UNIVERSITI PUTRA MALAYSIA

FRACTIONAL OPERATORS AND THEIR APPLICATIONS ON SPACES OF ANALYTIC AND UNIVALENT FUNCTIONS

ZAINAB ESA ABDULNABY

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FRACTIONAL OPERATORS AND THEIR APPLICATIONS ON SPACES OF ANALYTIC AND UNIVALENT FUNCTIONS

By

ZAINAB ESA ABDULNABY

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Doctor of Philosophy

May 2017

DEDICATIONS

To

My mother and my mother-in-law

For their loving

To

My father

For his trust, support and encouragement

and

To my husband Kasim and my beloved daughters, Maryam, Daniah, and Zahraa

For their patience and loving.

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Doctor of Philosophy

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May 2017

Chair: Professor Dr. Adem Kılıçman, PhD
Faculty: Science

Fractional calculus operators and linear (or convolution) operators have many interesting applications in the theory of analytic and univalent functions. These operators and their generalizations have been applied in obtaining the characterization properties, coefficient estimates and distortion inequalities for various subclasses of analytic and univalent functions.

The main objective of this thesis is to study certain different types of operators such as fractional differential, fractional integral and fractional mixed intgro-differential operators as well as convolution operators besides some classes defined by these operators. Our focus is on spaces of normalized analytic functions in the open unit disk, such as Banach spaces.

Firstly, a class of analytic functions with negative coefficients is investigated by constructing the fractional differential operator. Further, we illustrate some of its general geometric properties. We investigate the boundedness and the maximality of the extended fractional differential operator. The mixed integro-differential operator and its generalizations are established. In addition, applications are designed involving the per-Shwarzian derivatives. Moreover, the fractional integral operator is introduced for joining some special functions. Here, we concern about its univalency. Boundedness and compactness on a class of normalized Banach space are discussed. A generalized fractional differential operator and its normalized formula are developed and studied. The above studies are constructed in various types of Banach spaces in a complex domain.

Defining the new linear operators associated with the well known special functions such as Mittag-Leffler function are investigated by utilizing the convolution techniques in a complex domain. Studying some bounded transformation operators from Banach to another Banach spaces also, are considered. Extending the Moment-generating function in the complex plane $\mathbb{C}$ is applied to define new linear operators of convolution and fractional convolution types which are the generalization for the prominent operators in the open unit disk. Some applications involving these operators are obtained to solve some well known geometric problems such as Fekete–Szegö problem, while another are solved by using Jack's Lemma.

Finally, we defined a new fractional class of analytic and univalent functions in the open unit disk which can be considered as a generalization of *Koebe* function. As applications in this class, we defined subclasses of analytic and univalent functions of fractional powers, through employing a fractional differential operator. Then, some certain results on coefficient inequality, growth and distortion theorem and extreme points are studied.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

PENGOPERASI PECAHAN TERTENTU DAN PENGGUNAANNYA DALAM RUANG TERANALISIS DAN FUNGSI UNIVALEN

Oleh

ZAINAB ESA ABDULNABY

Mei 2017

Pengerusi: Profesor Dr. Adem Kılıçman, PhD
Fakulti: Sains

Pengoperasi kalkulus pecahan dan pengoperasi linear (atau konvolusi) mempunyai banyak aplikasi menarik dalam teori fungsi analisis dan univalen. Pengoperasi-pengoperasi ini dan pengitlakannya telah digunakan untuk mendapatkan sifat-sifat pencirian, anggaran pekali dan ketaksamaan herotan untuk pelbagai subkelas fungsi analisis dan univalen.

Objektif utama tesis ini adalah untuk mengkaji beberapa jenis pengoperasi yang berbeza seperti pengkamiran pecahan, pecahan integer dan pengoperasi pecahan tercampur pembezaan-pengkamiran serta pengoperasi konvolusi selain beberapa kelas yang ditakrifkan oleh pengoperasi-pengoperasi ini. Fokus kami adalah pada ruang normal bagi fungsi analisis dalam cakera unit terbuka, seperti ruang Banach.

Pertama, satu kelas fungsi teranalisis dengan pekali negatif dikaji dengan membina pengoperasi pembezaan pecahan. Selanjutnya, kami menggambarkan beberapa ciri umum geometri. Kami mengkaji batasan dan kemaksimalan pengoperasi pembezaan pecahan lanjutan. Pengoperasi pembezaan-kamiran tergabung dan keitlakkannya diterbitkan. Selain itu, aplikasi yang direka melibatkan setiap pembeza Shwarzian. Selanjutnya, pengoperasi kamiran pecahan diperkenalkan untuk menghubungkan beberapa fungsi khas. Di sini, kami menitikberatkan tentang univalensinya. Pembatasan dan kepadatan pada kelas ruang Banach ternormal dibincangkan. Satu pengitlakkan pengoperasi pembezaan pecahan dan formula kenormalannya dibangunkan dan dikaji. Kajian di atas dibina dalam pelbagai jenis ruang Banach dalam domain kompleks.

Definisi pengoperasi linear baru yang berkaitan dengan fungsi-fungsi khas yang telah diketahui seperti fungsi Mittag-Leffler dikaji dengan menggunakan teknik-teknik konvolusi dalam domain yang kompleks. Mengkaji sesetengah pengoperasi transformasi terbatas dari satu ruang Banach ke ruang Banach yang lain, juga dipertimbangkan. Melanjutkan fungsi janaan- Moment dalam satah kompleks $\mathbb{C}$ digunakan dan pengitlakkan pengoperasi jenis konvolusi dan konvolusi pecahan yang berkaitan dengan fungsi ini ditakrifkan. Beberapa aplikasi yang melibatkan pengoperasi ini diperolehi untuk menyelesaikan beberapa masalah geometri yang telah diketahui seperti masalah Fekete-Szegö, manakala yang lain diselesaikan dengan menggunakan Lema Jack.

Akhir sekali, kami mentakrifkan kelas pecahan baru bagi fungsi analisis dan univalen dalam cakera unit terbuka yang dianggap sebagai satu pengitlakkan bagi fungsi *Koebe*. Untuk aplikasi dalam kelas ini, kami takrifkan subkelas bagi fungsi analisis dan univalen bagi kuasa pecahan, dengan menggunakan satu pengoperasi pengkamiran pecahan. Kemudian, beberapa keputusan tertentu pada ketidaksamaan pekali, pertumbuhan dan penyelewengan teorem dan titik yang melampau dikaji.

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I certify that a Thesis Examination Committee has met on 17 May 2017 to conduct the final examination of Zainab Esa Abdulnaby on her thesis entitled "Fractional Operators and their Applications on Spaces of Analytic and Univalent Functions" in accordance with the Universities and University Colleges Act 1971 and the Constitution of the Universiti Putra Malaysia [P.U.(A) 106] 15 March 1998. The Committee recommends that the student be awarded the Doctor of Philosophy.

Members of the Thesis Examination Committee were as follows:

Fudziah binti Ismail, PhD
Professor
Faculty of Science
Universiti Putra Malaysia
(Chairman)

Nik Mohd Asri bin Nik Long, PhD
Associate Professor
Faculty of Science
Universiti Putra Malaysia
(Internal Examiner)

Isamiddin S.Rakhimov, PhD
Professor
Faculty of Science
Universiti Putra Malaysia
(Internal Examiner)

Ayhan Serbetci, PhD
Professor
Ankara University
Turkey
(External Examiner)

NOR AINI AB. SHUKOR, PhD
Professor and Deputy Dean
School of Graduate Studies
Universiti Putra Malaysia

Date: 6 July 2017

This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Doctor of Philosophy.

The members of the Supervisory Committee were as follows:

Adem Kılıçman, PhD
Professor
Faculty of Science
Universiti Putra Malaysia
(Chairperson)

Mat Rofa Ismail, PhD
Associate Professor
Faculty of Science
Universiti Putra Malaysia
(Member)

Sharifah Kartini Said Husain, PhD
Senior Lecturer
Faculty of Science
Universiti Putra Malaysia
(Member)

Rabha Waell Ibrahim, PhD
Senior Lecturer
Faculty of Science
University Malaya
(Member)

ROBIAH BINTI YUNUS, PhD
Professor and Dean
School of Graduate Studies
Universiti Putra Malaysia

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Name and Matric No: Zainab Esa Abdulnaby (GS38674)

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- supervision responsibilities as stated in the Universiti Putra Malaysia (Graduate Studies) Rules 2003 (Revision 2012-2013) are adhered to.

Signature: ______________________

Prof. Dr. Adem Kilicman
Pensyarah
Jabatan Matematik
Universiti Putra Malaysia
43400 UPM, Serdang
Selangor Darul Ehsan

Name of Chairman of Supervisory Committee: Professor Dr. Adem Kılıçman

Signature: ______________________

Prof. Dr. Adem Kilicman
Pensyarah
Jabatan Matematik
Universiti Putra Malaysia
43400 UPM, Serdang
Selangor Darul Ehsan

Name of Member of Supervisory Committee: b/p Associate Professor Dr. Mat Rofa Ismail

Signature: ______________________

DR. SHARIFAH KARTINI SAID HUSAIN
Senior Lecturer
Department of Mathematics,
Faculty of Science,
Universiti Putra Malaysia,
43400 UPM Serdang,
Selangor Darul Ehsan, Malaysia.

Name of Member of Supervisory Committee: Dr. Sharifah Kartini Said Husain

Signature: ______________________

Name of Member of Supervisory Committee: Dr. Rabha Waell Ibrahim

TABLE OF CONTENTS

LIST OF ABBREVIATIONS

$\mathbb{R}$	The set of real numbers
$\mathbb{N}$	The set of natural numbers
$\mathbb{N}_0$	The set of natural numbers including zero
$\mathbb{C}$	The set of complex numbers
$\mathbb{U}$	The open unit disk $z \in \mathbb{C} : \|z\| < 1$
$\overline{\mathbb{U}}$	The closed unit disk $z \in \mathbb{C} : \|z\| \leq 1$
$\mathscr{H}(\mathbb{U})$	The class of all analytic functions in the open unit disk
$\mathscr{H}[a,n]$	The subclass of analytic functions of $\mathscr{H}(\mathbb{U})$
$\mathscr{A}$	The class of analytic functions of $\mathscr{H}(\mathbb{U})$
$\mathscr{A}(n)$	The class of analytic functions of $\mathscr{H}[a,n]$
$\mathscr{S}$	The set of univalent functions
$\mathscr{P}$	The set of analytic functions with negative coefficients
$\mathscr{S}^*$	The set of starlike functions
$\mathscr{K}$	The set of convex functions
$\mathscr{P}^*$	The set of starlik functions with negative coefficients
$\mathscr{C}$	The set of convex functions with negative coefficients
$\mathbb{K}(z)$	The *Koebe* function
$\mathbb{K}_\theta(z)$	The rotation function
$\mathfrak{A}^p$	The Bergman space
$\mathfrak{A}^p_q$	The weighted Bergman space
$\mathscr{B}$	The Bloch space
$\mathscr{B}_q$	The weighted Bloch space
$\mathscr{B}^\tau_q$	The weighted τ-Bloch space
$\mathscr{B}^\tau_{q,\log}$	The weighted logarithmic τ-Bloch space
T_f	The pre-Schwarzian derivatives
$\|\|T_f\|\|$	The norm pre-Schwarzian derivatives
$*$	The convolution binary operation
$D^\alpha_z f(z)$	The fractional derivative Srivastava-Owa operator
$\mathscr{I}^\alpha_z f(z)$	The fractional integral Srivastava-Owa operator

CHAPTER 1

INTRODUCTION

Complex analysis is a deep and far-ranging study of the fundamental notions of complex differentiation and integration and has the attractive advantages not provable in the real domain. For instance, complex functions are necessarily analytic, meaning that they can be represented by convergent power series (Taylor series), and hence are infinitely differentiable. They may be of several variables. Nevertheless, our focus in this study is particularly on those functions which are of one complex variable.

The area of geometric function theory is one of the branches of complex analysis, which covers all geometrical properties of analytic and univalent functions. This theory has raised the interest of many researchers since the beginning of the 20th century when appeared the first important papers in this domain, due to Koebe in 1907 and Bieberbach in 1916 (see Goodman (1979) and Duren (1983)).

Often, operator theory plays an important role in geometric function theory. Because, studying linear operators use to define, improve and generalize many well known analytic function classes. Another reason, that this studying involving more geometric features for the investigated analytic function classes and preserve many of their properties. A further important reason, it leads to developing various applications and provides uncomplicated methods and tools to solve a number of problems in such theory.

On the other hand, operator theory is described as the branch of functional analysis that deals with bounded linear operators as well as their properties. It has growth and development with strong connections to pure and applied mathematics for covering the biggest spaces in such area.

However, there is a pretty company between each of the complex analysis, the operator theory, and the functional analysis can be found in the geometric function theory. This is fascinating area of study, for several reasons:

(i) This study leads to the formulation of many interesting problems in analytic functions.

(ii) The methods of functional analysis usually grant clarity and elegance to the proofs of classical theorems and make the results accessible in more common situations.

(iii) Improvement the operators and their properties on several Banach spaces of analytic functions.

As a result, a large number of applications in such area via the operator theory are evolved. Therefore, we utilize this study to define fractional differential and integral operators in a complex domain then determine their boundedness and compactness properties on complex Banach spaces.

This introductory chapter begins by providing the reader with some background information and general on the analytic and univalent functions theory. Then it follows by presenting some special classes of univalent functions and complex Banach spaces of analytic and univalent functions as well as fractional differential and integral operators defined in a complex domain which are the main part of this thesis. Special functions and some of their properties are given in the following section. Finally, this chapter illustrates the main problems that need to be solved and highlights on the most important motivations for writing this study, also provides a general overview of the outline and content of this study.

1.1 Analytic functions

Let $\mathbb{C}$ be the complex plane, then a subset $E \subset \mathbb{C}$ is called a domain if E is open and if any two points of E can be connected by a broken line segment in E. The domain E is said to be simply connected if it has no holes.

The complex-valued function (of one variable) f is said to be analytic (or regular, holomorphic) at a point z_0 if it is differentiable at every point in some neighborhood of the point z_0. An analytic function f in a complex domain E is a convergent power series that can be presented as a Taylor series formed in their domain as follows:

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k,$$

where the coefficient $a_k = f^{(k)}(z_0)/k!$, that means f converges in some open disk centred at z_0. It is easy to obtain $f^{(k)}(z_0)$ from the Cauchy integral formula that is given by

$$f^{(k)}(z_0) = \frac{k!}{2\pi i} \int_{\Lambda} \frac{f(z)}{(w-z)^{k+1}} dw,$$

where Λ is a rectifiable simple closed curve covering z and f is analytic indoors and on it.

Normalization: Let $f : E \to \mathbb{C}$ is analytic on E, then

(i) the function f is normalized if it takes the value zero at the origin, that is $f(0) = 0$, and

(ii) its derivative takes the value 1 at the origin, that is $f'(0) = 1$.

Example 1.1 *As a simple example for a normalized analytic function is $f(z) = z$ for all $z \in E$, that is*

$$f(0) = 0 \quad \text{and} \quad f'(0) = 1.$$

In this point there is an important question arises as follows:

Do all the analytic functions normalized?

Definitely not, for example the function $f(z) = z^2$ is analytic but not normalized. Clearly, the set of normalized analytic functions is non-empty, so there exist a subset of them which have a nice geometric property.

1.2 Univalent functions

The theory of univalent functions is the basic connection between the analytical structure and geometric behavior. The problem lies in finding a useful set of conditions on the sequence of a_k that are necessary and sufficient for $f(z)$ to be univalent in $\mathbb{U}$.

Definition 1.1 *A single-valued function f is said to be univalent (or one to one, schliht (German) or odnolistni (Russian)), in a domain $E \rightarrow \mathbb{C}$ if it never takes the same value twice; that is, if $f(z_1) \neq f(z_2)$ for all points z_1 and z_2 in E with $z_1 \neq z_2$.*

Definition 1.2 *The single-valued function f is said to be locally univalent at a point $z_0 \in E$ in some neighbourhood of z_0 if and only if $f'(z_0) \neq 0$.*

If f is univalent in a domain E then it is trivially locally univalent throughout E but the converse is not true.

Example 1.2 *Let $E = \mathbb{C} \setminus \{0\}$ and $f(z) = z^2 (z \in E)$. It is clear that the function f is analytic in E and locally univalent at any point of E since $f'(z_0) = 2z_0 \neq 0$, for all $z_0 = \frac{1}{2} \in E$. Nevertheless, the function f is not univalent in E since $f(z) = f(-z)$ for all $z \in E$.*

The open unit disk: The option of the open unit disk $\mathbb{U} := \{z \in \mathbb{C}; |z| < 1\}$ as a complex domain for the study of analytic and univalent functions theory is a matter of great benefit, to make the mathematical calculation more simple and leads to elegant formula. Adding to that, there is no loss of generality in this option, since the Riemann mapping theorem asserts that:

Theorem 1.1 (Gray, 1994). *If $E \subset \mathbb{C}$ is a simply connected domain and $z_0 \in E$, then there exists a unique function f, analytic and univalent in E onto the open unit disk $\mathbb{U}$ in such a way that $f(z_0) = 0$ and $f'(0) > 0$.*

Consequently, every univalent function in E is linked with a univalent function in the open unit disk $\mathbb{U}$ and therefore, the properties of the univalent function determined on the open unit disk $\mathbb{U}$ can be obviously translated into the properties of the original function defined in the simply connected domain E. For these reasons, the option of the open unit disk $\mathbb{U}$ as a domain for study in details analytic and univalent functions is the best matter in such theory.

Let $\mathscr{H}(\mathbb{U})$ be the class of all analytic functions whose domain is the open unit disk $\mathbb{U}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N}$ (set of natural numbers) let $\mathscr{H}[a,n]$, be the subclass of $\mathscr{H}(\mathbb{U})$, consisting analytic functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots .$$

Let $\mathscr{A}(n) \subset \mathscr{H}[a,n]$, be the class of function f defined by

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k, \qquad (z \in \mathbb{U}) \tag{1.1}$$

which is analytic in the open unit disk $\mathbb{U}$. In particular, we set $\mathscr{A}(1) = \mathscr{A} \subset \mathscr{H}(\mathbb{U})$, which is the class of normalized analytic functions defined by the following power series:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad (z \in \mathbb{U}). \tag{1.2}$$

Also, let $\mathscr{S}$ be the subclass of $\mathscr{A}$ of univalent and analytic functions $f : \mathbb{U} \to \mathbb{C}$ satisfied the normalization conditions $f(0) = 0$ and $f'(0) = 1$.

Example 1.3 *Here, we recall many examples of functions belonging to the class $\mathscr{S}$:*

(i) *the identity mapping: $f(z) = z$, we have $f(\mathbb{U}) = \mathbb{U}$,*

(ii) *the inverse of univalent function f^{-1} is also univalent in an open unit disk $\mathbb{U}$,*

(iii) *the function*

$$\begin{aligned} f(z) = \frac{z-a}{1-\overline{a}z} &= (z-a)(1-\overline{a}z+\overline{a}^2z^2+\overline{a}^3z^3+\cdots) \\ &= a+(1-\overline{a}a)z+(\overline{a}-\overline{a}^2a)z^2+(\overline{a}^2-\overline{a}^3a)z^3+\cdots \\ &= a+(1-|a|^2)z+\overline{a}(1-|a|^2)z^2+\overline{a}^2(1-|a|^2)z^3+\cdots \\ &= a+(1-|a|^2)\sum_{k=1}^{\infty}\overline{a}^{(k-1)}z^k, \end{aligned}$$

where $|a| < 1$ and

(iv) *the function*

$$f(z) = \frac{z}{1-z^2} = z + z^3 + z^5 + \cdots,$$

which maps $\mathbb{U}$ *onto the plane minus the two half-lines* $[1/2, \infty)$ *and* $(-\infty, -1/2]$.

We next present the most important example of a function of class $\mathscr{S}$.

Example 1.4 *Let* $\mathbb{K} : \mathbb{U} \to \mathbb{C}$ *be a function given by*

$$\mathbb{K}(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \cdots + kz^k + \cdots = z + \sum_{k=2}^{\infty} kz^k, \tag{1.3}$$

which maps $\mathbb{U}$ *onto* $\mathbb{C} \setminus \{\xi \in \mathbb{C} : \Re\{\xi\} \leq -1/4, \mathrm{Im}\xi = 0\}$ *where* $\mathbb{K}$ *is called the Koebe function.*

Furthermore, let $\mathbb{K}_\theta : \mathbb{U} \to \mathbb{C}$ be given by $\mathbb{K}_\theta(z) = \frac{z}{(1-e^{i\theta}z)^2}$, where $\theta \in \mathbb{R}$. We note that $\mathbb{K}_\theta$ is a rotation function of angle$-\theta$ of the function $\mathbb{K}$, since $\mathbb{K}_\theta = e^{-i\theta}\mathbb{K}(e^{i\theta}z)$, $z \in \mathbb{U}$. Therefore, $\mathbb{K}_\theta$ is univalent on $\mathbb{U}$ and maps the open unit disk $\mathbb{U}$ onto the complex plane except for a radial slit to ∞ which starts from the point $(-1/4)e^{-i\theta}$.

Analytically, the class of univalent functions $\mathscr{S}$ is not closed under addition (this follows trivially from the fact that any $f \in \mathscr{S}$ satisfies that $f'(0) = 1$). For example, $f = z/(1-z)$ and $g = z/(1+iz)$ are in the class of univalent functions $\mathscr{S}$, but $(f+g)$ is not in $\mathscr{S}$. However, the convolution (or Hadamard product), is the most important binary operation in the class $\mathscr{S}$, since the outcomes arise from the elementary observation that the convolution of two univalent functions in $\mathscr{S}$ is also in the class of univalent functions $\mathscr{S}$ (see Ruscheweyh (1975)).

Definition 1.3 *The convolution (or Hadamard product) of two normalized analytic functions f defined by* (1.2) *and*

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \qquad (z \in \mathbb{U}),$$

is denoted by $f * g$ and defined as follows

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k \qquad (z \in \mathbb{U}), \tag{1.4}$$

where the geometric analytic function

$$I(z) = \sum_{k=0}^{\infty} z^{k+1} = \frac{z}{1-z} \qquad (z \in \mathbb{U}),$$

is the identity analytic function under this Hadamard product, that means that, $(f * I)(z) = f(z)$ *in* $\mathbb{U}$.

The convolution technique $*$ is a pretty binary operation of univalent functions theory and it satisfies the following characteristics:

For all $z \in \mathbb{U}$, let f, g and h be normalized analytic functions in the class $\mathscr{S}$, then

(i) Inverse: $(f * f^{-1})(z) = I(z)$.

(ii) Commutative: $(f * g)(z) = (g * f)(z)$.

(iii) Associative: $(f * g) * h = (f * h) * g = (g * h) * f$.

The following lemma shows the derivative of convolution two normalized analytic functions in the open unit disk $\mathbb{U}$.

Lemma 1.1 (Ruscheweyh, 1975). *Let f and g be two normalized analytic functions in a complex domain $\mathbb{U}$. Then*

$$z(g * f)'(z) := g(z) * zf'(z) \Leftrightarrow (g * f)'(z) = \frac{g(z)}{z} * f'(z), \ (|z| < 1, z \neq 0).$$

The following theorems can be applied to obtain the sharp upper and lower bounds for $|f(z)|$ and $|f'(z)|$, respectively on $|z| < 1$ (see Goodman (1983)).

Theorem 1.2 (Growth Theorem). *Let $f \in \mathscr{S}$ on $|z| < 1$ with $f(0) = 0$ and $f'(0) = 1$, and let $r = |z| < 1$. Then*

$$\frac{1-r}{(1+r)^2} \leq |f(z)| \leq \frac{1+r}{(1-r)^2}. \tag{1.5}$$

The equalities hold if and only if a function f is a rotation of the Koebe function.

Theorem 1.3 (Distortion Theorem). *Let $f \in \mathscr{S}$ on $|z| < 1$ with $f(0) = 0$ and $f'(0) = 1$, and let $r = |z| < 1$. Then*

$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3}. \tag{1.6}$$

The equalities hold if and only if a function f is a rotation of the Koebe function.

Nehari (1975) proved that, the family $\mathscr{S}$ of the functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, $|z| < 1$ is compact. In another words, $\mathscr{S}$ is a compact subset of the space of all analytic functions on the open unit disk $\mathbb{U}$, because it is closed and locally bounded. Closed means that the limit of a convergent sequence of univalent functions is again an univalent function in $\mathscr{S}$. Locally bounded means that for every $r (0 < r < 1)$, there is a positive number

$M(r)$ so that $|f(z)| < M(r)$ whenever $|z| < r$ and f is in $\mathscr{S}$. These observations pushed Bieberbach to create the most popular conjecture in 1916 which asserts that the *Koebe* function has the largest coefficients in $\mathscr{S}$.

Theorem 1.4 (Bieberbach Conjecture). *If $f \in \mathscr{S}$, then the Koebe function has the largest coefficients, that is*

$$|a_k| \leq k, \qquad (k \geq 2).$$

Bieberbach in 1916 (see Duren (1983)) demonstrated that the inequality for $k = 2$, and conjectured that it is true for any k. Later, De Branges (1985) proved that this conjecture it applies for all coefficients $k \geq 2$.

In view of geometric functions, long gap between the formulation of the Bieberbach's conjecture in 1916 and its proof by De Branges (1985), led to creating new motivations for researchers in defining and studying some certain subclasses of analytic and univalent functions in the open unit disk $\mathbb{U}$.

1.3 Some special classes of univalent functions

In this section, we present some special subclasses of univalent functions with positive and negative coefficients which are completely characterized by natural geometric properties have been widely considered by many researchers.

Classes of Starlike functions: The starlike of univalent function is a subclass in $\mathscr{S}$ and it was investigated by Alexander (1915) as follows:

Definition 1.4 (Duren, 1983). *A function $f \in \mathscr{S}$ is said to be starlike in $\mathbb{U}$ if the image of $\mathbb{U}$ under f consisting of the point $\upsilon_0 = 0 \in \mathbb{U}$ is conformally starlike domain, $f(\mathbb{U})$ with respect to the origin $\upsilon_0 = 0$, that is any line segment joining $\upsilon_0 = 0$ to every other points in $f(\mathbb{U})$, lies entirely in $f(\mathbb{U})$. We denoted the set of all starlike functions by $\mathscr{S}^*$.*

Some results for function $f \in \mathscr{A}$ was obtained by Robertson (1936) as follows:

Theorem 1.5 *If $f \in \mathscr{A}$, then the necessary and sufficient condition for the function f to be in the class of normalized $\mathscr{S}^*$ in $\mathbb{U}$ is*

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \quad (z \in \mathbb{U}). \tag{1.7}$$

The next result shows that the Bieberbach conjecture in Theorem 1.4 holds for $f \in \mathscr{S}^*$.

Theorem 1.6 (Duren, 1983). *If $f \in \mathscr{S}^*$ then*

$$|a_k| \leq k, \qquad k = 2, 3, \cdots.$$

The equality occurs if and only if f is Koebe function.

Example 1.5 *The Koebe function $\mathbb{K}(z) = \dfrac{z}{(1-z)^2}$ is starlike function in $\mathbb{U}$, since it maps $\mathbb{U}$ onto the entire complex plane $\mathbb{C}$ minus the slit $-\infty < f(\mathbb{U}) < -\frac{1}{4}$.*

In general case, for $0 \leq \alpha < 1$, let $\mathscr{S}^*(\alpha)$ be a subclass of $\mathscr{A}$ defined as follows:

Definition 1.5 (Goodman, 1983). *A function f is said to be starlike function of order $\alpha\,(0 \leq \alpha < 1)$ in $\mathbb{U}$, if*

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \qquad (z \in \mathbb{U}), \tag{1.8}$$

this class is denoted by $\mathscr{S}^*(\alpha)$.

It is clear that, $\mathscr{S}^*(\alpha) \subset \mathscr{S}^*$ for $\alpha\,(0 \leq \alpha < 1)$ and $\mathscr{S}^*(0) = \mathscr{S}^*$.

Classes of Convex functions: The convex univalent functions is another subclass of $\mathscr{S}$, this subclass defined and studied by Robertson (1936) as follows:

Definition 1.6 *A function $f \in \mathscr{S}$ is said to be convex in $\mathbb{U}$ if the image of $\mathbb{U}$ under f is a conformally onto a convex domain $f(\mathbb{U})$, that is any line segment joining any two points in $f(\mathbb{U})$ lies entirely in $f(\mathbb{U})$. This set of all convex functions is denoted by $\mathscr{K}$.*

Here, we recall some results due to Robertson (1936) as follows:

Theorem 1.7 *Let f be given by* (1.2)*, then the necessary and sufficient condition for a function f to be in the class $\mathscr{K}$ is*

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0, \qquad (z \in \mathbb{U}). \tag{1.9}$$

Theorem 1.8 (Duren, 1983) *If $f = z + \sum_{k=0}^{\infty} a_k z^k$ is in $\mathscr{K}$ then*

$$|a_k| \leq 1, \qquad k = 2, 3, \cdots.$$

The equality occurs if and only if f has the following form:

$$f(z) = \frac{z}{1-e^{i\ell}z}, \qquad \ell \in \mathbb{R}.$$

Example 1.6 *The function $f(z) = \dfrac{z}{1-z}$ is convex in $\mathbb{U}$, because $f(z)$ maps $\mathbb{U}$ onto the half-plane $\Re\{f\} > -1/2$.*

In general case, for $\alpha\,(0 \le \alpha < 1)$, let $\mathscr{K}(\alpha)$ be subclass of $\mathscr{A}$ defined as follows.

Definition 1.7 *A function f is said to be convex function of order $\alpha\,(0 \le \alpha < 1)$ in $\mathbb{U}$, if*

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \quad (z \in \mathbb{U}), \tag{1.10}$$

this class is denoted by $\mathscr{K}(\alpha)$.

It is clear that,

$$f \in \mathscr{K}(\alpha) \Longleftrightarrow zf' \in \mathscr{S}^*(\alpha), \quad (0 \le \alpha < 1) \tag{1.11}$$

while the first connection between the classes $\mathscr{S}^*$ and $\mathscr{K}$ was provided by Alexander (1915) as in the following.

Theorem 1.9 (Alexander Theorem)(Duren, 1983). *Let f be a function given by* (1.2), *then*

$$f \in \mathscr{K} \Longleftrightarrow zf' \in \mathscr{S}^*.$$

Another important subclasses of normalized analytic functions introduced by Liu (2004) in the open unit disk are:

For some $\alpha\ (0 < \alpha \le 1)$, if $f \in \mathscr{A}$ satisfies

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2}, \quad z \in \mathbb{U}, \tag{1.12}$$

then f is said to be strongly starlike of order α in $\mathbb{U}$, and denoted by $f \in S\mathscr{S}^*(\alpha)$.

For some $\alpha\ (0 < \alpha \le 1)$, if $f \in \mathscr{A}$ satisfies

$$\left|\arg\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \frac{\alpha\pi}{2}, \quad z \in \mathbb{U}, \tag{1.13}$$

then f is said to be strongly convex of order α in $\mathbb{U}$, and denoted by $f \in S\mathscr{K}(\alpha)$.

Furthermore, for $\psi > 1$, let $\mathscr{N}(\psi)$ be the subclass of $\mathscr{A}$, consisting of the functions f, which satisfies the following condition

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) < \psi, \quad z \in \mathbb{U} \tag{1.14}$$

and let $\mathscr{M}(\psi)$ be the subclass of $\mathscr{A}$ consisting of the functions f which satisfies the following condition

$$\Re\left(\frac{zf''(z)}{f'(z)}\right) < \psi, \quad z \in \mathbb{U}. \tag{1.15}$$

Clearly,

$$f \in \mathscr{N}(\psi) \Longleftrightarrow zf' \in \mathscr{M}(\psi).$$

For $\psi > 1$, the classes $\mathscr{N}(\psi)$ and $\mathscr{M}(\psi)$ were introduced by Owa and Nishiwaki (2002), and Owa and Srivastava (2002).

Class of bounded turning: Let $\mathscr{S}\mathscr{B}$ be the set of functions whose derivatives have positive real parts, that

$$\Re\{f'(z)\} > 0, \qquad (z \in \mathbb{U}). \tag{1.16}$$

In general case, a function $f \in \mathscr{S}$ is said to be bounded turning functions of order $\alpha\,(0 \le \alpha < 1)$, if

$$\Re\{f'(z)\} > \alpha \qquad (z \in \mathbb{U}), \tag{1.17}$$

this class is denoted by $\mathscr{S}\mathscr{B}(\alpha) \subset \mathscr{A}$. They are entirely univalent functions as has preceded. Many results concerning this class can be found in (Miller and Mocanu (2002); Darus et al. (2009); Ibrahim and Darus (2011)).

Class of pre-Schwarzian derivative: For a locally univalent function f on $\mathbb{U}$, the class of pre-Schwarzian derivative (or rational nonlinearity) of function f, is defined by

$$T_f = \frac{f''(z)}{f'(z)}, \qquad (z \in \mathbb{U}) \tag{1.18}$$

this class studied by Goodman (1983), Kim and Sugawa (2006) and Ponnusamy and Sugawa (2008). Indeed, the class T_f is the logarithmic derivative of f', this quantity has several applications with function of operator theory in the theory of locally univalent functions. Further, we define the norm of pre-Schwarzian derivatives, which considered by (Kim and Sugawa, 2006) as follows:

$$||T_f|| = \sup_{z\in\mathbb{U}}(1-|z|^2)\left|T_f\right|<\infty. \tag{1.19}$$

Recently, these classes are widely used by a large number of researchers, to study starlikeness and convexity properties by employing different types of operators on their definitions.

Analytic functions with negative coefficients: Let $\mathscr{P}$ denote the subclass of $\mathscr{S}$ consisting of functions whose coefficients are negative and be expressed as

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \qquad a_k \geq 0, \tag{1.20}$$

which are considered by Silverman (1975). Moreover, for $0 \leq \alpha < 1$, we write

$$\mathscr{P}^*(\alpha) := \mathscr{S}^*(\alpha) \cap \mathscr{P} \qquad \text{and} \quad \mathscr{C}(\alpha) := \mathscr{K}(\alpha) \cap \mathscr{P}.$$

The convolution of two analytic functions f given by (1.20) and g defined by $g(z) = z - \sum_{k=2}^{\infty} b_k z^k$, $b_k \geq 0$ belong to $\mathscr{P}$ is denoted by $(f*g)(z)$ and expressed as

$$(f*g)(z) = z - \sum_{k=2}^{\infty} a_k b_k z^k, \qquad (z \in \mathbb{U}).$$

Principle of subordination: Here, we begin with the classical Schwarz's lemma which is considered one of the most interesting lemma in the theory of analytic and univalent functions, that have been formulated as follows:

Lemma 1.2 (Schwarz Lemma)(Nehari, 1975). *Let the analytic function $w(z)$ be regular in $\mathbb{U}$ and let $w(0) = 0$. If $|w(z)| \leq 1$ in $\mathbb{U}$, then*

$$|w(z)| \leq |z|, \qquad |z| < 1, \tag{1.21}$$

and $|w'(0)| \leq 1$. The equality in (1.21) realizes if and only if $w(z) = Mz$ and $|M| = 1$.

Definition 1.8 (Principle of subordination)(Duren, 1983). *Let the functions f and g in $\mathbb{U}$, the function f is subordinate to g if there exists a Schwarz function $w(z)$, analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| = 1$, $z \in \mathbb{U}$ such that $f(z) = g(w(z))$. In special case, if the function g is univalent in $\mathbb{U}$ the above subordination is equivalent to:*

$$f(0) = g(0) \quad \textit{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}),$$

then f is subordinate to $g(z) \in \mathbb{U}$ and we write

$$f \prec g \quad (z \in \mathbb{U}). \tag{1.22}$$

The following subordination lemma solves a problem in integral means.

Lemma 1.3 (Duren, 1983). *For $\nu > 0$ and $0 < r < 1$. If f and g be two analytic functions in $\mathbb{U}$ with $g \prec f$, then*

$$\int_0^{2\pi} |g(re^{i\theta})|^\nu d\theta \leq \int_0^{2\pi} |f(re^{i\theta})|^\nu d\theta. \tag{1.23}$$

1.4 Spaces of analytic functions

In this section, we supply the open unit disk $\mathbb{U}$ with complex Banach spaces to study some topological properties for operators of analytic and univalent functions. Let first give important inequalities in functional analysis that are used in this study. These notations are taken from Zhu (2007), Hedenmalm et al. (2012), Hoffman (2007) and Garnett (2007).

Minkowski inequalities: are the classical inequalities used in mathematical analysis and defined by

(i) Let f and ψ be two integrable functions in a domain $\mathbb{X} \subset [0, \infty)$. Then for $p > 1$:

$$\left[\int_{\mathbb{X}} |f(z) + \psi(z)|^p dz\right]^{1/p} \leq \left[\int_{\mathbb{X}} |f(z)|^p dz\right]^{1/p} + \left[\int_{\mathbb{X}} |\psi(z)|^p dz\right]^{1/p}. \tag{1.24}$$

(ii) Let $\mathbb{X}$ and $\mathbb{Y}$ be σ-finite measure spaces with measures t and z, respectively, and let Ψ be a complex function on $\mathbb{X} \times \mathbb{Y}$. Then for $p \geq 1$:

$$\left[\int_{\mathbb{Y}} \left|\int_{\mathbb{X}} \Psi(z,t) dt\right|^p dz\right]^{1/p} \leq \int_{\mathbb{X}} \left(\int_{\mathbb{Y}} |\Psi(z,t)|^p dz\right)^{1/p} dt, \tag{1.25}$$

the inequality in (1.25) holds only if $|\Psi(z,t)| = f(z)\psi(t)$, for some non-negative measurable functions ψ and f.

Definition 1.9 (Complex Banach space). *A vector space $\mathbb{X}(\mathbb{U})$ over the filed of complex numbers $\mathbb{C}$ is said to be a complex Banach space for the norm*

$$||f|| = \sup\{||f(z)|| : z \in \mathbb{U}\},$$

and for all $f \in \mathbb{X}(\mathbb{U})$, the complex Banach space $\mathbb{X}(\mathbb{U})$ has the following properties:

(i) $||f+g|| \leq ||f||+||g||$ for all f and g in $\mathbb{X}(\mathbb{U})$,

(ii) $||bf|| = |b|||f||$ for all $f \in \mathbb{X}(\mathbb{U})$ and for all $b \in \mathbb{C}$,

(iii) $||f|| = 0$ if and only if $f = 0$,

(iv) If $\{f_n\}$ is a Cauchy sequence in $\mathbb{X}(\mathbb{U})$, that is, $||f_n - f_m|| \to 0$ as $n \to \infty$ and $m \to \infty$, then there exists an element $A \in \mathbb{X}(\mathbb{U})$ such that $||f_n - A|| \to 0$ as $n \to \infty$.

The function $||\cdot||$ above is called a norm on a complex Banach space $\mathbb{X}(\mathbb{U})$. The condition (iv) above says that $\mathbb{X}(\mathbb{U})$ is complete in the norm $||\cdot||$.

We are going to work with complex Banach spaces whose elements are analytic and univalent functions in the open unit disk $\mathbb{U}$. Next, two examples for complex Banach spaces of analytic functions that will be used later in this study.

The Bergman space: For $0 < p < \infty$, the Bergman space $\mathfrak{A}^p(\mathbb{U})$, consisting of those functions f which are analytic in the open unit disk $\mathbb{U}$ of the complex plane for which p-norm is finite:

$$\| f \|_{\mathfrak{A}^p} := \left\{ \int_{\mathbb{U}} |f(z)|^p d_{\mathfrak{A}}(z) \right\}^{1/p} < \infty, \tag{1.26}$$

where $d_{\mathfrak{A}}(z) = 1/\pi dxdy$. For $1 \leq p < \infty$ the Bergman space $\mathfrak{A}^p$, has the following properties:

(i) $||cf||_{\mathfrak{A}^p} \leq |c|||f||_{\mathfrak{A}^p}$, where c is a real number.

(ii) $||f(z)||_{\mathfrak{A}^p} = 0$ if and only if $f(z) = 0$.

(iii) $||f(z)+g(z)||_{\mathfrak{A}^p} \leq ||f(z)||_{\mathfrak{A}^p} + ||g(z)||_{\mathfrak{A}^p}$ for $1 \leq p$ is satisfied.

The triangle inequality on $\mathfrak{A}^p$ is often equal to Minkowski's inequality in expression (1.24). However, if $0 < p < 1$, then the property of triangle inequality (iii) is replaced by

$$||f(z)+g(z)||^p_{\mathfrak{A}^p} \leq ||f(z)||^p_{\mathfrak{A}^p} + ||g(z)||^p_{\mathfrak{A}^p}.$$

As special case of $\mathfrak{A}^p(\mathbb{U})$, suppose that $q : (0,1] \to [0,\infty)$ is a weighted function which is integrable on $(0,1]$ and q on $\mathbb{U}$ is $q(z) = q(|z|)$, by assuming that q is normalized so $\int_{\mathbb{U}} q(z) d_{\mathfrak{A}}(z) = 1$. For $1 \leq p < \infty$, the weighted Bergman space $\mathfrak{A}^p_q(\mathbb{U})$ is the space of all analytic functions $f : \mathbb{U} \to \mathbb{C}$ defined by

$$\| f \|_{\mathfrak{A}^p_q} := \left\{ \int_{\mathbb{U}} |f(z)|^p q(z) d_{\mathfrak{A}}(z) \right\}^{1/p} < \infty, \tag{1.27}$$

holds true (see Baernstein et al. (2004)). We have the following inclusion:

$$\mathfrak{A}^p \subset \mathfrak{A}_q^p.$$

The Bloch space: The Bloch space $\mathscr{B}$ contains of all analytic functions $f : \mathbb{U} \to \mathbb{C}$, such that

$$\| f \|_{\mathscr{B}} = \sup_{z\in\mathbb{U}} \left\{ (1-|z|^2)|f'(z)| \right\} < \infty. \tag{1.28}$$

holds true. Let $q : (0,1] \to [0,\infty)$ and f be an analytic function in the open unit disk $\mathbb{U}$ is said to be in the weighted Bloch space $\mathscr{B}_q$ if

$$(1-|z|) \mid f'(z) \mid < \hbar q(1-|z|), \quad (z \in \mathbb{U})$$

for some $\hbar > 0$. Note that, if $q = 1$ then $\mathscr{B}_1 = \mathscr{B}$. Further, the weighted τ- Bloch space $\mathscr{B}_q^\tau$, covering of all $f \in \mathscr{B}_q^\tau$ defined by

$$\| f \|_{\mathscr{B}_q^\tau} = \sup_{z\in\mathbb{U}} \left\{ |f'(z)| \frac{(1-|z|)^\tau}{q(1-|z|)} \right\} < \infty. \tag{1.29}$$

It is easy to note that if an analytic function $g \in \mathscr{B}_q^\tau$, then

$$\sup_{z\in\mathbb{U}} \left\{ |\mathbb{k}g(z)| \frac{(1-|z|)^\tau}{q(1-|z|)} \right\} \le c < \infty, \tag{1.30}$$

where $\mathbb{k}$ is a *positive* number. For $0 \le q < \infty$ and $0 < \tau < \infty$, the weighted logarithmic τ-Bloch space $\mathscr{B}_{q,\log}^\tau$, of analytic functions f in $\mathbb{U}$ is defined by

$$||f||_{\mathscr{B}_{q,\log}^\tau} = \sup_{z\in\mathbb{U}} \left\{ |f'(z)| \frac{(1-|z|)^\tau}{q(1-|z|)} \log\left(\frac{1}{(1-|z|)} \right) \right\} \le c < \infty.$$

We have the following inclusions:

$$\mathscr{B} \subset \mathscr{B}_q \subset \mathscr{B}_q^\tau \subset \mathscr{B}_{q,\log}^\tau.$$

1.5 Fractional operators

The basic primer in classical calculus are started with derivatives and integrals functions and these two operations are inverse to each other in some sense. Fractional calculus is an operation covers these classical functions and their generalizations. There are various methods to define the fractional calculus in ordinary differential and integral equations (see Lovoie et al. (1976); Tremblay (1979); Samko et al. (1983); Hilfer (2000)). Specifically, fractional calculus operators play an important role in geometric function theory to define new generalized subclasses of analytic functions and then study their properties. For example, operators of the classical fractional calculus defined by Owa (1978) and Srivastava and Owa (1989) as follows.

Definition 1.10 *The fractional integral Srivastava–Owa operator of order* β *is defined for a function* f *by*

$$\mathscr{I}_z^{\beta} f(z) := \frac{1}{\Gamma(\beta)} \int_0^z f(\zeta)(z-\zeta)^{\beta-1} d\zeta, \tag{1.31}$$

where $0 < \beta$, *and the function* f *is analytic in simply-connected region of the complex* z*-plane containing the origin and the multiplicity of* $(z-\zeta)^{\beta-1}$ *is removed by requiring* $\log(z-\zeta)$ *to be real when* $(z-\zeta) > 0$.

Lemma 1.4 *For* $z \in \mathbb{U}$ *and* $f \in \mathscr{A}$, *then*

(i) $\mathscr{I}_z^0 f(z) = f(z)$.

(ii) $\mathscr{I}_z^{\beta}(b) = bz^{\beta}/\Gamma(\beta+1)$, *where b is a constant function.*

(iii) $\mathscr{I}_z^{\beta}\{z^m\} = \frac{\Gamma(m+1)}{\Gamma(m+\beta+1)} z^{m+\beta}, m > -1; 0 \leq \beta < 1, z \in \mathbb{U}$.

Definition 1.11 *The fractional derivative Srivastava–Owa operator of order* β *is defined by*

$$D_z^{\beta} f(z) := \frac{1}{\Gamma(1-\beta)} \frac{d}{dz} \int_0^z f(\zeta)(z-\zeta)^{-\beta} d\zeta, \tag{1.32}$$

where $0 \leq \beta < 1$, *and the function* f *is analytic in simply-connected region of the complex* z*-plane containing the origin and the multiplicity of* $(z-\zeta)^{-\beta}$. *is removed as in Definition 1.10 above.*

Lemma 1.5 *For* $z \in \mathbb{U}$ *and* $f \in \mathscr{A}$, *then*

(i) $D_z^0 f(z) = f(z)$.

(ii) $D_z^{\beta}(b) = bz^{-\beta}/\Gamma(1-\beta)$, *where b is a constant function.*

(iii) $D_z^{\beta}\{z^m\} = \frac{\Gamma(m+1)}{\Gamma(m-\beta+1)} z^{m-\beta}, m > -1; 0 \leq \beta < 1, z \in \mathbb{U}$.

Definition 1.12 *Under the conditions of Definition 1.11, the fractional derivative of order* β *is defined by*

$$D_z^{m+\beta} f(z) = \frac{d^m}{dz^m} D_z^{\beta} f(z). \tag{1.33}$$

1.6 Special functions

In this section, we will present some of special functions that used throughout this study. Most of these notions are taken from (Kilbas et al., 2006).

Gamma function: For complex argument with positive real, the Euler gamma function Γ is defined by

$$\Gamma(z) := \int_0^z e^{-t} t^{z-1} dt, \quad \Re(z) > 0. \tag{1.34}$$

Note that, the function Γ satisfies the functional equations

$$z\Gamma(z) = \Gamma(z+1) \quad \text{and} \quad z\Gamma(z-1) = \Gamma(z), \quad \Re\{z\} > 0.$$

In particular case, if $z = \omega$, then $\Gamma(\omega+1) = \omega!$, where $\omega \in \mathbb{N}_0 = \{0,1,2,\cdots\}$.

The binomial coefficients are represented by the gamma function for $\alpha, \omega \in \mathbb{C}$ by

$$\binom{\alpha}{\omega} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\omega+1)\Gamma(\omega+1)}, \quad \alpha \notin \{0,-1,-2,\cdots\}. \tag{1.35}$$

Beta function: The beta function $B(u,v)$ is a special function defined by

$$B(u,v) := \int_0^1 t^{u-1}(1-t)^{v-1} dt \quad \Re(u) > 0, \Re(v) > 0. \tag{1.36}$$

In terms of the gamma function, the beta function $B(.,.)$ satisfies the following identity

$$B(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)} \quad \Re(u) > 0, \Re(v) > 0. \tag{1.37}$$

The expression in (1.37) shows the close association between the analytical continuation of the beta and gamma functions to the entire complex plane $\mathbb{C}$.

Mittag-Leffler function: Mittag-Leffler type function is denoted by $E_{\alpha,\beta}(z)$, defined by Wiman in 1905 and studied by Humbert in 1953 (see Haubold et al. (2011)):

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}. \tag{1.38}$$

As important special cases of this function are mentioned below:

(i) when $\beta = 1$ with $\min\{\Re(\alpha)\} > 0$ the function (1.38) reduces to the one that has been defined by Mittag-Leffler in 1903 (see Kilbas et al. (2004))

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}, \tag{1.39}$$

(ii) when $\alpha = 1$ the function (1.39) reduces to the exponential function

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots.$$

The hypergeometric function and its generalization: Hypergeometric function and its generalization are considered the most important special functions, because of their many connections to other classes of special functions, and its numerous identities and expressions in terms of series and integrals. They were introduced by Gauss in 1866 (see Beukers (2014)), and he has proved to be of enormous significance in mathematics and the mathematical sciences elsewhere. Therefore, we describe some properties of hypergeometric functions which are useful for us to derive some of our main results.

For z, a, c in $\mathbb{C}$ and $c \neq \{0, -1, -2, \cdots\}$, the confluent hypergeometric (or Kummer) function is denoted by ${}_1F_1(a, c; z)$ and is defined by

$${}_1F_1(a,c;z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} \frac{z^k}{k!}. \tag{1.40}$$

If $\Re\{a\} > 0$ and $\Re\{c\} > 0$, the confluent hypergeometric can be represented as an integral

$${}_1F_1(a,c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 \nu^{a-1}(1-\nu)^{c-a-1} e^{z\nu} d\nu. \tag{1.41}$$

For a, b and c are complex numbers with c is neither zero nor a negative integer with $|z| < 1$, the Gaussian hypergeometric function is denoted by ${}_2F_1(a, b, c; z)$ and is defined by

$${}_2F_1(a,b,c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \cdot \frac{z^k}{k!}, \tag{1.42}$$

where $(a)_k$ is the *Pochhammer* symbol defined by

$$(\upsilon)_k := \frac{\Gamma(\upsilon+k)}{\Gamma(\upsilon)} = \begin{cases} \upsilon(\upsilon+1)...(\upsilon+k-1), & \text{if} \quad k = \upsilon \in \mathbb{N};\, \upsilon \in \mathbb{C}; \\ 1, & \text{if} \quad k = 0;\, \upsilon \in \mathbb{C} \setminus \{0\}, \end{cases} \tag{1.43}$$

and it follows that, $(1)_k = k!$ and $(\upsilon)_{k+1} = (\upsilon+k)(\upsilon)_k$.

Remark 1.1 *The series in equation* (1.42) *can be written as*

$${}_2F_1(a,b,c;z) = 1 + \frac{a \cdot b}{1 \cdot c} z + \frac{a(a+1)b(b+1)}{1 \cdot 2 \cdot c(c+1)} z^2 + \ldots \tag{1.44}$$

In special case, if $a=1$ and $b=c$, then the series (1.44) *takes the form*

$$_2F_1(1,b,c;z) = 1+z+z^2+z^3+\ldots,$$

which is a geometric series. Since (1.44) reduces to geometric series as a special case of (1.42), then (1.44) is called hypergeometric series. The series of the hypergeometric functions (1.42) is convergent for $|z|<1$. Moreover, the hypergeometric function is analytic in the open unit disk $\mathbb{U}$, and other useful properties of this function are

(i) If $0<b<c$, then the integral representation for the hypergeometric function is defined by

$$_2F_1(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt. \tag{1.45}$$

(ii) Differentiation of hypergeometric functions

$$\frac{d}{dz}{}_2F_1(a,b,c;z) = \frac{ab}{c}{}_2F_1(1+a,1+b,1+c;z). \tag{1.46}$$

(iii) The hypergeometric function can be given by

$$_2F_1(-a,b;c;1) = \frac{\Gamma(c-b+a)\Gamma(c)}{\Gamma(c-b)\Gamma(c+a)}$$

and from this, we have the Kummer's first formula (see Buchholz (2013)):

$$e^z{}_1F_1(a;c;-z) = {}_1F_1(c-a;c;z). \tag{1.47}$$

For complex parameters $\alpha_1,...,\alpha_p\,(\alpha_i \neq 0,-1,-2,...;i=1,...,p)$, and $\beta_1,...,\beta_s\,(\beta_j \neq 0,-1,-2,...;j=1,...,s)$, where $p,s\in\mathbb{N}_0 := \{0,1,...\}$ with $p\leq s+1$ and $|z|<1$, the generalized hypergeometric function denoted by ${}_pF_s(\alpha_1,\cdots,\alpha_p;\beta_1,\cdots,\beta_s;z)$ and defined by

$$_pF_s(\alpha_1,\alpha_2\ldots\alpha_p;\beta_1,\beta_2\cdots,\beta_s;z) = \sum_{k=0}^{\infty}\frac{(\alpha_1)_k,(\alpha_2)_k\cdots(\alpha_p)_k}{(\beta_1)_k,(\beta_1)_k\cdots(\beta_s)_k}\cdot\frac{z^k}{(k)!}. \tag{1.48}$$

Fox-Wright function ${}_p\Psi_s$: For complex parameters

$$\alpha_1\ldots\alpha_p\left(\frac{\alpha_i}{A_i}\neq\{0,1,2,..\};i=\{1,\ldots,p\}\right)$$

and

$$\beta_1\ldots\beta_s\left(\frac{\beta_j}{B_j}\neq\{0,1,2,..\};j=\{1,\ldots,s\}\right),$$

the generalized Wright function ${}_p\Psi_s$ is simply an extension of hypergeometric function

${}_pF_s$ function given in equation (1.48), defined by

$$
\begin{aligned}
{}_p\Psi_s\left[\begin{matrix}(\alpha_1,A_1),\ldots,(\alpha_p,A_p);\\(\beta_1,B_1),\ldots,(\beta_s,B_s);\end{matrix}z\right] &= {}_p\Psi_s\left[(\alpha_i,A_i)_{1,p},(\beta_j,B_j)_{1,s};z\right]\\
&= \sum_{k=0}^{\infty}\frac{\Gamma(\alpha_1+A_1k)\cdots\Gamma(\alpha_p+A_pk)}{\Gamma(\beta_1+B_1k)\cdots\Gamma(\beta_s+B_sk)}\frac{z^k}{k!}\\
&= \sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(\alpha_i+A_ik)}{\prod_{j=1}^{s}\Gamma(\beta_j+B_jk)}\frac{z^k}{k!}, \qquad (1.49)
\end{aligned}
$$

where $A_i>0$ for all $i=\{1,\ldots,p\}$, $B_j>0$ for all $j=\{1,\ldots,s\}$ and for suitable values $|z|$, such that

$$\sum_{j=1}^{s}B_j-\sum_{i=1}^{p}A_i\geq -1.$$

For the special case, where $A_i=1$ for all $i=\{1,\ldots,p\}$ and $B_j=1$ for all $j=\{1,\ldots,s\}$, where $p\leq s+1; p,s\in\mathbb{N}_0=\mathbb{N}\cup\{0\}; |z|<1$, then we have the following relationship:

$$ {}_pF_s(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_s;z)=\Delta_p\Psi_s[(\alpha_i,1)_{1,p};(\beta_j,1)_{1,s};z] \qquad (1.50)$$

where $\Delta=\frac{\Gamma(\alpha_1)\ldots\Gamma(\alpha_p)}{\Gamma(\beta_1)\ldots\Gamma(\beta_s)}$. Further, the generalized of Mittag-Leffler type function with two parameters $E_{\alpha,\beta}(z)$ can be viewed as a special case of the above function ${}_p\Psi_s$ in equation (1.49) that is

$$E_{\alpha,\beta}(z)={}_1\Psi_1\left[\begin{matrix}(1,1);\\(\alpha,\beta);\end{matrix}z\right].$$

Moment generating function: Let W be an one-dimensional random vector and let $F(w)$ be its distribution function. The Moment generating function of a random variable W defined in complex plane $\mathbb{C}$ by Curtiss (1942) as follows:

$$M_W(z)=E\left[e^{zW}\right]:=\int_{-\infty}^{\infty}e^{zw}dF(w)\quad z\in\mathbb{C}, \qquad (1.51)$$

in which $\Re(z)\geq 0$ and the integral is supposed to converge for z in some neighbourhood of the origin. In dealing with certain distribution problems, the function $M_W(z)$ has been widely used by statisticians.

1.7 Problem statement

This study addressed the following problems concerning fractional (differential, integral and mix integro-differential) operators with their generalizations, as well as linear operators in complex plane $\mathbb{C}$, which are summarized as follows:

On fractional operators:

Recently, Ibrahim and Jahangiri (2014) defined and extended fractional differential operator $\mathfrak{T}_z^{\alpha,\mu}$ in geometric function theory, and they left as an open problem in complex plane $\mathbb{C}$. Because the coefficient problem for various families of analytic and univalent functions is basically about the search for precise bounds of the coefficient a_k. So it is common to ask:

1. Is it possible to find the bounds coefficient of new classes involving a fractional differential operator $\mathfrak{T}_z^{\alpha,\mu}$ in the open unit disk $\mathbb{U}$?

Srivastava et al. (1989) proved that by using fractional calculus Srivastava–Owa operators and their properties, it could be defined new fractional operators which are the extension for some others, that leads to the following question.

2. Does the fractional differential operator $\mathfrak{T}_z^{\alpha,\mu}$ has a modification in $\mathscr{S}$?

It is well known that an operator is bounded in the open unit disk if and only if it is normalized (satisfied the conditions $f(0) = 0$ and $f'(0) = 1$), therefore, we have the following question

3. Is the modification of fractional differential operator bounded in some complex Banach spaces?

Breaz and Breaz (2002) proved an univalency condition of the integral operator introduced in Stanciu and Breaz (2014). May one ask

4. By utilizing mixed operators (differential and integral), can we impose a new operator; type the Breaz integral in the open unit disk $\mathbb{U}$?

Since every functions f in $\mathscr{S} \subset \mathscr{A}$ have inverse in $\mathscr{S}$, unquestionably, the following question is arises.

5. Can we extend the univalancy properties to cover some well known operators?

Ibrahim (2011) defined a generalization fractional calculus of Srivastava-Owa operators and studied some of their geometric properties. That leads to the following question.

6. Does the Srivistava-Owa operators define with two fractional parameters?

Moreover, the above question motivates us to some else.

7. Can we study the bounded and compact generalized fractional operators on some complex Banach spaces?

On linear operators (Convolution operators):

Carlson and Shaffer (1984) provided linear operator associated with Gaussian hypergeometric function. In recent study, Srivastava and Attiya (2007) have represented a linear operator with generalized Hurwizt-Lerch zeta function. Those new methods of defining linear operators associated with special functions in $|z| < 1$, motived us to ask

8. Can we apply the Mittag-Leffler function, to define a new linear operator?

The Moment generating function was extended in complex plane by Curtiss (1942). The following question raised

9. Can we apply the complex Moment generating function, to define new linear operators of analytic functions in the open unit disk?

On fractional analytic functions:

In geometric function theory there is a well known function, which is called *Koebe* function. Then may one ask

10. Is it possible to define the *Koebe* function with fractional power?

1.8 Research objectives

The objectives of this research are to study one of the substantive issues in many applications of geometric function theory in complex domain $\mathbb{U}$ to find solutions to the problem in section 1.7,

(i) to define new classes of analytic and univalent function with negative coefficients involving fractional differential operators.

(ii) to modify a fractional differential operator and study its geometrical and topological properties.

(iii) to derive new classes of analytic and univalent functions including mixed integro-differential operator type fractional and study their univalency properties.

(iv) to formulate a new fractional integral operator and to study its geometric interpolations.

(v) to study the boundedness and compactness properties on complex Banach spaces for new generalize certain fractional differential operator in the open unit disk.

(vi) to introduce a linear operator associated with Mittag-Leffler function and study its boundedness properties on weighted Bloch spaces.

(vii) to study some univalency properties for subclasses of univalent functions defined by making use the linear operators associated with Moment generating function.

(viii) to define new fractional classes of functional analytic univalent functions and study some their geometric properties in a complex domain $\mathbb{U}$.

1.9 Organization of the thesis

This thesis is organized in the following way. In Chapter 2, we provide an overview of topics, which are used to manage the research, the outcome, and contributions through this thesis

In Chapter 3, we present a systematic study of the various interesting geometric properties and applications of fractional differential operator $\mathfrak{T}_z^{\alpha,\mu}$ and prove several characterization theorems involving starlikeness and convexity of the function $\mathfrak{T}_z^{\alpha,\mu} f(z)$. Further, we define a modified fractional differential operator $\mathscr{T}^{\alpha,\mu}$ in the open unit disk $\mathbb{U}$ and estimate their upper bounds on some complex Banach spaces.

In Chapter 4, we provide a new integral operator involving modified fractional differential operator $\mathscr{T}_z^{\alpha,\mu} f(z)$ and study some its univalency properties such as starlikness and convexity in the open unit disk $\mathbb{U}$. Further, we present new classes of analytic univalent functions defined by making use a new definition of a generalized mixed integro-differential operator. Some applications on the norm pre-Schwarzian derivatives also introduced in the last of this chapter.

In Chapter 5, we define a fractional operator of analytic and univalent functions in terms of integral. We observe the relation between the fractional differential operator $\mathfrak{T}_z^{\alpha,\mu}$ defined in Chapter 3 with the fractional integral operator $\mathfrak{L}_z^{\alpha,\mu}$ defined in this chapter. As applications, we utilize some univalency properties to obtain examples associated with special functions. The boundedness and compactness properties for fractional integral operator on complex Bergman space of order $p\,(0 < p < 1)$ are studied. Furthermore, we introduce the univalency property for the modified fractional integral operator.

In Chapter 6, we define a new generalized fractional differential operator and study some its applications with several special functions in the open unit disk. Further, we prove the normalized fractional differential operator with its univalency (starlikeness and convexity) characteristics. In another hand, boundedness and compactness properties for the last mentioned operator on weighted Bloch spaces are introduced.

In Chapter 7, we define new linear operators associated with the well known special functions which are the Mittag-Leffler function and Moment generating function. Some applications such as boundedness on complex weighted Bloch spaces are studied. Further, finding the best estimate for analytic and univalent functions by using Fekete-Szegö method and Jack Lemma also are introduced.

In Chapter 8, we define fractional functional classes of analytic and univalent functions in the open unit disk $\mathbb{U}$. By utilizing this fractional functions, we define a new convolution operator, also we employ this operator to define new classes in such domain. The coefficient conditions with some applications for the new classes of analytic functions with negative coefficients are studied.

In Chapter 9, we provide a summary and future work of the research of this thesis.

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