UNIVERSITI PUTRA MALAYSIA

FRACTIONAL DIFFERENTIAL CALCULUS ON RIEMANNIAN MANIFOLDS AND CONVEXITY PROBLEMS

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By

AL-LEHABI WEDAD SALEH

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Doctor of Philosophy

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DEDICATIONS

To My Beloved Parents, Brothers and Sisters
To My Beloved Husband and Children

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Doctor of Philosophy

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February 2017

Chairman: Professor Adem Kılıçman, PhD
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A Riemannian manifold embodies differential geometry science. Moreover, it has many important applications in physics and some other branches of sciences. Based on the above perspectives, the present thesis focuses on the study of some new results relating concept geodesic-ray property, convexity and starshapedness in complete simply connected smooth Riemannian manifold without conjugate points. In addition, the above terms are studied in the Cartesian product of two complete simply connected smooth Riemannian manifolds without conjugate points. Furthermore, this thesis introduces the concept of geodesic strongly E-convex functions and geodesic E-b-vex functions and discusses some of their properties. Moreover, examples in nonlinear programming problems are used to illustrate the applications of the results.

Fractional calculus is a field of mathematics study that grows out of the traditional definitions of calculus integral and derivative operators in much the same way fractional exponents is an outgrowth of exponents with integer value. This thesis shows some results related to fractional Riemannian manifolds such as fractional connection, Torsion tensor of a fractional connection and difference tensor of two fractional connections. Moreover, area and volume on fractional differentiable manifolds are studied.

In conclusion, some new integral inequalities of generalized Hermite-Hadamard's type integral inequalities for generalized s-convex functions in the second sense on fractal sets are discussed. In addition, a new class of generalized s-convex functions in both senses on real linear fractal sets is defined. The definition of generalized s-convex functions in both senses on the co-ordinates on fractal sets and some of its properties

are studied. Some new inequalities for product of generalized s-convex functions on the co-ordinates on fractal sets are presented.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

KALKULUS PERBEZAAN PECAHAN KE ATAS MANIFOLD RIEMANNAN DAN MASALAH KECEMBUNGAN

Oleh

AL-LEHABI WEDAD SALEH

Februari 2017

Pengerusi: Profesor Adem Kılıçman, PhD
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Manifold Riemann merangkumi sains geometri pembezaan. Selain itu, ia mempunyai banyak aplikasi penting dalam fizik dan dalam beberapa cabang sains yang lain. Berdasarkan perspektif di atas, tesis ini memfokuskan kajian tentang beberapa dapatan baharu yang berkaitan dengan konsep sifat sinar geodesi, kecembungan, dan bentuk bebintang dalam manifold Riemann rata terkait mudah lengkap tanpa titik konjugat. Di samping itu, istilah di atas telah dikaji dalam produk Cartesian dua manifold Riemann rata terkait mudah tanpa titik konjugat. Seterusnya, tesis ini memperkenalkan konsep fungsi E-cembung terkuat geodesi dan fungsi E-b-vex geodesi dan membincangkan beberapa sifatnya. Selain itu, contoh-contoh dalam masalah pengaturcaraan tak linear digunakan untuk menggambarkan aplikasi dapatan tersebut.

Kalkulus pecahan merupakan bidang matematik yang berpunca daripada definisi tradisional mengenai kamiran kalkulus dan operator pembeza, seperti juga cara yang sama, eksponen pembezaan merupakan pertumbuhan keluar tentang eksponen dengan nilai integer. Tesis ini dikhususkan pada dapatan manifold Riemann, seperti perkaitan pecahan, tensor Torsion bagi perkaitan pecahan dan perbezaan tensor bagi dua perkaitan pecahan. Tambahan pula, kawasan dan isi padu ke atas manifold pembezaan pecahan telah dikaji.

Kesimpulannya, beberapa ketaksamaan kamiran baharu bagi ketaksamaan kamiran jenis Hermite-Hadamard teritlak bagi fungsi s-cembung teritlak dalam kedua-dua maksud tentang set fraktal linear sebenar telah dibincangkan. Di samping itu, kelas baharu fungsi s-cembung teritlak dalam kedua-dua maksud tentang set fraktal telah ditakrifkan.

Definisi fungsi s-cembung teritlak dalam kedua-dua maksud tentang kordinat ke atas set fraktal dan beberapa sifatnya telah dikaji. Beberapa ketaksamaan baharu untuk produkl fungsi s-cembung teritlak mengenai kordinat ke atas set fraktal telah dibentangkan.

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I certify that a Thesis Examination Committee has met on 7 February 2017 to conduct the final examination of Al Lehabi, Wedad Saleh Q on her thesis entitled "Fractional Differential Calculus on Riemannian Manifolds and Convexity Problems" in accordance with the Universities and University Colleges Act 1971 and the Constitution of the Universiti Putra Malaysia [P.U.(A) 106] 15 March 1998. The Committee recommends that the student be awarded the Doctor of Philosophy.

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Declaration by graduate student

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This is to confirm that:

- the research conducted and the writing of this thesis was under our supervision;
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LIST OF ABBREVIATIONS

$\mathbb{N}$	The natural numbers
$\mathbb{R}$	The real numbers
$\mathbb{R}^{\alpha}$	Real line numbers on a fractal space
$\mathbb{R}^{n}$	n-dimensional Euclidean space
$\mathbb{R}_{+}$	The set of positive real number
$\mathbb{C}$	The complex numbers
C^{1}	The set of all functions which have continuous first order derivatives
C^{2}	The set of all functions which have continuous second order derivatives
C^{∞}	The set of infinitely differentiable functions
S^{1}	1-dimensional sphere (unit circle)
S^{2}	2-dimensional sphere (unit sphere)
S^{n}	n-dimensional sphere
M,N	n-dimensional differentiable manifold
$M_1 \times M_2$	The Cartesian product of two C^{∞} Riemannian manifolds
TM	Tangent bundle
$T_pM = M_p$	Tangent space of differentiable manifold M at $p \in M$
$\Im(M)$	Set of differentiable vector fields on manifold M
$\Im^{\alpha}(M)$	Set of fractional vector fields on manifold M
$\Im^{\alpha}_{U}$	The fractional vector fields on U
$[X,Y]$	Lie bracket
$[X^{\alpha},Y^{\alpha}]$	Fractional bracket
$\bigtriangledown$	Connection
$\bigtriangledown^{\alpha}$	The fractional covariant derivative
$Tor(X,Y)$	Torsion tensor of $\bigtriangledown$
$\tilde{T}(X^{\alpha},Y^{\alpha})$	Torsion tensor of $\bigtriangledown^{\alpha}$
$B(X,Y)$	The difference tensor of $\bigtriangledown$
$\tilde{B}(X^{\alpha},Y^{\alpha})$	The difference tensor of $\bigtriangledown^{\alpha}$
$\langle X,Y \rangle$	The inner product of vectors X and Y
$Int(B)$	The interior of B
∂B	The boundary of B
B^{c}	The complement of B
$\bar{B}$	The closure of B
$CH(B)$	The convex hull of B
$kerS$	Kernel of S
∂_i	The partial derivative operators $\frac{\partial}{\partial x_i}$
∂_i^{α}	Fractional derivative operators $\frac{\partial^{\alpha}}{\partial x_i^{\alpha}}$
δ^{i}_{j}	Kronecker's delta
$\tilde{\Gamma}^{j}_{ik}$	Chistoffel symbols of $\bigtriangledown^{\alpha}$

W^n	A C^∞ complete simply connected n-dimensional Riemannian manifold without conjugate points
t-convex set	Totally convex set
g-convex function	Geodesic convex function
SEC	Strongly E-convex
$sSEC$	Semi strongly E-convex
GEC	Geodesic E-convex
$GsEC$	Geodesic semi E-convex
$GSEC$	Geodesic strongly E-convex
$epi(f)$	Epigraph of function f
$\Gamma(x)$	Gamma (Euler's Gamma) function
$\beta(x,y)$	Beta function
K_s^1	s-Convex function in the first sense
K_s^2	s-Convex function in the second sense
E_s^1	s-Convex function in the first sense on $[a_1,a_2]\times[b_1,b_2]$
E_s^2	s-Convex function in the second sense on $[a_1,a_2]\times[b_1,b_2]$
MWO_{s_1,s_2}^1	s-Convex function in the first sense with $s=\frac{s_1+s_2}{2}$
MWO_{s_1,s_2}^2	s-Convex function in the second sense with $s=\frac{s_1+s_2}{2}$
GK_s^1	Generalized s-convex function in the first sense
GK_s^2	Generalized s-convex function in the second sense
GE_s^1	Generalized s-convex function in the first sense on $[a_1,a_2]\times[b_1,b_2]$
GE_s^2	Generalized s-convex function in the second sense on $[a_1,a_2]\times[b_1,b_2]$
GK_{s_1,s_2}^1	Generalized s-convex function in the first sense with $s=\frac{s_1+s_2}{2}$
GK_{s_1,s_2}^2	Generalized s-convex function in the second sense with $s=\frac{s_1+s_2}{2}$
GRP	Geodesic-ray property
$L^1([a_1,a_2])$	The space of integrable functions on $[a_1,a_2]$
$d(x,A)$	The distance from a point to a set $d(x,A)=\inf_{y\in A} d(x,y)$
$A(.,.)$	The arithmetic mean
$G(.,.)$	The geometric mean
$K(.,.)$	The quadratic mean
$\sup(.,.)$	Supremum
$\max(.,.)$	Maximum
$\min(.,.)$	Minimum

CHAPTER 1

INTRODUCTION

1.1 General Introduction

Manifolds crop up every where in mathematics. These generalizations of curves and surfaces to arbitrarily many dimensional space and provide the mathematical context for understanding space in all of its manifestations. Today, the tools of manifold theory are indispensable in most major subfields of pure mathematics and are becoming increasingly important in such diverse fields as genetics, robotics, econometrics, statistics, computer, graphics, and of course, the undisputed leader a mong consumers of mathematics-theoretical physics. There are many types of manifolds such as manifolds without conjugate points, manifolds without focal points, manifolds with non-positive curvature and manifolds with positive curvature.

As in Euclidean geometry, the concept of convexity and starshapedness play an important role also in Riemannian geometry, since in a Riemannian manifold geodesic joining two given points is not necessarily unique, the situation is somewhat complicated.

Integer order derivative and integer order integral have geometrical interpretation. Also, they have been applied in every field of sciences, engineering and mathematics as a tool to solve the diverse problems. A different approach to geometrical interpretation of fractional integration and fractional differentiation have been suggested by several authors (Podlubny et al. (2007) and Machado (2003)).

1.2 Problem Statement

There is no doubt that a Riemannian manifold plays an important role in differential geometry science see (Do Carmo Valero, 1992). In addition, it has many important applications in physics and some other branches of sciences see (Gutkin, 2011) and (An et al., 2013). Due to these reasons, we introduce some new definitions which help us to focus on convexity in Riemannian manifold and we need more discussion on the importance of relationship between convexity and starshapedness in Riemannian manifold. Next, the fractional differential calculus on a differentiable manifold is studied in (Albu and Opris, 2008) and (Jumarie, 2013). Even though fractional calculus is a highly useful and important topic, however the research on the geometric interpretation and applications are limited in current literature. Thus, in this study we focus on the area and volume on fractional differentiable manifold and discuss some related properties. Finally, since there are some important inequalities in plane which have not been studied in fractal sets, then one of the problem statements in this thesis is devoted to the introduction and discussion of several integral inequalities of generalized Hermite-Hadamard's type for generalized s-convex functions on fractal sets.

1.3 Research Objectives

Based on the identified problem, the objectives of this investigation are:

1. To introduce a new definition of geodesic-ray property in a C^∞ complete Riemannian manifold and the relation of geodesic-ray property with convexity and starshapedness.

2. To study geodesic-ray property, convexity and starshapedness in the Cartesian product of two complete simple connected smooth Riemannian manifolds without conjugate points.

3. To establish and present the basic facts and results of convex and generalized convex functions defined on Riemannian manifold.

4. To show possible definition for fractional connection in differential manifold and find many new desirable properties of Torsion tensor of a fractional connection and difference tensor of two fractional connections.

5. To modify the definition of s-convex functions on the co-ordinates on set in order to generalize some results in the plane to fractal sets and obtain several new integral inequalities of generalized Hermite-Hadamard's type for generalized s-convex function on fractal sets.

1.4 Research Methodology

A description of the methodology is as follows:

1. Problem identification and literature review: A comprehensive literature review on the related problems.

2. Mathematical analysis of formulated problems: Different types of invariants will be created and their behaviors investigated.

3. Result and discussion: Papers have been prepared and published after completion of the analysis of the results and discussion.

1.5 Outline of the Thesis

Below is an outline of this thesis organization, which consists of seven chapters.

Chapter 1 gives a general introduction of the research work where the motivation and objectives are defined.

Chapter 2 focuses on the previous work done by other researchers. Some historical background on Riemannian geometry is presented in this chapter. Also, fractional differential calculus in manifold is mentioned. The development of ideas that leads to generalized s-convex functions in both senses of fractal sets is outlined.

In Chapter 3, geodesic-ray property is defined. Futhermore, this chapter proposes and proves some new results related to geodesic-ray property, convexity and starshapedness in the Cartesian product of two complete simple connected smooth Riemannian manifolds without conjugate points. A finite collection

$$\mathbb{A} = \{A_i(x) \colon x \in S \backslash kerS, A_i(x) \nsubseteq A_j(x) \ \forall 1 \leq i, j \leq n\}$$

whose union is S and intersection is $kerS$ is proved.

Chapter 4 is devoted to the study of geodesic strongly E-convex functions and geodesic E-b-vex functions. Motivated by Youness and Emam (2005b) and Mishra et al. (2011), this chapter introduces the above functions on Riemannian manifold. In addition, some of their basic properties are established. At the end of this chapter an optimization problem is considered.

Chapter 5 deals with fractional differential calculus on Riemannian manifold. In this chapter, several new definitions, properties and examples are added which are related to fractional differential calculus on Riemannian manifold.

Chapter 6 includes the proposal of some new definitions, which are: A new class of generalized s-convex functions in both senses on fractal sets, generalized s-convex functions in both senses on the co-ordinates on fractal sets. A number of new integral inequalities of generalized Hermite-Hadamard's type for generalized s-convex functions in the second sense on fractal sets are presented.

Chapter 7 concludes with some important points arising from the research describes in this thesis and provides directions for further investigations.

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