UNIVERSITI PUTRA MALAYSIA

INTERVAL ITERATIVE METHODS ON SIMULTANEOUS INCLUSION OF POLYNOMIAL ZEROS

SYAIDA FADHILAH BT MOHAMMAD RUSLI

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By

SYAIDA FADHILAH BT MOHAMMAD RUSLI

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Doctor of Philosophy

April 2017

DEDICATION

to

My dear and beloved husband;

Nujaidi Israr Abdul Hadi,

My adorable daughters and sons;

Afnan Irdina, Adni Surfina, Adam Husaini and Ammar Hadeef,

My amazing parents;

Mardhiyah Haidir and Mohammad Rusli Yahaya,

My sweet sisters;

Lina Sofiyah, Eiffah Rodhiyah and Eizzah Rufaydah

and

all my family members...

Thank you for always being there for me through my ups and downs...

Abstract of thesis presented to the senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Doctor of Philosophy

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SYAIDA FADHILAH BT MOHAMMAD RUSLI

April 2017

Chairman : Mansor Bin Monsi, PhD
Faculty : Sciences

The main aim of the thesis is to modified procedures of bounding real zeros of polynomials simultaneously. For this purpose the interval approach is used in order to obtain faster and more accurate results. The research is based on the existing procedures: the interval symmetric single-step ISS1 and the interval repeated single-step IRSS1. To begin with, the basic concepts of interval computations and some brief introductions on Newton's method are provided.

The modifications done in this thesis can be grouped into two types. The first is the repeated procedures and the second type is the Newton's modified procedures. The modified procedures proposed consist of four repeated procedures and two Newton's modified procedures. The algorithms of these modified procedures are elaborated to show the significance of each procedure.

Theoretically, the analyses of inclusions for all procedures are presented to ensure the inclusions property of the procedures. In order to find the rate of convergence of the procedures, the analyses of *R*-order of convergence are discussed in detail. To obtain the numerical results, coding for the procedures are developed and implemented using the MATLAB R2007a combined with the Intlab toolbox. Numerical results are presented in terms of CPU times, number of iterations and the widths of final intervals to indicate the accuracies of the procedure.

For the conclusion, faster computational time and good accuracies are achieved from the new modified procedures. Furthermore, they attained higher rate of convergences than the existing procedures.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk Ijazah Doktor Falsafah

KAEDAH LELARAN SELANG DALAM MEMERANGKAP PENSIFAR NYATA POLINOMIAL SECARA SERENTAK

Oleh

SYAIDA FADHILAH BT MOHAMMAD RUSLI

April 2017

Pengerusi : Mansor Monsi, PhD
Fakulti : Sains

Tujuan utama tesis ini adalah untuk mengubahsuai prosedur yang menghadkan punca nyata polinomial secara serentak. Untuk tujuan ini pendekatan selang digunakan bagi mendapatkan keputusan yang lebih pantas dan tepat. Kajian ini adalah berasaskan prosedur sedia ada, prosedur selang langkah-tunggal bersimmetri ISS1 dan prosedur selang berulang langkah-tunggal bersimmetri IRSS1. Sebagai permulaan, konsep-konsep asas pengiraan selang dan sedikit pengenalan tentang kaedah Newton diberikan.

Pengubahsuaian yang dilakukan dalam tesis ini dapat dikumpulkan kepada dua jenis iaitu prosedur berulang dan prosedur Newton terubahsuai. Prosedur terubahsuai yang dicadangkan terdiri daripada empat prosedur berulang dan dua prosedur Newton terubahsuai. Algoritma bagi prosedur-prosedur terubahsuai ini diterangkan bagi menunjukkan sifat ketara bagi setiap prosedur.

Secara teori, analisis rangkuman bagi setiap prosedur dibentangkan bagi memastikan perangkuman setiap prosedur tersebut. Untuk mendapatkan kadar penumpuan sesuatu prosedur, analisis kadar penumpuan peringkat R dibincangkan dengan teliti. Bagi mendapatkan keputusan berangka, pengekodan algoritma bagi prosedur-prosedur dibina dan diimplimentasikan dengan menggunakan “MATLAB R2007a” digabungkan dengan “Intlab V5.5 toolbox”. Keputusan berangka dibentangkan dalam bentuk masa CPU, jumlah lelaran dan kelebaran selang terakhir yang menunjukkan kejituan sesuatu prosedur.

Sebagai kesimpulan, masa pengiraan yang lebih pantas dan kejituan yang baik dapat dicapai daripada prosedur-prosedur terubahsuai baru. Tambahan lagi, prosedur-prosedur ini mempunyai kadar penumpuan yang lebih tinggi berbanding prosedur sedia ada.

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I certify that a Thesis Examination Committee has met on 26 April 2017 to conduct the final examination of Syaida Fadhilah bt Mohammad Rusli on her thesis entitled "Interval Iterative Methods on Simultaneous Inclusion of Polynomial Zeros" in accordance with the Universities and University Colleges Act 1971 and the Constitution of the Universiti Putra Malaysia [P.U.(A) 106] 15 March 1998. The Committee recommends that the student be awarded the Doctor of Philosophy.

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF ABBREVIATIONS

R	Real numbers.
$I(R)$	Set of real numbers.
x^*	Zeros.
ξ	Zeros.
TP	Test polynomial.
n	Degree of polynomial.
m	Inner iteration.
k	Iteration.
r	Step number in algorithm.
t	CPU times.
i	Component.
$x_i^{(k)}$	Point x of component i at iteration k.
$X_i^{(k)}$	Interval X of component i at iteration k.
$X_i^{(0)}$	Initial interval X of component i.
$X^{(k,r)}$	Interval of iteration k, step r
x_{i_I}	Infimum of interval X of component i.
x_{i_S}	Supremum of interval X of component i.
$X \cap Y$	Intersection of interval X and Y.
$X \subseteq Y$	Inclusion of intervals.
$w_i^{(k)}$	Width of interval of component i at iteration k.
$m(X)$	Midpoint of interval X.
$p(x)$	Polynomial of x.
$P'(x_i^{(0)})$	Interval gradients of the polynomials of $x_i^{(0)}$.
$O_R(I, x^*)$	R-order of procedure I which converge to x^*.
$R_p(w^{(k)})$	R-factor of a null sequence $w^{(k)}$.
max	Maximum

min	Minimum
IT	Interval total step
IS	Interval single-step
ISS1	Interval symmetric single-step
IMSS1	Interval midpoint symmetric single-step
IZSS1	Interval zoro symmetric single-step
IMZSS1	Interval midpoint zoro symmetric single-step.
IRSS1	Interval repeated symmetric single-step.
IRMSS1	Interval repeated midpoint symmetric single-step.
IRZSS1	Interval repeated zorro symmetric single-step.
IRMZSS1	Interval repeated midpoint zorro symmetric single-step.
IMW	Interval symmetric Monsi-Wolfe.
INSMW	Interval Newton symmetric Monsi-Wolfe.
INMSMW	Interval Newton midpoint symmetric Monsi-Wolfe.
IRNSMW	Interval repeated Newton symmetric Monsi-Wolfe.

CHAPTER 1

INTRODUCTION

Polynomials are expressions that consist of constants, coefficients and variables or also called the indeterminate. Polynomials may involve the operations of additions, subtractions, multiplications and non-negative integer exponents. Polynomial zeros have many applications such as in control engineering, finance, economics and theoretical computer science. Many cases in our real life can be formed into polynomial form, from a simple daily life problem to some complicated situation. Thus, finding the zeros of the polynomial is very important to solve various kinds of problems. Basically, zeros finding method is a method for finding a value x such that $p(x) = 0$ for a given function p. Therefore the concept of finding zeros of a polynomial is synonym with solving an equation.

There are many studies on various methods used for finding the zeros of polynomials found in literature. In this study, we are focusing on iterative methods to find the zeros of the polynomial as this approach is effective and accurate.

1.1 Methods of Estimating Polynomial Zeros and Concept of Interval Computations

We consider a polynomial of degree $n > 1$ defined by

$$p_n(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0 \quad , \tag{1.1.1}$$

where $a_i \in R^1 (i = 1, \dots, n)$ are given.

This polynomial can be expressed as

$$p_n(x) = \prod_{i=1}^{n}(x - x_i). \tag{1.1.2}$$

Suppose that $x_i^* \in R$, for $i = 1, \dots, n$, $p_n(x)$ has n distinct zeros. This means we consider cases only for polynomials with distinct zeros and multiple zeros are not included. In addition, we only consider polynomials with real zeros and we use interval analysis to trap the zeros of the polynomials with great accuracies. Since our approach is in interval, we do not need to worry about the error because the zeros are bound in the final results.

In this research, the procedures start with some disjoint intervals $X_1^{(0)}, X_2^{(0)}, X_3^{(0)}, \dots, X_n^{(0)}$ as the initial intervals which contain the polynomial zeros. These initial intervals are obtained from the approximation values of the zeros. In order to obtain the approximate values of the zeros, there are several approaches we can apply. One of them is by graphical method where we can approximate the location of the zeros by plotting the graph of the polynomials. For the test polynomials, we use some characteristic polynomials in which the diagonal values represent the approximation of the zeros. For other standard form polynomials, we use MATLAB solver to locate where the zeros lie. These including intervals $X_j^{(0)} = \left[x_{jI}^{(0)}, x_{jS}^{(0)}\right] \ni \xi_j, 1 \le j \le n$ are pair wise disjoints, that is

$$X_j^{(0)} \cap X_k^{(0)} = \emptyset\,, 1 \le j < k \le n. \tag{1.1.3}$$

When the procedure is run over the initial intervals, smaller bounded closed intervals are determined where each of them is guaranteed to still contain the roots. In other words, the intervals sequences generated by the procedures are always converging to the zeros that is

$$X_i^{(0)} \supset X_i^{(1)} \supset X_i^{(2)} \supset \cdots \text{ with } \lim_{k\to\infty} X_i^{(k)} = \xi_i.$$

According to Monsi (1988), the interval arithmetic approach can be a principal and a necessary tool to determine the narrow computationally rigorous bounds on polynomial zeros as the widths of intervals are limited only by the precision of the machine floating point arithmetic.

In the next section, we will introduce the operation of the interval computation which are used in this thesis as well as the properties of the basic operations on real intervals. The aim is to understand the system of rules for calculating with intervals, sufficient for the application used in the thesis.

1.1.1 Operations of Interval Computations

$A = [a_1, a_2]$ is called a bounded closed real interval $I(R)$ where $a_1 \in R$ is the infimum of A or $i(A) = a_1$, and $a_2 \in R$ is the supremum of A or $s(A) = a_2$. The set of all closed real interval $I(R)$ is defined by

$$I(R) = \{A = [a_1, a_2] | a_1, a_2 \in R, a_1 \le a_2\}\ . \tag{1.1.1.1}$$

The members of $I(R)$ are denoted by the capital letters $A, B, C, \dots, X, Y, Z$. Real numbers $x \in R$ may be considered the special members $[x, x]$ from $I(R)$, and they are called point intervals.

Definition 1.1.1.1: (Alefeld and Herzberger, 1983)
Two intervals $A = [a_1, a_2]$ and $B = [b_1, b_2]$ are called equal that is $A = B$ if they are equal in the set theoretic sense. ■

From Definition 1.1.1.1, it follows that $A = B \Leftrightarrow a_1 = b_1\,, a_2 = b_2$.

The relation "=" between the two elements from $I(R)$ is reflexive, symmetric and transitive.

Definition 1.1.1.2: (Alefeld and Herzberger, 1983)
Let $* \in \{+,-,\cdot,/\}$ be a binary operation on the set of real numbers R. If $A, B \in I(R)$, then

$$A * B = \{z = a * b \mid a \in A, b \in B\} \tag{1.1.1.2}$$

defines a binary operation on the real interval $I(R)$ ■

From Definition 1.1.1.2, the operations on intervals $A = [a_1, a_2]$ and $B = [b_1, b_2]$ may be calculated explicitly as

$$A + B = [a_1 + b_1, a_2 + b_2] \quad , \tag{1.1.1.3}$$

$$A - B = [a_1 - b_2, a_2 - b_1] \quad , \tag{1.1.1.4}$$

$$A \cdot B = [\min\{a_1b_1, a_2b_2, a_2b_1, a_1b_2\}, \max\{a_1b_1, a_2b_2, a_2b_1, a_1b_2\}] \ , \tag{1.1.1.5}$$

and if $0 \notin B$ then

$$\frac{A}{B} = [a_1, a_2] \cdot \left[\frac{1}{b_2}, \frac{1}{b_1}\right],$$

$$= [\min\{a_1/b_2, a_1/b_1, a_2/b_2, a_1/b_1\}, \max\{a_1/b_2, a_1/b_1, a_2/b_2, a_1/b_1\}]. \tag{1.1.1.6}$$

Definition 1.1.1.3: (Monsi, 1988)
An interval $A \in I(R)$ is a degenerate (or is a point interval) if and only if $a_1 = a_2$. ■

The set $I_D(R)$ of degenerate intervals and the set R of real numbers are isomorphic. This permits a meaning to be given to $a * B$ ($a \in R$, $B \in I(R)$, $* \in \{+,-,\cdot,/\}$).

Definition 1.1.1.4: (Monsi, 1988)
If $a \in R$ and $B \in I(R)$ then

$$a + B = [a + b_1, a + b_2],$$
$$a - B = [a - b_2, a - b_1],$$
$$a \cdot B = [\min\{ab_1, ab_2\}, \max\{ab_1, ab_2\}],$$

and if $0 \notin B$,then

$$a/B = [a, a] \cdot \left[\frac{1}{b_2}, \frac{1}{b_1}\right] = [\min\{\frac{a}{b_2}, \frac{a}{b_1}\}, \max\{\frac{a}{b_2}, \frac{a}{b_1}\}]. \quad ■$$

Proposition 1.1.1.1: (Monsi, 1988)
Interval arithmetic is *inclusion monotonic* that is to say, if $A, B, C, D \in I(R)$ then ($\forall * \in \{+,-,\cdot,/\}$)

$$(A \subseteq C, B \subseteq D) \Rightarrow (A * B \subseteq C * D).$$

Proof
By Definition 1.1.1.2

$$\begin{aligned} A * B &= \{a * b \mid a \in A , b \in B \} \\ &\subseteq \{c * d \mid c \in C , d \in D \} \\ &= C * D \ \blacksquare \end{aligned}$$

Definition 1.1.1.5: (Monsi, 1988)
Let $A, B \in I(R)$ be given. Then the intersection $A \cap B$ of A and B is defined by

$$A \cap B = \{x \in R \mid x \in A , x \in B\}. \ \blacksquare$$

Proposition 1.1.1.2: (Monsi, 1988)
$(a)\big(\forall A, B \in I(R)\big)\, A \cap B = B \cap A$;
$(b)\big(\forall A, B \in I(R)\big)\, A \cap B \subseteq A, A \cap B \subseteq B$;
$(c)(A \cap B = A \Leftrightarrow A \subseteq B), (A \cap B = B \Leftrightarrow B \subseteq A)$.

Proof of (a)
By Definition 1.1.1.5

$$\begin{aligned} A \cap B &= \{ x \in R \mid x \in A, x \in B\} , \\ &= \{ x \in R \mid x \in B, d \in A\} , \\ &= B \cap A. \end{aligned}$$

Proof of (b)
By Definition 1.1.1.5

$$A \cap B = \{ x \in R \mid x \in A, x \in B\} .$$

So

$$(x \in A \cap B) \Rightarrow x \in A,$$

whence

$$A \cap B \subseteq A,$$

and

$$(x \in A \cap B) \Rightarrow x \in B,$$

whence

$$A \cap B \subseteq B,$$

Proof of (c)
By (b) $A \cap B \subseteq B$, so

$$(A \cap B = A) \Longrightarrow (A \subseteq B).$$

Conversely, if $A \subseteq B$ then by Definition 1.1.1.5

$$\begin{aligned} A \cap B &= \{ x \mid x \in A, x \in B\} \\ &= \{ x \mid x \in A\} \\ &= A. \end{aligned}$$

Therefore

$$(A \cap B = A) \Leftrightarrow (A \subseteq B).$$

Interchanging A and B and using (a), give

$$(A \cap B = A) \Leftrightarrow (B \subseteq A). \blacksquare$$

1.1.2 Properties of Interval Computations

Proposition 1.1.2.1: (Alefeld and Herzberger, 1983)
If A, B and C are members of the real interval $I(R)$. Then it follows that

(a) $(A+B)+C = A+(B+C)$ (associativity of addition);

(b) $(A \cdot B) \cdot C = A \cdot (B \cdot C)$ (associativity of multiplication);

(c) $A+B = B+A$ (commutativity of addition);

(d) $A \cdot B = B \cdot A$ (commutativity of multiplication);

Proposition 1.1.2.2: (Alefeld and Herzberger, 1983)
If $\underline{0} = [0,0]$ and $\underline{1} = [1,1]$, then X and Y are the unique neutral elements with respect to addition and multiplication, that is

$$(a)\ A = X + A = A + X \quad for\ all\ \ A \in I(R)\ \Leftrightarrow X = \underline{0}\,.$$

$$(b)\ A = Y \cdot A = A \cdot Y \quad for\ all\ \ A \in I(R)\ \Leftrightarrow Y = \underline{1}\,.$$

Proposition 1.1.2.3: (Alefeld and Herzberger, 1983)

$$(\big(A, B \in I(R)\big), (AB = 0)) \Rightarrow \big(A = \underline{0}\ \ or\ \ B = \underline{0}\big).$$

Proposition 1.1.2.4: (Alefeld and Herzberger, 1983)
Interval arithmetic is subdistributive; that is to say $(\forall A, B, C \in I(R))$

$$A(B+C) \subseteq AB + BC.$$

Proposition 1.1.2.5: (Alefeld and Herzberger, 1983)
Let$(A, B, C \in I(R))$be given. Then

$$(a)\ \ (A \pm C = B \pm C) \Rightarrow (A = B);$$
$$(b)\ \ (AC = BC) \nRightarrow (A = B);$$
$$(c)\ \ (A/C = B/C) \Rightarrow (A = B).$$

Definition 1.1.2.1: (Alefeld and Herzberger, 1983)
The distance between two intervals $A = [a_1, a_2]$, $B = [b_1, b_2] \in I(R)$ is defined as

$$q(A,B) = \max\{|a_1 - b_1|, |a_2 - b_2|\}. \blacksquare$$

Definition 1.1.2.2: (Alefeld and Herzberger, 1983)
The absolute value of an interval $A = [a_1, a_2] \in I(R)$ is defined as

$$|A| = q(A, [0,0]) = \max\{|a_1|, |a_2|\}. \blacksquare$$

Theorem 1.1.2.1 (Alefeld and Herzberger, 1983)
Let $A = [a_1, a_2],\ B = [b_1, b_2],\ C = [c_1, c_2],\ D = [d_1, d_2]\ \in I(R)$.Then
$(a)\ q(A + B = A + C) = q(B, C);$
$(b)\ q(A + B = C + D) \leq q(A, C) + q(B, D);$
$(c)\ q(aB, aC) = |a|q(B, C), \qquad a \in R;$
$(d)\ q(AB + AC) = |A|q(B, C).\ \blacksquare$

Definition 1.1.2.3: (Alefeld and Herzberger, 1983)
The width $w(A)$ of an interval $A = [a_1, a_2]$ where $A \in I(R)$ is defined by $w(A) = a_2 - a_1$. $\blacksquare$

Definition 1.1.2.4: (Alefeld and Herzberger, 1983)
The midpoint $m(A)$ of $A \in I(R)$ is defined by $m(A) = \frac{1}{2}(a_1 + a_2)$ $\blacksquare$

1.1.3 Interval Evaluation and Range of Real Functions

In this section, we consider f is a continuous function and an expression $f(x)$ is a calculating procedure that will determine a value of the function f for every argument x.

Alefeld and Herzberger (1983) stated that the expression

$$W\big(f, X; A^{(0)}, \dots, A^{(m)}\big) = \{f(x; a^{(0)}, \dots, a^{(m)}) | x \in X, a^{(k)} \in A^{(k)}, 0 \leq k \leq m\}$$
$$= [\min_{\substack{x \in X \\ a^{(k)} \in A^{(k)}, 0 \leq k \leq m}} f\big(x; a^{(0)}, \dots, a^{(m)}\big), \max_{\substack{x \in X \\ a^{(k)} \in A^{(k)}, 0 \leq k \leq m}} f(x; a^{(0)}, \dots, a^{(m)})],$$

denote the interval of all values of the function f when $x \epsilon X$, and $a^{(k)} \in A^{(k)}$, $0 \leq k \leq m$, are considered independent of each other. This definition is independent of the expression for f.

Theorem 1.1.3.2 (Alefeld and Herzberger, 1983)
Let f be the continuous function of the real variable $x^{(1)}, \dots, x^{(n)}$, and let $f(x^{(1)}, \dots, x^{(n)}; a^{(0)}, \dots, a^{(m)})$ be an expression for f. Also assume that the interval evaluation $f\big(Y^{(1)}, \dots, Y^{(n)}; B^{(0)}, \dots, B^{(m)}\big)$ is defined for the intervals $Y^{(1)}, \dots, Y^{(n)}; B^{(0)}, \dots, B^{(m)}$.

It then follows that

(a) for all

$$X^{(k)} \subseteq Y^{(k)},\ A^{(j)} \subseteq B^{(j)},\ 1 \le k \le n\ , \qquad 0 \le j \le m\,,$$

it holds that

$$(i) \quad W\big(f, X^{(1)}, \dots, X^{(n)}; A^{(0)}, \dots, A^{(m)}\big) \subseteq f\big(X^{(1)}, \dots, X^{(n)}; A^{(0)}, \dots, A^{(m)}\big) \text{(inclusion property);}$$

(b) for all

$$X^{(k)} \subseteq Z^{(k)} \subseteq Y^{(k)}\ ,\ A^{(j)} \subseteq C^{(j)} \subseteq B^{(j)}\,, \qquad 1 \le k \le n\ , \qquad 0 \le j \le m\,,$$

it holds that

$$(ii) \quad f\big(X^{(1)}, \dots, X^{(n)}; A^{(0)}, \dots, A^{(m)}\big) \subseteq f\big(Z^{(1)}, \dots, Z^{(n)}; C^{(0)}, \dots, C^{(m)}\big) \text{(inclusion monotonicity).} \ \blacksquare$$

1.2 Newton's Method

Newton's method is a well-known method widely used for solving equations. Whilst it might not be the most efficient nor the most robust method by itself, it is usually used together with other method that is globally convergent. The combination would help in giving the value of the initial guess, x_0 closer to the root ξ.

The Newton's method is given by

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} (i = 0,1, \dots, n). \qquad (1.2.1)$$

The iteration starts with an initial guess of the root of the function,x_0, then a function f defined over the real numbers x and the function's derivatives f'. In our case, the function f is the polynomial with degree *n*.

In this thesis, we will use this Newton's method at the beginning of the algorithm in Chapter 6, Chapter 7 and Chapter 8.

1.3 Problem Statement

In this research, there are two main ideas used to develop new improved procedures. First is by introducing the concept of inner iterations to the existing procedures in the literature. We include the process of inner iterations to the algorithm where the values of the inner iterations depend on the value of m. The value of m is set initially before the algorithm starts. The conventional approach is to add more steps to the algorithm of existing procedures like the idea of the interval symmetric single-step procedure ISS1 and the interval zorro symmetric single-step procedure IZSS1. Therefore, instead of adding more steps to the algorithm, we just set the value of m to its optimum value, for the algorithm to perform best. This idea is expected to save computational time because when the inner iteration is used, more calculated values can be re-used in the inner loop.

For the second idea, we add the Newton's method at the beginning of the procedure. Note that the Newton's method is calculated only once throughout the algorithm. This idea is expected to accelerate the process of bounding the intervals closer to the zeros. Therefore, the procedures will converge faster and lessen the computational time.

1.4 Objective of the Research

The main objective of the studies is to propose new procedures in bounding the polynomials zeros simultaneously by using interval arithmetic approach. By using the interval computation, we tend to achieve good accuracy as we can ensure a narrow computationally rigorous bound on the polynomial zeros. The objective of the thesis can be accomplished by:

1. constructing the interval repeated midpoint symmetric single-step procedure IRMSS1 based on the interval midpoint symmetric single-step procedure IMSS1 and the interval repeated single-step procedure IRSS1.
2. constructing the interval repeated zorro symmetric single-step procedure IRZSS1, based on the interval zorro symmetric single-step procedure IZSS1 and the interval repeated single-step procedure IRSS1.
3. constructing the interval repeated midpoint zorro symmetric single-step procedure IRMZSS1, based on the interval midpoint zorro symmetric single-step procedure IMZSS1 and the interval repeated single-step procedure IRSS1.
4. formulating the interval Newton symmetric Monsi-Wolfe procedure INSMW, by introducing the Newton's method into the interval symmetric single-step ISS1.
5. composing the interval repeated Newton symmetric Monsi-Wolfe procedure IRNSMW, based on the interval Newton symmetric Monsi-Wolfe procedure INSMW.

6. formulating the interval Newton midpoint symmetric Monsi-Wolfe procedure INMSMW, by introducing the Newton's method into the interval midpoint symmetric single-step IMSS1.
7. performing the analysis of inclusions for all proposed procedures to ensure the
convergences of the procedures and analysing the R-order of convergence for each modified procedure for comparisons.

1.5 Thesis Outline

In this thesis, we discuss the interval iterative procedures in bounding roots of polynomials simultaneously. We start this thesis with the introduction on the basic concept of interval computations that is used in obtaining the numerical results. We include the important operations and properties involved in the analysis process in this research as they are the keys for us to analyzing the R-order of convergence of the procedures and also to proceed with the modified procedures.

In chapter 2, we discuss the general idea of the R-order of convergence as explained in Ortega and Rheinboldt (1970). By referring to the idea, we analyze the R-order of convergence of all the modified procedures. We also include the brief discussions regarding the previous work which are the sources of the modified procedures.

Chapter 3 to 8 are the chapters that contain the detailed discussions on all six modified procedures. In chapter 3, we developed the first modified procedure named the interval repeated midpoint symmetric single-step IRMSS1. In this procedure, we apply the idea of using the updated midpoints in the procedure IRSS1 by Monsi (1988) and this procedure appears to be the repeated version of the procedure IMSS1. The value of inner iteration is dependent on the value of m and can be set at the beginning of the procedure.

Chapter 4 covers the modified procedure namely the interval repeated zorro symmetric single-step IRZSS1. For this modification, we apply the idea of using algorithm with zorro pattern likewise in procedure IZSS1 and include the inner iteration in the algorithm so that it will be the repeated version of the procedure IZSS1.

In chapter 5, the same concept of repeated version is applied but this time is on the algorithm of IMZSS1 and it is called the interval repeated midpoint zorro symmetric single-step procedure denoted as IRMZSS1.

In chapter 6, we derived the modified method namely the interval Newton symmetric Monsi-Wolfe procedure INSMW. For this procedure, we improve the procedure ISS1 by Monsi (1988), by adding the Newton method at the beginning of the procedure on the algorithm. This step will only be computed once in the procedure.

In Chapter 7, we will construct the repeated version INSMW which is called the interval repeated Newton symmetric Monsi-Wolfe procedure IRNSMW. Similar to the concept used in Chapter 3, Chapter 4 and Chapter 5, the inner iteration will be included in the algorithm of INSMW where the inner iteration depends on the value of m.

Next in chapter 8, we apply the idea of using the Newton method at the beginning of the IMSS1 procedure. In this procedure, we always use the updated midpoints for every step in the algorithm. This modification is called the interval Newton midpoint symmetric single-step procedure INMSMW.

In each chapter, we present the algorithms and the analysis of R-order of convergence for the modified procedures. All six modified procedures show improvements as they have better rate of convergence and are supported by CPU times and number of iterations. The numerical results obtained are shown in the form of tables and graphs at the end of each chapter. The stopping criteria used is $w_i^{(k)} \leq 10^{-12}$.

Finally in chapter 9, we conclude the results of our research and listed down some of the future work regarding this study.

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