t)^{1-\alpha}}dt, \;\; \alpha > 0, \;\; x > 0, \\ J^{0} f(x) &= f(x). \end{aligned}$$

Definition 1.11 : *(The Gamma Function)*
The most important function is Gamma function. The Gamma function has definition with some unique properties, by using its recursion relations, obtain formulas:

$$\Gamma(x) = \int_{0}^{\infty} e^{-t} t^{x-1} dt, \;\; x \in R^{+}.$$

Some properties of the Gamma function:

(1) The Gamma function is:

$$\begin{aligned}
\Gamma(x+1) &= x\Gamma(x), x \in R^+ \\
\Gamma(x+1) &= \int_0^\infty e^{-t}t^x dt \\
&= \lim_{z\to\infty, y\to 0^+} \int_y^z e^{-t}t^x dt \\
&= \lim_{z\to\infty, y\to 0^+} \left(-e^t t^x|_y^z + x\int_y^z e^{-t}t^{x-1}dt \right) \\
&= x\int_0^\infty e^{-t}t^{x-1}dt = x\Gamma(x).
\end{aligned}$$

(2) $\Gamma(x) = (x-1)!,\ \ x \in N$, from the above equation in part (1), we get: $\Gamma(x) = (x-1)\Gamma(x-1) = (x-1)(x-2)\Gamma(x-2) = (x-1)(x-2)(x-3)...1 = (x-1)!$.

(3) Gamma function is never zero.

(4) The formula $\Gamma(ax+b)\ \sqrt{2\pi}e^{-ax}(ax)^{ax+b-\frac{1}{2}}$ is known as Asymptotic formula. By the definition of Gamma function and by letting $t = y^2 \Rightarrow dt = 2ydy$, we get

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-y^2}dy.$$

If we replace x by y in the above equation, then:

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-x^2}dx.$$

Now, by multiplying together two above equations to get:

$$\Gamma\left(\frac{1}{2}\right)^2 = 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dxdy,$$

this is the double integral over the first quadrant.
Now, by polar coordinates we have:

$$\Gamma\left(\frac{1}{2}\right)^2 = 4\int_0^{\frac{\pi}{2}}\int_0^\infty e^{-r^2}rdrd\theta = \pi.$$

Thus, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

The following definition is the Beta function.

Definition 1.12 : *(The Beta Function)*
The Beta function is defined by a definite integral, its definition is given by:

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt.$$

The Beta function can also be defined in terms of the Gamma function

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \;\; x,y \in R^+.$$

Some properties of the Beta function:

(1) The solution of the Beta function by a definite integral is:

$$\begin{aligned} \beta(x,y) &= \int_0^1 t^{x-1}(1-t)^{y-1}dt \\ &= \int_0^\infty {\frac{t^{x-1}}{1+t}}^{x+y} dt \\ &= 2\int_0^{\frac{\pi}{2}} (\sin(t))^{2x-1}(\cos(t))^{2y-1}dt. \end{aligned}$$

(2) The solution of the Beta integral is $\beta(x+1,y+1)$.

(3) The beta function is symmetric that is $\beta(x,y) = \beta(y,x)$.

(4) $\beta(x,y) = \beta(x+1,y) + \beta(x,y+1)$.

(5) $\beta(x,y+1) = \frac{x}{y}\beta(x+1,y) = \frac{x}{x+y}\beta(x,y)$.

The following definition is the Error function.

Definition 1.13 : *(The Error Function)*
The definition of the Error function is given by:

$$Erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt, \;\; x \in R.$$

The complementary Error function (Erfc) is a closely related function that can be written in terms of the Error function as:

$$Erfc(x) = 1 - Erf(x),$$

note that, $Erf(0) = 0$ *and* $Erf(\infty) = 1$.

The following definition is the Mellin–Ross function.

Definition 1.14 : *(The Mellin–Ross Function)*
The definition of the Mellin-Ross function is given by:

$$E_t(v,a) = t^v e^{at}\Gamma^*(v,t),$$

It can be written also as:

$$E_t(v,a) = t^v \sum_{k=0}^{\infty} \frac{(at)^k}{\Gamma(k+v+1)} = t^v E_{1,v+1}(at).$$

The following definition is the Mittag–Leffler function.

Definition 1.15 : *(The Mittag-Leffler Function)*
The definition of the Mittag-Leffler function is given by:

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k+\beta)}, \ \ \alpha > 0, \ \ \beta > 0, \ \ x \in R^+.$$

In particular if $\alpha = 1$, *and* $\beta = 1$ *then we can get:*

$$E_{1,1}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x.$$

If $\alpha = 1$, *and* $\beta = 2$, *then we obtain:*

$$E_{1,2}(x) = \sum_{k=0}^{\infty} \frac{x^k}{(k+1)!} = \frac{e^x - 1}{x}.$$

1.3 Problem statements

This study addresses the following problems regarding FIT with some applications, which are summarized as follows:

(1) What is the relation between the FIT and linear canonical transformation?

Fractional Laplace transform is a special case of complex linear canonical transformation. The one-dimensional fractional Laplace transformation is defined for functions of only one independent variable. It was first introduced in 1970, and proven to be useful in many applications. It was introduced to solve ordinary fractional calculus, as well as multi-dimensional fractional calculus. Therefore, it is important to solve the initial and boundary value problem, which consists of the linear ordinary fractional calculus. Fractional derivative is also used in many branches of sciences, mathematics, physics, chemistry, and engineering. Fractional calculus has gained importance due to its applicability in diverse fields of science and engineering Petras et al. (2002).

The integrals of an integer-order clear physical interpretations, engineering and geometry, solves applied problems in various fields of science via significant simplification. It was touched upon in the introduction that the formulation of the concept for fractional integrals was a natural outgrowth of integer order integrals in much the same way that the fractional exponent follows the more traditional integer order exponent, see Loverro (2004). Integral equations systems are reported in applied sciences, chemistry, physics, engineering and populations growth models. Studies of integral equations systems have attracted attention in scientific applications, see Wazwaz (2011) and Debnath and Bhatta (2006). The intuitive idea of fractional order calculus is as old as integer order calculus, which is evident in a letter penned by Leibnitz to L'Hopital. The fractional order calculus is a generalization of the integer order calculus to a real or complex number.

Fractional integro-differential equations are present in mathematical modeling of various physical phenomena such as heat conduction in materials with memory and diffusion processes. It appears in various research and scientific applications as well.

(2) Is it possible to use the VIM to find the exact solution for most problems can't be solved analytically?

The MDLT and VIM have been studied by several researchers and have many applications in applied mathematics, mathematical physics, and engineering. In the present work, we implement the multi-dimensional Laplace transform method to solve linear multi-dimensional fractional equations. We also modified the multi-dimensional modification of He's variational iteration method to solve multi-dimensional fractional order integro-differential equations with the given conditions. We used this method MVIM since most of the problems cannot be solved analytically and it is difficult to find the exact solution. Finding good approximate solutions using numerical methods prompted us to apply MVIM to solve these problems. This method can solve some problems while constructing semi analytical solutions in the form of polynomials.

(3) What is the differences between numerical methods for finding good approximate solutions for problems?

There are several numerical methods, for example, Tatari and Dehghan (2007) and Odibat (2010) discussed the convergence of VIM systematically .

While Momani and Noor (2006) used Adomian decomposition method to solve the problem and find a good approximation solution. Also, the fractional differential transform method used by Arikoglu and Ozkol (2007), and the collocation method by Rawashdeh (2006), was used to find an approximate solution for problems that cannot be solved analytically. The collocation method was used to find numerical solutions for fractional integro-differential equation, and it was used exact solution for the linear fractional integro-differential equation with the boundary conditions using one iteration. Ghorbani and Saberi-Nadjafi (2009); Molliq et al. (2009); Dal (2009); Jafari and Tajadodi (2010) introduced the VIM, spline collection method El-Hawary and El-Shami (2009), Laplace transform method Kreysig (1983); Farjo (2007); Kimeu (2009); Aghili and Masomi (2013, 2014), fractional Laplace transform Jumarie (2009), homotopy perturbation method and variational iteration method Nawaz (2011) and many others.

(4) Is the fractional Laplace transform defined for only one variable functions?

In 2009, Jumarie G. defined the fractional Laplace transformation of only one independent variable with his inversion and analyzed some of their properties for this definition.

Dahiya and Nadjafi (1999) defined the multi-dimensional Laplace transform for functions of more than one independent variable. Also, they studied some of the theorems and properties concerned with this definition.

Anwar et al. (2013) established the double Laplace transform formulas for the partial fractional integrals. They found the double Laplace formulas of partial fractional derivatives in the context of Caputo.

(5) Is it possible to use ITs to discuss the fundamental solutions of space-time diffusion equation of fractional order?

Zhang and Liu (2007) considered the space-time Riesz fractional partial differential equations with periodic conditions, and they obtained equations from the integro- partial differential equation by replacing the time derivative with a Caputo fractional derivative, and the space derivative with Riesz potential. Also, they derived the explicit expressions of the fundamental solutions for the space Riesz fractional partial differential equation and the space-time Riesz fractional partial differential equation. Ebadian et al. (2015) used the algorithm depends on triangular function method to solve the fractional diffusion-wave equation.

Kumar (2014); Goufo et al. (2015); Atangana (2016) successfully derive singular kernels, which find many applications in real world problems and applied to the fields of groundwater and thermal science. Caputo and Fabrizio obtained results from the version based upon the Riemann-Liouville approach developed by Atangana,Caputo and Fabrizio (2016).

Issues was pointed out by Riemann-Liouville, where the kernel was nonlocal and the related integral is not a fractional operator, but the average of the function and its integral, and the solution of the following equation $\left(\frac{d^{\alpha}y}{dx^{\alpha}} = ay\right)$. That is an exponential equation instead of a non-local function. Therefore, some researchers concluded that the fractional parameter can be viewed as filter regulator. Riemann-Liouville kernel, although non-local, is singular and useful for a problem when modeling real world problems.

Atangana and Baleanu (2016b) suggested a new operator with fractional order based upon the Riemann-Liouville having all the benefits of Riemann-Liouville with non-singular kernel to deal with these problems. It should also be pointed out that observation was based on the fractional average of the Riemann-Liouville fractional integral of the given function and the function itself. Moreover, the derivative was found to be very useful in thermal and material sciences, see Atangana (2016). The new derivative with fractional orders are simultaneously filters and fractional derivatives.

(6) Is possible to develop an efficient algorithm for solving fractional matrix equations?

Bhrawy et al. (2016d) adapted an operational matrix formulation of the collocation method for one and two-dimensional non-linear fractional subdiffusion equations. They also used both double and triple shifted Jacobi polynomials as basis functions to elucidate approximate solutions of one and two-dimensional cases. The space-time fractional derivatives given in the underline problems are expressed by Jacobi operational matrices and helps to investigate spectral collocation schemes for both temporal as well as spatial discretizations.

The multi-dimensional fractional calculus as the origination of integer order multi-dimensional fractional calculus have been used to style problems in applications and mathematical tools by models for various phenomena in sciences and engineering. Fractional calculus has been used to model the physical and engineering processes, which are best described by fractional differential equations.

Thus the present study aims to find the exact solution and reliable tool for solving fractional linear partial differential equations, space-time partial fractional derivative, multi-dimensional integral equations and integro-differential equations of the Volterra type. We extend new fractional derivative and integral including Riemann-Liouville have a non-local and non-singular kernel to multi-dimensional fractional and find the exact solution of system of matrix fractional differential equation in Riemann-Liouville sense. The applications of multi-dimensional fractional calculus can be made to be powerful, effective, and possess a high-level trust. Furthermore, it accelerates the rate of convergence and the formation of degenerates. In this thesis, we focus on the Riemann-Liouville sense to present new solutions to problems by certain applications for solving them.

1.4 Research objectives

We list the objectives of study as follows:

(i) to extend the study of one-dimensional fractional to the multi-dimensional fractional calculus and study the multi-dimensional Laplace transformation with their respective applications.

(ii) to investigate theorems and properties of the multi-dimensional fractional integral transforms, to solve the partial differential equations, space-time partial fractional derivative, multi-dimensional integral and integro-differential equations by using the linearity and convolution properties where the fractional integrals are used in the Riemann-Liouville type. Then, to solve the fractional order integro-differential equations with non-local boundary conditions by constructing an initial trial-function, so that iterations will lead to an exact solution.

(iii) to extend and find the exact solution of space-time fractional derivative also to define and study a new fractional derivative and integral, including Riemann-Liouville for linear operators with non-local and non-singular kernels.

(iv) to extend the study to solve a system of matrix fractional differential equations via the integral transformation method and convolution product to the Riemann-Liouville fraction of matrices.

In the next section we provide the details how these objectives are organized in this study.

1.5 Organisation of the Thesis

This thesis consist of eight chapters. The first chapter describes a general outline of the research work, where the stimulation and objectives are defined.

In Chapter 2, we report a brief historical literature review and the application of fractional calculus, Laplace transform method, and Variation iteration method in integer and fractional order calculus. Some existing analytical solution methods will also be discussed.

In Chapter 3, a brief history pertaining to fractional calculus using Laplace transform and modification of He's Variation iteration method in fractional integro-differential equation will be provided. Some special mathematical functions will be displayed with their respective properties. Different types of fractional calculus, including definitions, theorems, lemma, properties and some examples will be discussed as well. A list of definitions and properties of fractional Laplace transform, theorems and properties with inversion will be touched upon.

In Chapter 4, will cover some basic methods for fractional calculus and describe the Laplace transform method used to solve linear ordinary fractional calculus, and the modification of Hes variational iteration method for solving fractional integro-differential equations.

In Chapter 5, we extend the topics into the multi-dimensional realm using the multi-dimensional Laplace transform to find the exact solution of the linear fractional partial differential equations of n-th order with initial and boundary value problems, integral equations and integro-differential equations of the second kind, and multi-dimensional modification of He's variation iteration method in fractional partial integro-differential equation. A list of definitions and properties of multi-dimensional fractional Laplace transform, theorems, and properties with inversion of multi-dimensional will be discussed as well.

In Chapter 6, will detail the study of the multi-dimensional space-time fractional derivative and their potential applications, while developing the mathematical foundations of those operators. The multi-dimensional space-time fractional method is developed to augment equations for anomalous diffusion to employ fractional in space-time. We present a new fractional derivative, including Riemann-Liouville with Laplace transform, with a non-local and non-singular kernel proposed by Atangana and Baleanu (2016b) and extended to fractional partial derivative using multi-dimensional Laplace transform. We intend to answer some outstanding questions posed by many researchers in the field of fractional calculus. The results are presented.

In Chapter 7, will discuss the Laplace transform method based on operational matrices of fractional derivatives in the context of Riemann-Liouville to solve several kinds of linear fractional differential equations. We will also show the theorem of non-homogeneous matrix fractional partial differential equation using some illustrative examples to demonstrate the effectiveness of the new methodology. We would also present the operational matrices of fractional derivatives using Laplace transform in control systems. We will also discuss an analytical technique to solve the fractional order multi-term fractional differential equation. The thesis is summarized in Chapter 8, and we provide the future research recommendations as well.

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