UNIVERSITI PUTRA MALAYSIA

NUMERICAL METHODS FOR SOLVING OSCILLATORY AND FUZZY DIFFERENTIAL EQUATIONS

ALI KARIMI DIZICHEH

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By

ALI KARIMI DIZICHEH

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Doctor of Philosophy

June 2014

DEDICATIONS

To Hazrate FATEMEH ZAHRA (Salamollah alayha),
daughter of Prophet MOHAMMAD (Salavatollah alayhe va Aleh)

To Hazrate ABALFAZL ABAS (Alyhesalam),
son of Hazrate ALI (Salavatollah alayh)

To My Mother and Father

To My Teachers

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Doctor of Philosophy

NUMERICAL METHODS FOR SOLVING OSCILLATORY AND FUZZY DIFFERENTIAL EQUATIONS

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Chair: Professor Fudziah Ismail, Ph.D.

Faculty: Institute of Mathematical Research

In this thesis we develop five numerical schemes for solving ordinary differential equations. These include exponentially-fitted Runge-Kutta method, trigonometrically fitted hybrid method, Legendre wavelet method on large intervals as well as an iterative spectral collocation method, exponentially-fitted fuzzy Runge-Kutta method, exponentially-fitted system of fuzzy Runge-Kutta method. The stability analysis, estimation of local truncation errors and the efficiency of the methods' implementation in computer programs are discussed.

An exponentially-fitted explicit Runge-Kutta method of algebraic order 4 is formulated for the first-order ordinary differential equations

$$y' = f(x, y),\ y(x_0) = y_0.$$

It integrates exactly the first-order systems where their solutions are expressed as linear combinations of $\{\exp(wx), \exp(-wx)\}$ or $\{\cos(\lambda x), \sin(\lambda x)\}$ where $w = \lambda i$. Stability analysis of our approach as well as a good estimation for the local truncation errors are presented. The efficiency of the exponentially-fitted Runge-Kutta method is tested via some numerical experiments and a comparison with other existing methods.

A trigonometrically fitted explicit hybrid three-stage method is derived for the second-order initial value problems with oscillatory solutions. We compare our results with the classical hybrid method and the trigonometrically fitted explicit Runge-Kutta method through several examples. Our results indicate that trigono-

metrically fitted explicit hybrid method is more efficient than the classical hybrid method. we analyze the stability, phase- lag (dispersion) and dissipation.

An iterative spectral collocation method are introduced for solving initial value problems defined on large intervals. Indeed, the Legendre wavelet method is extended and proved valid for large interval. Then, the Legendre-Guass collocation points of the Legendre wavelets are computed. By employing an interpolation based on Legendre wavelet, we find approximate solution for any order (first-order and second-order) differential equations. Using this strategy the iterative spectral method converts the differential equation to a set of algebraic equations. Solving this set of algebraic equations yields an approximate solution.

Using exponentially-fitted Runge-Kutta (EFRK) method, we develop a method for numerically solving fuzzy first order linear and nonlinear differential equations under generalized differentiability. In addition, this method is applied for the system of first order fuzzy differential equations with uncertainty. The generalized Hukuhara differentiability are applied to estimate the solutions. For solving the fuzzy problems, the exponentially-fitted Runge-Kutta method is applied.

Finally, some examples are solved to illustrate our proposed approaches. The results are compared with those in the literature. We show that our proposed methods are simple and more accurate than the other existing methods.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

KAEDAH BERANGKA UNTUK PENYELESAIAN AYUNAN DAN PERSAMAAN PEMBEZAAN SAMAR

Oleh

ALI KARIMI DIZICHEH

Jun 2014

Pengerusi: Professor Fudziah Ismail, Ph.D.
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Dalam tesis ini kita akan membangunkan lima skim berangka bagi menyelesaikan masalah pembezaan biasa persamaan. Ini termasuk pesat dipasang kaedah Runge-Kutta, secara trigonometri kaedah hibrid dipasang, Legendre kaedah ombak kecil di selang besar serta kaedah penempatan bersama spektrum lelaran, pesat dipasang kabur Runge-Kutta kaedah, sistem pesat dipasang kabur kaedah Runge-Kutta.

Analisis kestabilan, anggaran ralat pangkasan setempat dan kecekapan pelaksanaan kaedah dalam program komputer yang dibincangkan. Satu yang jelas kaedah Runge-Kutta pesat dipasang perintah algebra 4 dirumus untuk pertama-perintah persamaan pembezaan biasa

$$y' = f(x, y),\ y(x_0) = y_0.$$

Ia menggabungkan betul-betul sistem tertib pertama di mana penyelesaian mereka dinyatakan sebagai gabungan linear bagi $\{eks(wx), eks(-wx)\}$ atau $\{kos(\lambda x), sin(\lambda x)\}$ di mana $w = i\lambda$.

Analisis kestabilan pendekatan kami dan juga anggaran yang baik bagi tempatan ralat pangkasan dibentangkan. Keberkesanan kaedah Runge-Kutta eksponen dipasang diuji melalui beberapa ujikaji berangka dan perbandingan dengan kaedah lain yang sedia ada. A secara trigonometri dipasang jelas hibrid kaedah tiga peringkat berasal bagi tertib kedua masalah nilai awal dengan penyelesaian ayunan. Kami membandingkan keputusan kami dengan kaedah hibrid klasik dan jelas kaedah Runge-Kutta yang secara trigonometri dipasang melalui beberapa contoh. Keputusan kami menunjukkan bahawa secara trigonometri dipasang kaedah hibrid eksplisit adalah lebih cekap daripada kaedah hibrid klasik. Kita menganalisis kestabilan, fasa lag (penyebaran) dan pelepasan.

Satu kaedah penempatan bersama spektrum lelaran diperkenalkan untuk menyelesaikan nilai awal masalah ditakrifkan pada selang yang besar. Malah, kaedah wavelet Legendre dilanjutkan dan terbukti sah untuk tempoh yang besar. Kemudian, Legendre-Guass titik penempatan bersama daripada riak Legendre dikira. Dengan menggunakan satu interpolasi berdasarkan Legendre ombak kecil, kita mencari penyelesaian anggaran untuk perintah (pertama-perintah dan tertib kedua) persamaan pembezaan. Menggunakan strategi ini kaedah spektrum lelaran menukarkan persamaan pembezaan untuk satu set persamaan algebra. Menyelesaikan set persamaan algebra menghasilkan penyelesaian hampir. Menggunakan pesat dipasang Runge-Kutta (EFRK) kaedah, kami membangunkan satu kaedah yang untuk menyelesaikan secara berangka kabur linear peringkat pertama dan persamaan pembezaan linear bawah kebolehbezaan umum. Di samping itu, kaedah ini digunakan untuk sistem perintah pertama persamaan pembezaan kabur dengan ketidakpastian.

Teritlak Hukuhara kebolehbezaan digunakan bagi menganggarkan penyelesaian. Untuk menyelesaikan masalah kabur, kaedah Runge-Kutta eksponen dipasang digunakan. Akhir sekali, beberapa contoh diselesaikan untuk menggambarkan pendekatan dicadangkan kami. Keputusan berbanding dengan mereka dalam kesusasteraan. Kita menunjukkan bahawa kaedah yang dicadangkan kami adalah mudah dan lebih tepat berbanding kaedah lain yang sedia ada.

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I certify that a Thesis Examination Committee has met on 17 June 2014 to conduct the final examination of Ali Karimi Dizicheh on his thesis entitled "Numerical Methods for Solving Oscillatory and Fuzzy Differential Equations" in accordance with the Universities and University Colleges Act 1971 and the Constitution of the Universiti Putra Malaysia [P.U.(A) 106] 15 March 1998. The Committee recommends that the student be awarded the Doctor of Philosophy.

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This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Doctor of Philosophy.

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DECLARATION

Declaration by graduate student

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CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF ABBREVIATIONS

IVP	Initial Value Problem
LTE	Local Truncation Error
EFRKMB	Exponentially-Fitted Runge-Kutta Method
RKMS	Exponentially-Fitted using Simos's technique
ODEs	Ordinary Differential Equations
PDEs	Partial Differential Equations
FDE	Fuzzy Differential Equation
HMC	Hybrid Classical Method
HMB3	Hybrid Method three stage using Vanden Berghe Technique
HMB2	Hybrid Method two stage using Vanden Berghee Technique
RKM	Fourth-order Runge-Kutta Method
RK44M	Runge-Kutta Fourth-order Four-stage Method
LWSM	Legndre Wavelets Spectral Method
FCC	Fuzzy Constant Coefficients
VIM	Variational Iteration Method

CHAPTER 1

INTRODUCTION

1.1 Background

Many research results have been reported in the area of the numerical solution of initial value problems in the last century. Particularly, problems exhibit a pronounced oscillatory property have taken a special attention. Such problems often arise in the filed of celestial mechanics, astrophysics, electronics and molecular dynamic, etc. There are many different approaches to deal with such problems. However, it is of central importance for an approach to specifically address the structure of the problem and its physical solutions.

Numerous numerical methods have been developed to approximate solutions of differential systems while a challenging important task is to preserve reasonable bounds for errors. Recent developments in computer sciences and technology have given the theory of numerical analysis a given momentum.

There exist ways for finding analytical solutions for certain simple ordinary differential equations. However, most models of real life problems are modeled by nonlinear differential equations which there does not exist any approach for finding analytic solutions. Furthermore, any analytic solution must be implemented in a computer for most real applications. Therefore, the most and final practical solutions are the numeric solutions.

Generally, there are two types of numerical methods, single-step methods and multistep methods. The single-step method assumes an initial point and approximates a new solution in a one step process. However, the multistep methods comprise of consecutive phases such that in each phase, the previous solutions and their associated derivatives are used. Therefore, the multistep methods require more initial points in which they are usually determined via a one-step approach.

Recently fuzzy systems and their associated solutions have been introduced in order to study problems with real conditions. The systems of first order differential equations under fuzzy conditions, have been treated in this thesis under generalized Hukuhara differentiability.

1.1.1 Initial value problem

We deal with the initial value problem of second order ordinary differential equations

$$y'' = f(x, y), \quad y(x_0) = y_0, \quad y'(x_0) = y'_0, \tag{1.1}$$

where y stands for the vector valued function with $y = [y_1, y_2, \ldots, y_n]$ and $y' = [y'_1, y'_2, \ldots, y'_n]$.

Theorem 1.1.1 *(Lambert 1991) Assume that the function $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous for all (x, y) in the region D; where D is given by $a \leq x \leq b$, $-\infty < y < \infty$. Further, assume that f satisfies the Lipschitz condition corresponding with the Lipschitz constant L, i.e.,*

$$\|f(x, y) - f(x, \hat{y})\| \leq L\|y - \hat{y}\| \tag{1.2}$$

holds for every $(x, y) \in D$, $(x, \hat{y}) \in D$. Then, there exists a unique continuous and differentiable solution $y(x)$ for all $(x, y) \in D$ associated with the initial problem (1.1).

For the proof of Theorem (1.1.1) see Henrici (1962) and Lambert (1991).

1.2 Runge-Kutta method

The general s-stage Runge-Kutta method is defined by

$$y_{n+1} = y_n + h \sum_{i=1}^{s} b_i f(x_n + c_i h, Y_i) \tag{1.3}$$

where

$$\begin{aligned} Y_1 &= y_n; \\ Y_i &= y_n + h \sum_{j=1}^{s} a_{ij} f(x_n + c_j h, Y_j), \quad i = 2, \ldots, s \\ c_i &= \sum_{j=1}^{i-1} a_{ij} \end{aligned}$$

In this research, the explicit Runge-Kutta method of concern have the form

$$y_{n+1} = y_n + h \sum_{i=1}^{4} b_i f(x_n + c_i h, Y_i) \tag{1.4}$$

where

$$\begin{aligned} Y_1 &= y_n; \\ Y_2 &= y_n + h a_{21} f(x_n + c_1 h, Y_1) \\ Y_3 &= y_n + h(a_{31} f(x_n + c_1 h, Y_1) + a_{32} f(x_n + c_2 h, Y_2)) \\ Y_4 &= y_n + h(a_{41} f(x_n + c_1 h, Y_1) + a_{42} f(x_n + c_2 h, Y_2) + +a_{43} f(x_n + c_3 h, Y_3)) \end{aligned}$$

Table 1.1: Butcher tableau fourth order explicit Runge-Kutta method

$$\begin{array}{c|cccc}
0 & & & & \\
\frac{2}{5} & \frac{2}{5} & & & \\
\frac{3}{5} & -\frac{3}{20} & \frac{3}{4} & & \\
1 & \frac{19}{44} & -\frac{15}{44} & \frac{10}{11} & \\
\hline
 & \frac{11}{72} & \frac{25}{72} & \frac{25}{72} & \frac{11}{72}
\end{array} \tag{1.5}$$

The method chosen in this case is the fourth order explicit Runge-Kutta method by Dormand (1996) which is given in Table 1.1.

1.2.1 Local truncation error and order conditions

Using Table 1.1, we have $c_1 = 0, c_2 = \frac{2}{5}, c_3 = \frac{3}{5}, c_4 = 1$;
$a_{21} = \frac{2}{5}; a_{31} = -\frac{3}{20}; a_{32} = \frac{3}{4}$; $a_{41} = \frac{19}{44}; a_{42} = -\frac{15}{44}; a_{43} = \frac{10}{11}$
$b_4 = \frac{11}{72}$; $b_1 = b_4$, $b_3 = b_2$; $b_2 = \frac{1}{2} - b_4$;
Consider in Taylor series around x_n, i.e.

$$y(x_{n+1}) = y(x_n) + hy'(x_n) + \frac{h^2}{2}y''(x_n) + \frac{h^3}{6}y'''(x_n) + \frac{h^4}{24}y^{(4)} + \dots$$

where

$$\begin{aligned}
y'(x_n) &= f \\
y''(x_n) &= f_x + f.f_y \\
y'''(x_n) &= f_{xx} + 2f.f_{xy} + f^2.f_{yy} + f_y.(f_x + f.f_y) \\
y^{(4)}(x_n) &= f_{xxx} + 3f.f_{xxy} + 3f_x.f_{xy} + 5f.f_y.f_{xy} + 3f^2.f_{xyy} + 3f.f_x.f_{yy} \\
&\quad +4f^2.f_y.f_{yy} + f^3.f_{yyy} + f_y.f_{xx} + f_x.f_y^2 + f_y^3.f
\end{aligned}$$

Also we used Taylor expansion for a function of two variables around the point $(x_n,\ y_n)$. At first we put

$$T(s,r) = \frac{s^3}{6}f_{xxx} + \frac{s^2 r}{2}f_{xxy} + \frac{sr^2}{2}f_{xyy} + \frac{r^3}{6}f_{yyy}$$

then

$$\begin{aligned}
k_1 &= f \\
k_2 &= f + hc_2 f_x + (Y_2 - y_n)f_y + \frac{1}{2}c_2^2 h^2 f_{xx} + c_2 h(Y_2 - y_n)f_{xy} + \frac{1}{2}(Y_2 - y_n)^2 f_{yy} \\
&\quad +T(c_2 h, Y_2 - y_n) \\
&= f + c_2 h f_x + ha_{21} f.f_y + \frac{1}{2}c_2^2 h^2 f_{xx} + (ha_{21}f)hc_2 f_{xy} + \frac{1}{2}(ha_{21}f)^2 f_{yy} \\
&\quad +T(c_2 h, ha_{21} f)
\end{aligned}$$

and

$$\begin{aligned}
k_3 &= f + hc_3 f_x + (Y_3 - y_n) f_y + \frac{1}{2} c_3^2 h^2 f_{xx} + c_3 h (Y_3 - y_n) f_{xy} \\
&+ \frac{1}{2} (Y_3 - y_n)^2 f_{yy} + T(c_3 h, Y_3 - y_n)
\end{aligned}$$

where

$$Y_3 = y_n + h(a_{31} k_1 + a_{32} k_2).$$

also

$$\begin{aligned}
k_4 &= f + hc_4 f_x + (Y_4 - y_n) f_y \\
&+ \frac{1}{2} c_4^2 h^2 f_{xx} + c_4 h (Y_4 - y_n) f_{xy} + \frac{1}{2} (Y_4 - y_n)^2 f_{yy} \\
&+ T(c_4 h, Y_4 - y_n)
\end{aligned}$$

where

$$Y_4 = t_4 y_n + h(a_{41} k_1 + a_{42} k_2 + a_{43} k_3).$$

Finally we have

$$y_{n+1} = y_n + h(b_1 k_1 + b_2 k_2 + b_3 k_3 + b_4 k_4).$$

Now

$$\begin{aligned}
LTE &= y_{n+1} - y(x_{n+1}) \qquad (1.6) \\
&= (-f + b_1 f + b_2 f + b_3 f + b_4 f) h \\
&+ [-\frac{f_x}{2} - \frac{f.f_y}{2} + b_2 (c_2 f_x + a_{21} f.f_y) \\
&+ b_3 (c_3 f_x + (a_{31} f + a_{32} f) f_y) + b_4 (c_4 f_x + (a_{41} f + a_{42} f + a_{43} f) f_y)] h^2 \\
&+ [-\frac{f_{xx}}{6} - \frac{f f_{xy}}{3} - \frac{f^2.f_{yy}}{6} - \frac{f_x.f_y}{6} - \frac{f.f_y^2}{6} + b_2 (\frac{c_2^2 f_{xx}}{2} + c_2 a_{21} f.f_{xy} \\
&+ \frac{a_{21}^2 f^2.f_{yy}}{2}) + b_3 (a_{32} (c_2 fx + a_{21} f.fy) f_y + \frac{c_3^2 f_{xx}}{2} + c_3 (a_{31} f + a_{32} f) f_{xy} \\
&+ \frac{1}{2} (a_{31} f + a_{32} f)^2 f_{yy}) + b_4 ((a_{42} (c_2 fx + a_{21} f.f_y) \\
&+ a_{43} (c_3 f_x + (a_{31} f + a_{32} f) f_y)) f_y + \frac{1}{2} c_4^2 f_{xx} \\
&+ c_4 (a_{41} f + a_{42} f + a_{43} f) f_{xy} + \frac{1}{2} (a_{41} f + a_{42} f + a_{43} f)^2 f_{yy})] h^3 + \ldots.
\end{aligned}$$

Let

$$\begin{aligned}
s_1 &= \sum_{i=1}^{4} b_i - 1 \,, \\
s_2 &= \sum_{i=2}^{4} b_i c_i - \frac{1}{2} \,, \\
s_3 &= \sum_{i=2}^{4} b_i c_i^2 - \frac{1}{3} \,, \\
s_4 &= b_3 a_{32} c_2 + b_4 a_{42} c_2 + b_4 a_{43} c_3 - \frac{1}{6}, \\
s_5 &= \sum_{i=2}^{4} b_i c_i^3 - \frac{1}{4} \,,
\end{aligned}$$

$$
\begin{aligned}
s_6 &= b_3a_{32}a21 + b_4a_{42}a_{21} + b_4a_{43}a_{31} + b_4a_{43}a_{32} - \frac{1}{6};\\
s_7 &= b_3a_{32}c_3c_2 + b_4a_{42}c_2c_4 + b_4a_{43}c_3c_4 - \frac{1}{8};\\
s_8 &= b_3a_{32}c_2^2 + b_4a_{42}c_2^2 + b_4a_{43}c_3^2 - \frac{1}{12};\\
s_9 &= b_4a_{43}a_{32}c_2 - \frac{1}{24}.
\end{aligned}
$$

Setting $s_i, i = 1, \ldots, 9$ to zero, we obtain the order conditions:

$$
\begin{aligned}
\textstyle\sum_{i=1}^4 b_i &= 1\ ,\\
\textstyle\sum_{i=2}^4 b_ic_i &= \frac{1}{2}\ ,\\
\textstyle\sum_{i=2}^4 b_ic_i^2 &= \frac{1}{3}\ ,\\
\textstyle\sum_{i=2}^4 b_ic_i^3 &= \frac{1}{4}\ ,\\
b_3a_{32}c_2 + b_4a_{42}c_2 + b_4a_{43}c_3 &= \frac{1}{6},\\
b_3a_{32}a21 + b_4a_{42}a_{21} + b_4a_{43}a_{31} + b_4a_{43}a_{32} &= \frac{1}{6};\\
b_3a_{32}c_3c_2 + b_4a_{42}c_2c_4 + b_4a_{43}c_3c_4 &= \frac{1}{8};\\
b_3a_{32}c_2^2 + b_4a_{42}c_2^2 + b_4a_{43}c_3^2 &= \frac{1}{12};\\
b_4a_{43}a_{32}c_2 &= \frac{1}{24}.
\end{aligned}
$$

Here we use the values of Table 1.1 in (1.6). Then, using Maple code, the coefficients of h^3, h^4 are zero. Now, the coefficient of h^5 is obtained as follow:

$$
\begin{aligned}
C_5 = {} & \frac{1}{30}.f_{yy}.f.f_{xx} + \frac{13}{1800}.f_y.f_{xxx} + \frac{1}{15}.f.f_{xy}^2 + \frac{5}{176}.f_{yy}.f_x^2\\
& + \frac{1}{30}.f_{xy}.f_{xx} + \frac{163}{1320}f_{yy}.f_x.f.f_y + \frac{91}{880}.f_{yy}.f^2.f_y^2\\
& + \frac{103}{1800}.f_y.f^3.f_{yyy} + \frac{43}{600}.f_y.f.f_{xxy} + \frac{73}{600}.f_y.f^2.f_{xyy}\\
& + \frac{1}{10}.f_{xy}.f^2.f_{yy} + \frac{1}{30}.f^3.f_{yy}^2 + \frac{1}{15}.f_y.f_x.f_{xy} + \frac{1}{12}.f.f_y^2.f_{xy}\\
& + \frac{1}{20}.f_{xxy}.f_x + \frac{1}{120}.f_y^2.f_{xx} + \frac{1}{10}.f.f_{xyy}.f_x + \frac{1}{20}.f^2.f_{yyy}.f_x.
\end{aligned}
$$

1.2.2 Stability

Assume that $H = \lambda h$ and

$$y' = -\lambda y, \quad \lambda > 0. \tag{1.7}$$

Hence,

$$y = c_1 \exp(-\lambda x),$$

for a constant c_1. We use the Runge-Kutta method (1.4) for equation (1.7) and obtain

$$\begin{aligned} y_{n+1} &= y_n - H\sum_{i=1}^{s} b_i Y_i, \\ Y_i &= y_n - H\sum_{j=1}^{s} a_{i,j} Y_j, \quad i = 1, \dots, s. \end{aligned}$$

Alternatively, we have

$$y_{n+1} = y_n - Hb^t Y, \tag{1.8}$$
$$Y = ey_n - HAY, \tag{1.9}$$

where $e = (1, 1, \dots, 1)^t$ and $Y = (Y_1, Y_2, \dots, Y_s)^t$. From (1.8), we have

$$Y = (I + HA)^{-1} e y_n. \tag{1.10}$$

Next

$$\begin{aligned} y_{n+1} &= y_n - Hb^t Y = y_n - Hb^t (I + HA)^{-1} e y_n \\ &= (1 - Hb^t (I + HA)^{-1} e) y_n = R(H) y_n, \end{aligned}$$

for

$$R(H) = (1 - Hb^t (I + HA)^{-1} e).$$

The associated sequence is bounded if and only if

$$|R(H)| \le 1. \tag{1.11}$$

We define the corresponding stability region by

$$S = \{H \in \mathcal{C} | \ |R(H)| \le 1\}. \tag{1.12}$$

1.3 Hybrid method

Coleman (2003) studied the following class of hybrid method:

$$y_{n+1} = 2y_n - y_{n-1} + h^2 \sum_{i=1}^{s} b_i f(x_n + c_i h, Y_i), \quad x_n = x_0 + nh \tag{1.13}$$

$$Y_i = (1 + c_i) y_n - c_i y_{n-1} + h^2 \sum_{j=1}^{s} a_{i,j} f(x_n + c_j h, Y_j), \quad i = 1, \dots, s.$$

The Butcher tableau represents the following where $b^t = [b_1, b_2, \dots, b_s]$, $A = [a_{ij}]$ and $c^t = [c_1, \dots, c_s]$.

Table 1.2: Butcher tableau fourth order explicit Runge-Kutta method

$$\begin{array}{c|c} c & A \\ \hline & b^t \end{array}$$

In this research, our hybrid method follow

$$\begin{aligned} Y_1 &= y_n,\ Y_2 = y_{n-1} \\ Y_i &= (1+c_i)y_n - c_i y_{n-1} + h^2 \sum_{j=3}^{s} a_{i,j} f(x_n + c_j h, Y_j), \quad i = 3, \ldots, s, \\ y_{n+1} &= 2y_n - y_{n-1} + h^2 \sum_{i=1}^{s} b_i f(x_n + c_i h, Y_i). \end{aligned} \tag{1.14}$$

Here, f_{n-1} and f_n stands for $f(t_{n-1}, y_{n-1})$ and $f(t_n, y_n)$, respectively. These approaches are explicit and have $s-1$ function evaluations or phases in each stages of integration. Franco (2006) was the first who introduced this class of explicit hybrid method. They are a subclass of methods defined in equation (1.13), by taking $c_1 = -1$, $c_2 = 0$, $a_{21} = 0$ and $a_{ij} = 0$ for $j > i$.

1.3.1 Local truncation error and order conditions

Consider the autonomous scalar differential equation

$$y'' = f(y). \tag{1.15}$$

Taking a few consecutive differentiations gives rise to

$$\begin{aligned} y''' &= y' f'(y), \\ y^{(4)} &= y'' f'(y) + {y'}^2 f''(y), \\ y^{(5)} &= y''' f'(y) + 3y' f(y) f''(y) + {y'}^3 f'''(y), \\ y^{(6)} &= f'(y)f'(y)f(y) + 5f''(y)f'(y)(y')^2 + 3y'f(y)f''(y) + 5f''(y)f(y)(y')^2 \\ &\quad + f^{(4)}(y)(y')^4. \end{aligned} \tag{1.16}$$

The class of hybrid method (1.14) is applied to solve (1.15), and thereby,

$$y_{n+1} = 2y_n - y_{n-1} + h^2\Phi.$$

Here,

$$\Phi = b_1 f(y_{n-1}) + b_2 f(y_n) + \sum_{i=3}^{s} b_i f(Y_i).$$

Let the exact solution of (1.15)

$$y(x_{n+1}) = 2y(x_n) - y(x_{n-1}) + h^2\Delta.$$

Next, the local truncation error (LTE) follows

$$LTE = h^2(\Phi - \Delta). \tag{1.17}$$

Order conditions are defined as relationships between coefficients of a method which cause successive terms in a Taylor expansion of the local truncation error to vanish (Coleman 2003).
We demonstrate on how to get the order conditions by means of an example. Suppose that we want to obtain the order conditions of the problem (1.15) for hybrid method defined in (1.14) with two-stage. We expand $f(y_{n-1})$ and $f(Y_3)$ as a Taylor series about $h = 0$ giving

$$\begin{aligned}
\Phi &= (b_1+b_2+b_3+b_4)f(y_n) + [b_4 f'(y_n)c_4 y_n' + b_3 f'(y_n)c_3 y_n' \\
&\quad -b_1 f'(y_n)y_n']h + \{b_3[-\frac{1}{2}f'(y_n)c_3 y'' + f'(y_n)a_{31}f(y_n) \\
&\quad +f'(y_n)a_{32}f(y_n) + \frac{1}{2}f''(y_n)c_3^2(y_n')^2] + b_4[-\frac{1}{2}f'(y_n)c_4 y'' \\
&\quad +f'(y_n)a_{41}f(y_n) + f'(y_n)a_{42}f(y_n) \\
&\quad +f'(y_n)a_{43}f(y_n) + \frac{1}{2}f''(y_n)c_4^2(y_n')^2] \\
&\quad +b_1[\frac{1}{2}f'(y_n)y_n'' + \frac{1}{2}f''(y_n){y_n'}^2]\}h^2 + \ldots \qquad (1.18) \\
f(y_{n-1}) &= f(y_n) - f'(y_n)y_n'h + [\frac{1}{2}f'(y_n)y_n'' + \frac{1}{2}f''(y_n){y_n'}^2]h^2 \\
&\quad -\frac{1}{6}(y^{(3)}f'(y) + 3y'f(y)f''(y) + {y'}^3 f'''(y))h^3 + \ldots \\
f(Y_4) &= f(y_n) + f'(y_n)c_4 y_n'h + [-\frac{1}{2}f'(y_n)c_4 y'' \\
&\quad +f'(y_n)a_{41}f(y_n) + f'(y_n)a_{42}f(y_n) + f'(y_n)a_{43}f(y_n) \\
&\quad +\frac{1}{2}f''(y_n)c_4^2(y_n')^2]h^2 + \ldots
\end{aligned}$$

$$\begin{aligned}
f(Y_3) &= f(y_n) + f'(y_n)c_3 y_n'h + [-\frac{1}{2}f'(y_n)c_3 y'' + f'(y_n)a_{31}f(y_n) \\
&\quad +f'(y_n)a_{32}f(y_n) + \frac{1}{2}f''(y_n)c_3^2(y_n')^2]h^2 + \ldots
\end{aligned}$$

Taylor series expansions of $y(x_{n-1})$ and $y(x_{n+1})$ about $h = 0$ are given by

$$\begin{aligned}
y(x_{n+1}) &= y(x_n) + hy'(x_n) + \frac{h^2}{2}y''(x_n) + \frac{h^3}{6}y'''(x_n) + \frac{h^4}{24}y^{(4)}(x_n) + \ldots \\
y(x_{n-1}) &= y(x_n) - hy'(x_n) + \frac{h^2}{2}y''(x_n) - \frac{h^3}{6}y'''(x_n) + \frac{h^4}{24}y^{(4)}(x_n) + \ldots
\end{aligned}$$

and thus, the corresponding expansion for $y(x_{n+1}) - 2y(x_n) + y(x_{n-1})$ is

$$y(x_{n+1}) - 2y(x_n) + y(x_{n-1}) = h^2 y''(x_n) + \frac{h^4}{12}y^{(4)}(x_n) + \frac{h^6}{360}y^{(6)}(x_n)\ldots$$

and hence,

$$\Delta = y''(x_n) + \frac{h^2}{12}y^{(4)}(x_n) + \frac{h^4}{360}y^{(6)}(x_n)\ldots \qquad (1.19)$$

Assume that $y(x_n) = y_n$. Then (1.18) and (1.19) are substituted into the right side of equation (1.17). Next, (1.15) and derivative terms such as those given (1.16) are used to obtain

$$
\begin{aligned}
LTE &= h^2(b_1+b_2+b_3+b_4-1)f(y_n)+h^3(b_4c_4+b_3c_3-b_1)f'(y_n)y_n' \\
&\quad +h^4[(-\frac{b_4c_4}{2}-\frac{b_3c_3}{2}+b_4a_{41}+b_4a_{42}+b_4a_{43}+b_3a_{31}+b_3a_{32} \\
&\quad +\frac{b_1}{2}-\frac{1}{12})f'(y_n)f(y_n)+(\frac{b_4c_4^2}{2}+\frac{b_3c_3^2}{2}+\frac{b_1}{2}-\frac{1}{12})f''(y_n)(y_n')^2]
\end{aligned} \tag{1.20}
$$

Let

$$
\begin{aligned}
t_1 &= b_1+b_2+b_3+b_4-1, \\
t_2 &= b_4c_4+b_3c_3-b_1, \\
t_3 &= -\frac{b_4c_4}{2}-\frac{b_3c_3}{2}+b_4a_{41}+b_4a_{42}+b_4a_{43}+b_3a_{31}+b_3a_{32} \\
&\quad +\frac{b_1}{2}-\frac{1}{12}, \\
t_4 &= \frac{b_4c_4^2}{2}+\frac{b_3c_3^2}{2}+\frac{b_1}{2}-\frac{1}{12}.
\end{aligned}
$$

Setting t_1 and t_2 to zero, we obtain the order conditions:

$$
\begin{aligned}
b_1+b_2+b_3+b_4 &= 1, \\
b_4c_4+b_3c_3-b_1 &= 0
\end{aligned}
$$

If the above conditions are fulfilled, then t_3 reduces to

$$t_3 = b_4a_{41}+b_4a_{42}+b_4a_{43}+b_3a_{31}+b_3a_{32}-\frac{1}{12}$$

Setting t_3 to zero, we get the order condition:

$$b_4a_{41}+b_4a_{42}+b_4a_{43}+b_3a_{31}+b_3a_{32}-\frac{1}{12}=0.$$

Finally, by assuming that

$$
\begin{aligned}
b_1+b_2+b_3+b_4 &= 1, \\
b_4c_4+b_3c_3-b_1 &= 0, \\
b_4a_{41}+b_4a_{42}+b_4a_{43}+b_3a_{31}+b_3a_{32}-\tfrac{1}{12} &= 0.
\end{aligned}
$$

The following order conditions of order 2 to 9 are listed as given by Coleman (2003).

Order 2:

$$\sum_{i=1}^{s} b_i = 1 \tag{1.21}$$

Order 3:

$$\sum_{i=1}^{s} b_ic_i = 0 \tag{1.22}$$

Order 4:

$$\sum_{i=1}^{s} b_i c_i^2 = \frac{1}{6} \tag{1.23}$$

$$\sum_{j=1}^{s}\sum_{i=1}^{s} b_i a_{ij} = \frac{1}{12} \tag{1.24}$$

Order 5:

$$\sum_{i=1}^{s} b_i c_i^3 = 0 \tag{1.25}$$

$$\sum_{j=1}^{s}\sum_{i=1}^{s} b_i c_i a_{ij} = \frac{1}{12} \tag{1.26}$$

$$\sum_{j=1}^{s}\sum_{i=1}^{s} b_i a_{ij} c_j = 0 \tag{1.27}$$

Order 6:

$$\sum_{i=1}^{s} b_i c_i^4 = \frac{1}{15} \tag{1.28}$$

$$\sum_{j=1}^{s}\sum_{i=1}^{s} b_i c_i^2 a_{ij} = \frac{1}{30} \tag{1.29}$$

$$\sum_{j=1}^{s}\sum_{i=1}^{s} b_i c_i a_{ij} c_j = \frac{-1}{60} \tag{1.30}$$

$$\sum_{i=1}^{s}\sum_{j=1}^{s}\sum_{k=1}^{s} b_i a_{ij} a_{ik} = \frac{7}{120} \tag{1.31}$$

$$\sum_{j=1}^{s}\sum_{i=1}^{s} b_i a_{ij} c_j^2 = \frac{1}{180} \tag{1.32}$$

$$\sum_{i=1}^{s}\sum_{j=1}^{s}\sum_{k=1}^{s} b_i a_{ij} a_{jk} = \frac{1}{360} \tag{1.33}$$

Order 7:

$$\sum_{i=1}^{s} b_i c_i^5 = 0 \tag{1.34}$$

$$\sum_{j=1}^{s}\sum_{i=1}^{s} b_i c_i^3 a_{ij} = \frac{1}{30} \tag{1.35}$$

$$\sum_{j=1}^{s}\sum_{i=1}^{s} b_i c_i^2 a_{ij} c_j = 0 \tag{1.36}$$

$$\sum_{i=1}^{s}\sum_{j=1}^{s}\sum_{k=1}^{s} b_i c_i a_{ij} a_{ik} = \frac{1}{30} \tag{1.37}$$

$$\sum_{j=1}^{s}\sum_{i=1}^{s} b_i c_i^2 a_{ij} c_j = 0 \tag{1.38}$$

$$\sum_{i=1}^{s}\sum_{j=1}^{s}\sum_{k=1}^{s} b_i c_i a_{ij} a_{jk} = \frac{-1}{720} \quad (1.39)$$

$$\sum_{j=1}^{s}\sum_{i=1}^{s} b_i c_i a_{ij} c_j^2 = \frac{1}{72} \quad (1.40)$$

$$\sum_{i=1}^{s}\sum_{j=1}^{s}\sum_{k=1}^{s} b_i a_{ij} a_{ik} c_k = \frac{-1}{120} \quad (1.41)$$

$$\sum_{j=1}^{s}\sum_{i=1}^{s} b_i a_{ij} c_j^3 = 0 \quad (1.42)$$

$$\sum_{i=1}^{s}\sum_{j=1}^{s}\sum_{k=1}^{s} b_i a_{ij} c_j a_{jk} = \frac{1}{360} \quad (1.43)$$

$$\sum_{i=1}^{s}\sum_{j=1}^{s}\sum_{k=1}^{s} b_i a_{ij} a_{jk} c_k = 0. \quad (1.44)$$

1.3.2 Phase-lag and stability analysis

Let $H = \lambda h$ and consider the test problem

$$y'' = -\lambda^2 y, \quad \lambda > 0, \quad (1.45)$$

where the exact solution is given by

$$y = c_1 \exp(\lambda x) + c_2 \exp(-\lambda x)$$

with c_1 and c_2 are constants. Applying the hybrid method (1.13) to equation (1.45) yields

$$\begin{aligned} y_{n+1} &= 2y_n - y_{n-1} - H^2 \sum_{i=1}^{s} b_i Y_i, \\ Y_i &= (1 + c_i) y_n - c_i y_{n-1} - H^2 \sum_{j=1}^{s} a_{i,j} Y_j, \quad i = 1, \ldots, s, \end{aligned}$$

which can be expressed in vector form as

$$y_{n+1} = 2y_n - y_{n-1} - H^2 b^t Y \quad (1.46)$$

$$Y = (e + c) y_n - c y_{n-1} - H^2 A Y, \quad (1.47)$$

where $e = (1, 1, \ldots, 1)^t$ and $Y = (Y_1, Y_2, \ldots, Y_s)^t$. From (1.46), we get

$$Y = (I + H^2 A)^{-1} (e + c) y_n - (I + H^2 A)^{-1} c y_{n-1} \quad (1.48)$$

Then, substituting (1.48) into (1.47) give us

$$y_{n+1} - S(H^2) y_n + P(H^2) y_{n-1} = 0 \quad (1.49)$$

where $S(H^2) = 2 - H^2 b^t (I + H^2 A)^{-1} (e + c)$ and $P(H^2) = 1 - H^2 b^t (I + H^2 A)^{-1} c$.

The phase-lag and stability properties of the hybrid method (1.13) are determined by the characteristic equation

$$\xi^2 - S(H^2)\xi + P(H^2) = 0 \tag{1.50}$$

which is associated with (1.49). The study of phase-lag has been initiated by Brusa and Nigro (1980) in which the phase-lag was introduced as the truncation error on exponentials.

According to Van der Houwen and Sommeijer (1987), in the phase analysis, one compares the phase(s) (or argument(s)) of $\exp(\pm iH)$ with the principal root(s) of the characteristic polynomial. Thus, phase-lag is defined as the difference

$$t = H - \theta(H)$$

where H is the phase of $\exp(\pm iH)$ and is the phase of the principal root of (1.50). In order to determine , assume that the characteristic polynomial (1.50) has complex roots. Thus, the discriminant should be negative which means that:

$$(S(H^2))^2 - 4P(H^2) < 0.$$

It is noted that

$$\sqrt{(S(H^2))^2 - 4P(H^2)} = i\sqrt{4P(H^2) - (S(H^2))^2}$$

and therefore the principal root may be written as

$$r_1 = \sqrt{S(H^2)}2 + i\sqrt{4P(H^2) - (S(H^2))^2}.$$

Thus,

$$\tan(\theta(H)) = \frac{\sqrt{4P(H^2) - (S(H^2))^2}}{S(H^2)}$$

and

$$\cos(\theta(H)) = \frac{S(H^2)}{2\sqrt{P(H^2)}}.$$

Hence the phase of r_1 is

$$\theta(H) = \arccos\left(\frac{S(H^2)}{2\sqrt{P(H^2)}}\right).$$

The following is the definition according to the formula and concept given by Franco (2006).

Definition 1.3.1 *For the hybrid method corresponding to the characteristic polynomial* (1.50), *the quantity*

$$\phi(H) = H - \arccos\left(\frac{S(H^2)}{2\sqrt{P(H^2)}}\right)$$

is called phase-lag (or dispersion error) while the quantity

$$d(H) = 1 - \sqrt{P(H^2)}$$

is called dissipation (or amplification) error. A hybrid method is said to have the phase-lag of order n *if* $\phi(H) = O(H^{n+1})$. *If* $P(H^2) = 1$ *then* $d(H) = 0$ *and the method having this property is said to be zero dissipative or dissipative of order infinity. Conversely, if* $P(H^2) \neq 1$ *then we should have* $d(H) = O(H^{m+1})$. *The method with this property is said to be dissipative of order* m.

Definition 1.3.2 *For the hybrid method corresponding to the characteristic polynomial* (1.50)*, the interval* $(0, H_p)$ *is called the interval of periodicity if*

$$P(H^2) = 1 \text{ and } |S(H^2)| < 2, \quad \forall\ H \in (0, H_p).$$

Definition 1.3.3 *For the hybrid method corresponding to the characteristic polynomial* (1.50)*, the interval* $(0, H_a)$ *is called the interval of absolute stability if*

$$|P(H^2)| < 1 \text{ and } |S(H^2)| < 1 + P(H^2), \quad \forall\ H \in (0, H_a)$$

For the hybrid method defined in (1.14)*,* $S(H^2)$ *and* $P(H^2)$ *are polynomials in* H^2.

1.4 Legendre polynomial

In Kajani and Vencheh (2004), The Legendre polynomials are obtained by the recursive formulas:

$$\begin{aligned} L_0(t) &= 1, \\ L_1(t) &= t, \\ L_{m+1}(t) &= \frac{2m+1}{m+1} t L_m(t) - \frac{m}{m+1} L_{m-1}(t), \quad m = 1, 2, \ldots. \end{aligned} \tag{1.51}$$

In Chapter V, the Legendre polynomials are applied for solving differential equations with initial values on large intervals.

1.5 Fuzzy differential equations

Here, we recall some basic definitions and results by Georgiou et al. (2005) which will be used later. The set of all real numbers is indicated by $\mathbb{R}$. A fuzzy number is a mapping $u : \mathbb{R} \rightarrow [0, 1]$ with the following properties:

1. u is upper semi-continuous,
2. u is fuzzy convex, i.e., $u(\lambda x + (1-\lambda)y) \geq min\{u(x), u(y)\}$, $\forall\ x, y \in \mathbb{R}$, $\lambda \in [0, 1]$,
3. u is normal, i.e., $\exists x_0 \in \mathbb{R}$ for which $u(x_0) = 1$,

4. *supp* $u = \{x \in \mathbb{R} | u(x) > 0\}$ is the support of the u, and its closure cl(supp u) is compact.

Let $\mathbb{E}$ be the set of all fuzzy number on $\mathbb{R}$. The α-level set of a fuzzy number $u \in \mathbb{E}, 0 \le \alpha \le 1$, denoted by $[u]_\alpha$, is defined as

$$[u]_\alpha = \begin{cases} \{x \in \mathbb{R} | u(x) \ge \alpha\} & if\ 0 < \alpha \le 1 \\ cl(supp\ u) & if\ \alpha = 0. \end{cases}$$

It is clear that the α-level set of a fuzzy number is a closed and bounded interval $[\underline{u}(\alpha), \overline{u}(\alpha)]$,where $\underline{u}(\alpha)$ denotes the left-hand endpoint of $[u]_\alpha$ and $\overline{u}(\alpha)$ denotes the right-hand endpoint of $[u]_\alpha$. Since each $y \in \mathbb{R}$ can be regarded as a fuzzy number $\tilde{y}$ defined by

$$\tilde{y}(t) = \begin{cases} 1 & if \quad t = y, \\ 0 & if \quad t \neq y, \end{cases}$$

$\mathbb{R}$ can be embedded in $\mathbb{E}$.

Indeed, for Hausdorff distance we have the following metric properties: $D : \mathbb{E} \times \mathbb{E} \longrightarrow \mathbb{R}_+ \bigcup 0$,

$$D(u, v) = \sup_{\alpha \in [0,1]} \max\{|\underline{u}(\alpha) - \underline{v}(\alpha)|, |\bar{u}(\alpha) - \bar{v}(\alpha)|\},$$

It is easy to see that D is a metric in $\mathbb{E}$ and has the following properties

(i) $D(u \oplus w, v \oplus w) = D(u, v), \ \forall u, v, w \in \mathbb{E}$,

(ii) $D(k \odot u, k \odot v) = |k| D(u, v), \ \forall k \in \mathbb{R}, u, v \in \mathbb{E}$,

(iii) $D(u \oplus v, w \oplus e) \le D(u, w) + D(v, e), \quad \forall u, v, w \in \mathbb{E}$,

(iv) $(D, \mathbb{E})$ is a complete metric space.

Definition 1.5.1 *Let $x, y \in \mathbb{E}$. If there exists $z \in \mathbb{E}$ such that $x = y \oplus z$, then z is called the H-difference of x and y, and it is denoted by $x \ominus y$.*

In this thesis we consider the following definition of differentiability for fuzzy-valued functions which was introduced by Bede and Gal (2005).

Definition 1.5.2 *Let $f : (a, b) \to \mathbb{E}$ and $x_0 \in (a, b)$. We say that f is strongly generalized differential at x_0. If there exists an element $f'(x) \in \mathbb{E}$, such that*
(i) for all $h > 0$ sufficiently small, $\exists f(x_0 + h) \ominus f(x_0)$, $\exists f(x_0) \ominus f(x_0 - h)$ and the limits(in the metric d)

$$\lim_{h \to 0} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h \to 0} \frac{f(x_0) \ominus f(x_0 - h)}{h} = f'(x_0),$$

(ii) for all $h > 0$ sufficiently small, $\exists f(x_0) \ominus f(x_0 + h)$, $\exists f(x_0 - h) \ominus f(x_0)$ and the limits(in the metric d)

$$\lim_{h \to 0} \frac{f(x_0) \ominus f(x_0 + h)}{-h} = \lim_{h \to 0} \frac{f(x_0 - h) \ominus f(x_0)}{-h} = f'(x_0),$$

(iii) for all $h > 0$ sufficiently small, $\exists f(x_0 + h) \ominus f(x_0)$, $\exists f(x_0 - h) \ominus f(x_0)$ and the limits(in the metric d)

$$\lim_{h\to 0} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h\to 0} \frac{f(x_0 - h) \ominus f(x_0)}{-h} = f'(x_0),$$

(iv) for all $h > 0$ sufficiently small, $\exists f(x_0) \ominus f(x_0 + h)$, $\exists f(x_0) \ominus f(x_0 - h)$ and the limits(in the metric d)

$$\lim_{h\to 0} \frac{f(x_0) \ominus f(x_0 + h)}{-h} = \lim_{h\to 0} \frac{f(x_0) \ominus f(x_0 - h)}{h} = f'(x_0).$$

Remark 1.5.1 *In this thesis, only cases (i)- and (ii)-differentiability in Definition 1.5.2 will be used.*

Theorem 1.5.1 *(see Bede (2006)). Let $f : (a,b) \to \mathbb{E}$ be a function and denote $[F(t)]^\alpha = [f_\alpha(t), g_\alpha(t)]$, for each $\alpha \in [0,1]$. Then*

(1) If f is (i)-differentiable, then $f_\alpha(t)$ and $g_\alpha(t)$ are differentiable functions and

$$[F'(t)]^\alpha = [f'_\alpha(t), g'_\alpha(t)],$$

(2) If f is (ii)-differentiable, then $f_\alpha(t)$ and $g_\alpha(t)$ are differentiable functions and

$$[F'(t)]^\alpha = [g'_\alpha(t), f'_\alpha(t)].$$

1.6 Objectives of the thesis

In this thesis, we propose some new efficient methods for numerically solving linear and nonlinear first order differential equations and the system of first order differential equations with deterministic and fuzzy conditions based on the exponentially-fitted Runge-Kutta method. In particular, the objectives of the thesis are:

1. Applying exponentially-fitted Runge-Kutta method to solve second order differential equations that has been reduced to first order ODE and obtaining the local truncation error and the stability regions.
2. To derive an explicit trigonometrically fitted hybrid method for solving oscillatory second order ordinary differential equations.

3. To propose a reliable algorithm based on Legendre wavelets-spectral method for solving first order and second order nonlinear oscillatory differential equations.

4. To solve first order ordinary differential equations with uncertainty, represented by fuzzy numbers and fuzzy-valued functions, involving characterization theorem and some other new results under generalized Hukuhara differentiability.

5. Solving system of first order fuzzy differential equations under generalized Hukuhara differentiability both exactly and numerically.

1.7 Outline of the thesis

In Chapter I, some preliminaries and basic concepts of Runge-Kutta method are given. Hybrid method and its Phase-lag and stability analysis is also discussed. Chapter I also covers some basic concepts on Legendre wavelets and fuzzy differential equations.

Chapter II is devoted to some discussion on earlier research on Runge-Kutta method, exponentially-fitted Runge-Kutta method, hybrid-type methods, exponentially-fitted hybrid-type methods and Legendre wavelet spectral method and fuzzy differential equations, system of first order fuzzy differential equations.

Chapter III focuses on the derivation of the exponentially Runge-Kutta method using the techniques introduced by Simos (1998) and Berghe et al. (1999) for solving oscillatory first order ordinary differential equations. Stability analysis and the local truncation error of the methods are also given.

In Chapter IV, we develop the trigonometrically fitted hybrid method based on the hybrid method of order five given in Franco (2006) for solving oscillatory second order ordinary differential equations. The stability region of the method when applied to linear second order ODEs is also depicted.

In Chapter VI, we solve first order fuzzy differential equations, firstly the fuzzy differential equation are transformed to ordinary differential equations using characterization theorem. The equations are first order then solved using exponentially-fitted Runge-Kutta method and numerical comparisons with other existing methods are made.

Chapter VII, we solve the fuzzy linear system of first order differential equations analytically under the generalized Hukuhara differentiability. Then the same fuzzy differential equations are solved using exponentially-fitted Runge-Kutta method and comparisons are made between the exact values and the computed values.

Finally, Chapter VIII summarizes the conclusion of the research and recommendation for future research will be suggested.

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